be attributed to the relative amplitudes which are clearly dependent on focusing conditions. Extensive tests were performed to establish that only a minor portion of the incident beam missed the target and that the results shown are characteristic of the reflection from the target and have no other origin.

In the great majority of slow-electron diffraction units operating at normal incidence, the specularly reflected beam cannot be observed because it is lost in the structure of the gun. However, Sproull,[2] using a magnetic deflection method, has studied this beam from the (112) and (100) faces of tungsten. Sproull's analysis was within the framework of conventional diffraction theory and he viewed the specularly reflected beam as a special case (colatitude angle zero) in which the

volume interference condition was satisfied. In this case a specularly reflected spot splits into two spots which diverge along a principal azimuth as the incident energy is raised. There appears no doubt that Sproull's observations on the (112) face fit this interpretation. However, in our experiments, even though we could visually observe the central spot almost continuously for incident energies from 0 to 180 ev, we observed no tendency for this spot to split. We suggest therefore that there exist cases in which the specularly reflected beam should not be considered as a limiting case of beams diffracted at other colatitude angles. The theory of MacColl[3] on the reflection of electrons by metallic crystals appears to be a good starting point for these cases. We have not yet attempted to apply this theory to the results of Fig. 2.

[2] W. T. Sproull, Phys. Rev. 43, 516 (1933).

[3] L. A. MacColl, Bell System Tech. J. 30, 888 (1951).

Cyclotron Resonance and de Haas-van Alphen Oscillations of an Interacting Electron Gas*

WALTER KOHN
University of California at San Diego, La Jolla, California
(Received April 5, 1961)

An electron gas with short-range interactions is considered in the presence of a uniform magnetic field. It is shown that (1) the cyclotron resonance frequency is independent of the interaction; (2) for a two-dimensional gas, the de Haas–van Alphen period is independent of the interaction. The low-lying excited states are briefly discussed.

THERE has been considerable interest in recent months in the effects of the electron-electron interaction on the cyclotron resonance frequency and de Haas–van Alphen oscillations of a gas of electrons. As some of the theoretical treatments of these problems use very sophisticated methods,[1] and others are based on incorrect qualitative reasoning, we wish here to present some simple considerations which we think shed some light on what has been a rather confusing situation.

In the present paper we restrict ourselves to the case of a short range electron-electron interaction, deferring specific effects of the long-range Coulomb force to a later account.

We write the Hamiltonian of our system, in a uniform magnetic field $\mathfrak{K}$ in the z direction, as

$$H = \frac{1}{2m}\sum_{i=1}^{N} \mathbf{P}_i^2 + U, \tag{1}$$

where

$$\mathbf{P}_i = [p_{i,x}, \ p_{i,y} + (e\mathfrak{K}/c)x_i, \ p_{i,z}], \tag{2}$$

and the interaction U is

$$U = \sum_{i,j} u(\mathbf{r}_i - \mathbf{r}_j). \tag{3}$$

CYCLOTRON RESONANCE

We verify directly that if we define the kinetic momentum of the whole system as

$$\mathbf{P} \equiv \sum_i \mathbf{P}_i, \tag{4}$$

then

$$\frac{d\mathbf{P}}{dt} = \frac{i}{\hbar}[H, \mathbf{P}] = -\frac{e}{mc}\mathbf{P} \times \mathfrak{K}, \tag{5}$$

which is the Lorentz equation for the whole system in operator form. We now define

$$P_\pm \equiv P_x \pm iP_y, \tag{6}$$

and the cyclotron frequency,

$$\omega_c = e\mathfrak{K}/mc. \tag{7}$$

Then by (5) we find that

$$[H, P_\pm] = \pm \hbar \omega_c P_\pm. \tag{8}$$

Now let Ψ_0 be the true ground state of the system (or, for that matter, any other eigenstate), and operate on Ψ_0 with Eq. (8). Denoting the energy of Ψ_0 by E_0, we obtain

