Analytic Properties and Rescattering Correction to the Born Approximation for Transition Matrix Elements*

B. Bosco†
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California

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In this paper the analytic properties of a matrix element of a general operator between a bound state and a scattering state are studied in the framework of Schrödinger theory. It is shown that the singularities of such a matrix element are easily inferred from those of the Born approximation. Finally, using the fact that the possible singularities which are not contained in the Born approximation are located far apart from those included in the lowest approximation, a simple formula is derived which allows one to obtain the rescattering correction to the Born approximation using the phase shifts explicitly.

I. INTRODUCTION

THE purpose of this paper is to point out some properties of the transition matrix elements in Schrödinger theory. These properties, which are the consequence of the analyticity of the matrix element in the energy variable of the final particles, may be interesting in different respects. For example, from these properties, relations can be deduced which display the close connection between the matrix element for the photodisintegration of a nucleus and the scattering of the resulting particles. These relations also exhibit to high degree the analogy between similar phenomena in wave mechanics and in field theory. For example, the equations for the photodisintegration of nuclei are structurally very similar to the Chew-Low equations for pion photoproduction. It is well known that the photoproduction amplitude has two branch cuts; one on the real positive axis which is connected with the scattering, and one on the negative axis which is connected to the crossing symmetry. Analogously, the photodisintegration amplitude has similar cuts; but the cut due to the crossing symmetry has to be replaced by cuts located in other regions and due to the anomalous thresholds.

The locations of these singularities and their connections with the anomalous thresholds have been recently studied for the model of the photodisintegration of a scalar deuteron.[1] We shall see that for every matrix element of two-body disintegration, the singularities due to final-state interaction can easily be factored out and the remaining singularities can be foreseen by simple inspection of the Born approximation. Of course, the type of such singularities, if they exist, depends upon the particular matrix elements in question, but, and this is a very important point, they consist in isolated singularities contained in the Born approximation and in lines of singularities of the same structure whose threshold is beyond the isolated singularities by a distance which is the inverse of the range of the nuclear force. In the nuclear phenomena such a distance is generally very large, and the lines of singularity discussed above are therefore usually far apart from the region of interest; so one is led to the approximation in which these singularities are neglected. In this approximation an integral equation can be obtained for the matrix element in terms of the scattering of final particles.

We further notice that in the case in which one is not allowed to make such an approximation, also, the approach from the point of view of the dispersion relations is very useful, since better approximations can be suggested, as we shall discuss later.

II. DEFINITIONS AND GENERAL REMARKS

In this section we state the definition of the matrix element with which we have to deal, and some observations from which our statement that the singularities due to the final-state interaction factorized out will follow at once. The matrix element we will consider is defined as[2]

$$M_k{}^l(q)=(\Psi_B(r),O(k,r)\Psi_l{}^{\text{out}}(q,r)),\tag{1}$$

where $\Psi_B(r)$ is a bound-state wave function; $O(k,r)$ is a general operator; and $\Psi_l{}^{\text{out}}(q,r)$ is a two-body wave function belonging to the continuum spectrum defined by the asymptotic behavior

$$\Psi_l{}^{\text{out}}(q,r)\sim\frac{e^{i\delta_l}}{qr}\sin(qr-\tfrac{1}{2}\pi l+\delta_l)\quad\text{as}\quad r\rightarrow\infty,\tag{2}$$

where δ_l is the scattering phase shift of the final particles in the lth angular momentum state.

In definition (1) we have explicitly indicated the dependence of k like a parameter since in the following we will be interested in the q dependence only. The main point to be noticed is that $r\Psi_l{}^{\text{out}}(q,r)$ can be written as a ratio of two functions which exhibit very simple analytic properties[3]:

$$r\Psi_l{}^{\text{out}}(q,r)=[q^l/f_l(-q)]\varphi_l(q^2,r),\tag{3}$$

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† Permanent address: Istituto di Fisica dell'Università, Torino, Italy.

[1] V. De Alfaro and C. Rossetti, Nuovo cimento **18**, 783 (1960).

[2] In the literature this matrix element is sometimes defined in a different way, i.e., $M_k{}^l(q)=\int\psi_e{}^{\text{in}*}(q,r),\,O(k,r)\psi_B(r)rdr$.

[3] R. Newton, J. Math. Phys. **1**, 319 (1960).

where $\varphi_l(q^2,r)$ is for every value of r an entire function of $q^2=E$,[4] and $f_l(q)$ has the following properties[3,5]:

(a) It is a regular analytic function when $\mathrm{Im}q<0$.

(b) The only zeros it has are on the imaginary negative axis and correspond to the bound states.

(c) As q approaches infinity on the lower half plane and on the real axis, $f_l(q)$ approaches one.

From Eq. (3) and the properties of φ_l and f_l we can immediately see that if we write

$$M_k{}^l(q)=\frac{R^l(k,q)}{f_l(-q)}, \tag{4}$$

all the singularities which are due to the final-state interactions (for example, poles due to bound states in the final-state system) are completely included in $f_l(-q)$ since the function φ_l which determines $R^l(k,q)$ is an entire function. We wish, however, to emphasize that by no means can one assert that the function $R^l(k,q)$ is an analytic function of q^2 in the complete plane; the point is that if $R^l(k,q)$ has singularities, they come from phenomena which are different from the final-state interaction. We also will notice that very weak assumptions are needed on the potential in order that the functions φ_l and $f_l(q)$ with the stated properties exist.[6] We will see in the next section that we have a direct method of analyzing the function $R^l(k,q)$ only if we restrict ourselves to a particular class of potentials. Finally, there is a very important remark to be made. Usually $R^l(k,q)$ is real, since, as we shall see in the next section, φ_l is real, and also $O(k,r)$ is generally real. On the other hand,[7]

$$f_l(q)=|f_l(q)|\exp i\delta_l(q). \tag{5}$$

From (4) and (5) it follows that

$$M_k{}^l(q)=\rho(k,q^2)e^{i\delta_l(q)}, \tag{6}$$

where $\rho(k,q)$ is a real function of the two variables k and q^2. Equation (6) is still valid under the more general condition that the operator $O(k,r)$ is only invariant under time reversal. The proof is given in Appendix A. Equation (6) is the expression for the matrix element $M_k{}^l(q)$ of the so-called "final state theorem".

It is worthwhile to observe that in Schrödinger theory this is a rigorous statement, while in second quantization due to the possibility of creation of new particles, its validity is limited to the energies for which such a creation is impossible.

[4] Our system of units is such that $\hbar=M=1$.

[5] R. Jost and W. Kohn, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, 9 (1953).

[6] See for example reference 3. Indeed, the weakest assumption one can make for such existence is that the first and second moments of the potential exist.

[7] See for example reference 3.

III. SINGULARITIES OF $R_l(k,q)$

In this section we analyze what kind of singularities, if any, are associated with the functions $R_l(k,q)$. We recall that from (1) and (3) R_l is defined:

$$R_l(k,q^2)=q^l\int_0^\infty rdr\,\Psi_B(r)O(k,r)\varphi_l(q^2,r). \tag{7}$$

It is also very important at this point to express $\varphi_l(q^2,r)$ by means of the Jost functions[3]:

$$\varphi_l(q^2,r)=\tfrac{1}{2}iq^{-l-1}\times[f_l(-q)f_l(q,r)-(-)^l f_l(q)f_l(-q,r)], \tag{8}$$

where $f_l(\pm q,r)$ are solutions of the Schrödinger equations defined by the boundary conditions:

$$\lim_{r\to\infty}e^{iqr}f_l(q,r)=i^l, \tag{9}$$

and

$$f_l(q)=\lim_{r\to 0}(qr)^l\frac{f_l(q,r)}{(2l-1)!!}.$$

We are therefore led to study the analytic properties of the function

$$F_l(k;q)=\int_0^\infty rdr\,\psi_B(r)O(k;r)f_l(q,r), \tag{10}$$

since from (7) and (8) we have

$$R_l(k,q^2)=\frac{i}{2}q^{-1}[f_l(-q)F_l(k;q)-(-)^l f_l(q)F_l(k,-q)]. \tag{11}$$

It is possible to analyze $F_l(k;q)$ if we limit ourselves to the case in which the potential which generates the scattering is a continuous superposition of exponential functions:

$$V(r)=\int_0^\infty \rho(\alpha)e^{-\alpha r}d\alpha,\quad \rho(\alpha)=0,\quad \alpha<\mu. \tag{12}$$

In such a case, which is a very important one from the physical point of view, a method which was first described by Martin[8] for the continuum spectrum, and for the bound states by Bertocchi *et al.*[9] can be usefully applied. For the sake of completeness we recall here briefly the Martin method and its extension to the bound-state wave function. For simplicity we limit ourselves to the S wave.