$$HP_+\Psi_0 - E_0P_+\Psi_0 = \hbar\omega_c P_+\Psi_0. \qquad (9)$$

Hence, if we call

$$\Psi_1 \equiv P_+\Psi_0, \qquad (10)$$

we see that Ψ_1 is an exact excited eigenstate of H with energy

$$E_1 = E_0 + \hbar\omega_c. \qquad (11)$$

Now if the system is placed in a *homogeneous* rotating microwave field,[2] we must add to H the perturbation

$$H' = -\frac{e}{i\omega m}\mathcal{E}_-P_+e^{-i\omega t}. \qquad (12)$$

We see that the perturbation (12) connects the state Ψ_0 with, and only with, the state Ψ_1, so that there results a sharp absorption at the frequency $\omega = \omega_c$.

In summary: Cyclotron resonance is not affected by the interaction U.

THE DE HAAS-VAN ALPHEN EFFECT

For simplicity we restrict ourselves here to a two-dimensional gas of fermions,[3] confined to a rectangle, $L_1 \times L_2$, in the x-y plane.

In the absence of the interaction U, the Hamiltonian (1) becomes the sum of single-particle Hamiltonians,

$$h = \frac{1}{2m}\left[p_x^2 + \left(p_y + \frac{e\mathcal{H}}{c}x\right)^2\right], \qquad (13)$$

whose eigenfunctions are

$$\psi_{n,k} = e^{iky}u_n(x+k/s), \qquad (14)$$

where

$$s \equiv e\mathcal{H}/\hbar c, \qquad (15)$$

and $u_n(x)$ is the nth harmonic oscillator eigenfunction of the Hamiltonian,

$$\frac{1}{2m}[p_x^2 + \hbar^2 s^2 x^2]. \qquad (16)$$

The energies associated with $\psi_{n,k}$ are

$$\epsilon_{n,k} = (n+\tfrac{1}{2})\hbar\omega_c, \qquad (17)$$

and each level has a degeneracy

$$g = (s/\pi)L_1L_2 \qquad (18)$$

associated with the following possible values of k,

$$k = 2\pi/L_2, \quad 4\pi/L_2, \quad \cdots sL_1, \qquad (19)$$

and the two possible spin orientations.

Let N be the total number of particles. Then if $\mathcal{H}$ has a value, denoted by $\mathcal{H}_\nu$, such that

$$\nu g = N, \qquad (20)$$

the ground state of the system will have the ν lowest levels (17) completely filled and all higher ones empty. These values of $\mathcal{H}$ are given by

$$1/\mathcal{H}_\nu = \nu\frac{2\pi e}{\hbar c}\frac{1}{A}, \quad \nu = 1, 2, \cdots, \qquad (21)$$

where $A = 2\pi^2 N/L_1L_2$ is the cross-sectional area of the Fermi "surface."

At absolute zero, all physical properties of the noninteracting gas show a quasi-periodic behavior—with the same period—as function of $1/\mathcal{H}$. For our purposes, we note especially that when $\mathcal{H} = \mathcal{H}_\nu$ the ground state of the system is *isolated*, the lowest excited states being at an energy $\hbar\omega_c$ above it. At intermediate values of $\mathcal{H}$, when the highest occupied single-particle level is not completely occupied, the ground state is not isolated but degenerate.

Now let $\mathcal{H} = \mathcal{H}_\nu$ and imagine the interaction U turned on. Then clearly, to all orders in perturbation theory,[4] the ground state will remain nondegenerate. Therefore, at the same sequence of fields $\mathcal{H}_\nu$ as without interaction, the system passes through the stage of having an isolated ground state.[5]

Hence in summary: The de Haas–van Alphen period of a two-dimensional electron gas is not altered by an interaction, to all orders in perturbation theory.