Let us consider the function $f_0(q,r)$; it is a solution of the scattering equation

$$f_0''(q,r)+q^2f_0(q,r)=V(r)f_0(q,r), \tag{13}$$

and satisfies the boundary condition

$$\lim_{r\to\infty}e^{iqr}f_0(q,r)=1. \tag{14}$$

[8] A. Martin, Nuovo cimento **14**, 403 (1959).

[9] L. Bertocchi, C. Ceolin, and M. Tonin, Nuovo cimento **18**, 770 (1960).

Let us write

$$f_0(q,r)=\Lambda(q,r)e^{-iqr}. \tag{15}$$

Then (14) implies:

$$\lim_{r\to\infty}\Lambda(q,r)=1. \tag{16}$$

From the ansatz (15) one gets for $\Lambda(q,r)$:

$$\Lambda''(q,r)-2iq\Lambda'(q,r)=V(r)\Lambda(q,r), \tag{17}$$

where $V(r)$ now has expression (12).

We now try to find a solution of (17) of the form:

$$\Lambda(q,r)=\int_0^\infty g(q,\sigma)e^{-\sigma r}d\sigma. \tag{18}$$

By substitution of (18) into (17) and after identification of the terms, we get

$$g(q,\sigma)[\sigma^2-2iq\sigma]=\int_0^{\sigma-\mu} d\alpha\, g(q,\alpha)\rho(\sigma-\alpha), \tag{19}$$

or

$$g(q,\sigma)=\delta(\sigma)+\frac{1}{\sigma(\sigma-2iq)}\int_0^{\sigma-\mu} g(q,\alpha)\rho(\sigma-\alpha)d\alpha, \tag{20}$$

the constant which multiplies the δ function having been determined by the boundary condition (16). We will not discuss such a solution in further detail; what is important to notice is that the support of the function g is the point $\sigma=0$ and the continuum $\sigma\geqslant\mu$. The bound state can be handled in a similar way. Let us suppose it is an S bound state; then $r\psi_B(r)$ is a solution of the Schrödinger equation:

$$[r\psi_B(r)]''-\chi^2[r\psi_B(r)]=V(r)[r\psi_B(r)], \tag{21}$$

where χ^2 is the binding energy of the bound state. We suppose that the potential is again given by (12).[10] If we attempt a solution of the form

$$r\psi_B(r)=\int_0^\infty G(\eta)e^{-\eta r}d\eta, \tag{22}$$

we get for $G(\eta)$ the equation

$$G(\eta)=N\delta(\eta-\chi)+\frac{1}{\eta^2-\chi^2}\int_0^{\eta-\mu} d\beta G(\beta)\rho(\eta-\beta). \tag{23}$$

Again we note that the support of the function $G(\eta)$ is the point $\eta=\chi$ and the continuum $\eta\geq\chi+\mu$. This brief analysis is sufficient for discussing the analytic properties of the function $F_0(k;q)$. From Eq. (10) we see that if we define

$$A^k(\alpha;q)=\int_0^\infty \exp[-(\alpha+iq)r]O(k;r)dr, \tag{24}$$

it follows from Eqs. (10), (19), and (23)

$$F_0(k;q)=NA^k(\chi;q)+N\int_{\chi+\mu}^\infty g(q;\eta)A^k(\eta;q)d\eta$$
$$+\int_{\chi+\mu}^\infty G(\eta)A^k(\eta;q)d\eta$$
$$+\int_{\chi+\mu}^\infty d\sigma\, G(\sigma)\int_\mu^\infty g(q,\eta)A^k(\eta+\sigma;q)d\eta. \tag{25}$$

This equation contains all we need in order to locate the singularities of $F_0(k;q)$. $F_0(k;q)$ will contain two kinds of singularities: those which are in the spectral functions g and G, and those which came from $A^k(\alpha;q)$. Since the integrals involving G start at $\chi+\mu$, from (23) we see that there are no singularities coming from this spectral function. Equation (20) tells us that g has a cut which starts at $q=i\mu/2$ and goes to $i\infty$. This is a well-known singularity of the function $f_0(q,r)$. As far as the singularities of $A^k(\alpha;q)$ are concerned, we can deduce from (25) the following rules (a) to compute explicitly $A^k(\alpha;q)$, that is, $F_0(k,q)$, using the zero-range approximation for the bound-state wave function and the Born approximation for the continuum wave function $f_0(q,r)$, and (b) to analyze the possible singularities of $A^k(\alpha;q)$.