LOW-LYING EXCITED STATES

We conclude with some remarks concerning the very-low-lying excited states. Consider the case where the field has a value $\mathcal{H}_\nu$, such that the ground state of the whole system is isolated. In the absence of an interaction the first excited level of the system has an energy $\hbar\omega_c$. It corresponds to creating a hole in the previously filled state ν, k and an electron in the excited state $\nu+1$, k'. The energy is independent of the values of k and k', so that this level has a degeneracy g^2. Note that by Eq. (14) the values of k and k' determine where the locations of the hole and electron along the x axis are.

---

[2] The practically important case of an inhomogeneous microwave field will be discussed in a later publication.

[3] For further details of such a gas see F. Seitz, *Modern Theory of Solids* (McGraw-Hill Book Company, New York, 1940), Sec. 183.

[4] Even if perturbation theory should have a vanishing radius of convergence, the nondegeneracy of the ground state—and hence the de Haas–van Alphen period—is unaffected by the interaction. For all that is used is continuity as function of the strength of U, not analyticity.

[5] The situation has a strong formal analogy to the shell structure of atoms and nuclei: A number of particles which is magic (i.e., leads to completely filled shells) in the absence of interparticle interaction, remains magic when the interaction is included.

Now consider the situation when the interaction has been turned on. We note first that the spatial extent D of $u_n(x)$, for $n \approx \nu$, is of the order of

$$D \approx k_F/s, \tag{22}$$

where k_F is the wave number at the top of the Fermi distribution. Hence, by (14), we expect that if

$$k/s - k'/s \gg D, \quad \text{or} \quad k - k' \gg k_F, \tag{23}$$

the hole and electron will be outside each other's range of interaction. For in this case their wave functions do not overlap, and we are postulating short-range forces whose range R will be negligible compared to D, for the fields of interest ($s \ll k_F^{-2}$). For such an electron-hole pair we must have the excitation energy

$$\Delta E = \epsilon_{\nu+1}{}^{(e)} - \epsilon_\nu{}^{(h)}, \tag{24}$$

where the ϵ's are defined as follows

$$\begin{aligned}
\epsilon_{\nu+1}{}^{(e)} &= E_0(N+1) - E_0(N), \\
\epsilon_\nu{}^{(h)} &= E_0(N) - E_0(N-1),
\end{aligned} \tag{25}$$

$E_0(M)$ denoting the ground-state energy for a total number of M electrons. That is, the excitation energy of the pair is the sum of the individual quasi-electron and quasi-hole energies.

The number of electron-hole pairs for which (23) does *not* hold is by (19) and (23) of the order of $g k_F L_2$ which is negligible compared to the total number of electron-hole pairs, namely $g^2 = g(L_1 s/\pi)L_2$. Hence, for the overwhelming number of electron-hole pairs, the excitation energy is practically given by ΔE.

Now we know from our discussion of cyclotron resonance that there is an excited state with energy $\hbar\omega_c$ above the ground state. However, we shall see that this state is one of the "exceptional" ones and that in general $\Delta E \neq \hbar\omega_c$.

Consider the "cyclotron state" in the absence of the perturbation U. It is obtained by operating on the ground state with the operator

$$P_+ = \sum_i \left[p_{i,x} + i \left(p_{i,y} + \frac{e\mathcal{3C}}{c} x_i \right) \right]. \tag{26}$$

Denoting the state with a hole in ν, k and an electron in the state $\nu+1$, k' by $\Psi_{k,k'}$, we find

$$P_+\Psi_0 = \text{const}\sum_k \Psi_{k,k}. \tag{27}$$

Thus we see that this state is a coherent linear combination of states in which the electron and hole are in the same location. When the interaction U is turned on, the energy of this state remains $\hbar\omega_c$, by our previous argument, but will be shifted relative to those states in which the electron and hole are far apart.

Slightly more elaborate considerations show that in fact the level which is the lowest excited level when $U=0$, broadens into a band when U is turned on. This band has a very high density near $E=E_0+\Delta E$, but also encompasses the cyclotron level at $E=E_0+\hbar\omega_c$. A study of the detailed nature of this band is in progress.