Suppose $A^k(\alpha;q)$ has n singular points. Since $A^k(\alpha;q)$ is a function of only the two arguments $\alpha+iq$ and k, these singular points q_ν will be given by the n relations:

$$\alpha+iq_\nu=\tau_\nu(k),\quad \nu=1,2\cdots n. \tag{26}$$

Then Eq. (25) tells us that $F_0(k;q)$ will have, in addition, lines of the same singularities which are given by the equations:

$$(\alpha+\xi)+iq=\tau_\nu(k), \tag{26'}$$

with ξ running from μ to ∞.

We wish to remark here that the singularities due to the spectral function g and G are of quite different origin from those which are contained in $A^k(\alpha;q)$, since the latter depends explicitly on the form of the operator $O(k,r)$. Going back to Eq. (11) we see that $R_0(k;q)$ is defined by means of a function which is regular and analytic in the whole plane (φ_l is an entire function). Therefore, the singularities due to the spectral function cannot be present in $R_0(k;q)$. Therefore we can say that the singularities in the Born approximation determine completely the singularities of the exact matrix element. Furthermore, we see from (11) that, given the singularities of the type discussed above, similar ones exist which can be obtained from Eqs. (26) and (26′) by changing q to $-q$. We proved our results only in the

[10] We wish to remark here that there is neither obligation nor need that the potential which generates the bound state be the same as that which generates the scattering. The only reason we choose the same potential in both phenomena is for more simplicity.

case of S waves. However, these results can be generalized to any angular momentum.[11]

IV. DISPERSION RELATION FOR THE MATRIX ELEMENT

The results derived in the previous section enable us to write down immediately the dispersion relation, using the Cauchy theorem. Let us call $B_k{}^l(q)$ the Born approximation, i.e., $\lim_{q\to\infty} M_k{}^l(q)$. The function $M_k{}^l(q) - B_k{}^l(q)$ will be regular and analytic in the upper half plane if we exclude the points on the imaginary axis for which $f_0(-i\alpha_\rho)=0$ and we cut the plane along the lines $i[\chi+\xi-\tau_\nu]$ with ξ running from μ to ∞. The points for which $f_0(-i\alpha_\rho)=0$ represent the poles due to bound states of the final particles. We can therefore apply the Cauchy theorem to the path of Fig. 1 and we obtain

$$\begin{aligned} M_k{}^l(q)-B_k{}^l(q) &= -\frac{1}{2\pi i}\sum_\rho \oint_{q'=i\alpha_\rho} \frac{M_k{}^l(q')dq'}{q'-q} \\ &+\frac{1}{2\pi i}\sum_\nu \int_{C_\nu} \frac{M_k{}^l(q')-B_k{}^l(q')}{q'-q}dq' \\ &+\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{M_k{}^l(q')-B_k{}^l(q')}{q'-q-i\epsilon}dq', \quad (27)\end{aligned}$$

since $B_k{}^l(q')$ has no poles due to the final-particle bound states and therefore does not contribute to the first integral in the right-hand side. Furthermore, $B_k{}^l(q)$ is real, and for q real we also have

$$M_k{}^{l*}(-q)=M_k{}^l(q)(-)^l. \quad (28)$$

If l is even,

$$\begin{aligned} M_k{}^l(q) &= B_k{}^l(q)+\mathrm{Im}\left\{\frac{1}{\pi}\sum_\rho \oint_{q'=i\alpha_\rho} \frac{M_k{}^l(q')dq'}{q'-q}\right\} \\ &+\mathrm{Im}\left\{\frac{1}{\pi}\sum_\nu \int_{q_\nu+i\mu}^{q_\nu+i\infty} \frac{[M_k{}^l(q')-B_k{}^l(q')]dq'}{q'-q}\right\} \\ &+\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}M_k{}^l(q')dq'^2}{q'^2-q^2-i\epsilon}. \quad (29)\end{aligned}$$

If l is odd, one gets a similar equation, the only change being that in the last integral dq'^2 has to be changed to $lqdq'$. In this equation $q_\nu = i[\chi-\tau_\nu(k)]$ and $[M_k{}^l(q') - B_k{}^l(q')]$ means the discontinuity of $M_k{}^l(q')-B_k{}^l(q')$ across the line of integration. The contour integrals around the poles can be expressed by means of the bound-state wave functions of the final system. This will be shown in Appendix B. Relation (29) as it stands may be of very little use since we have no way of knowing the discontinuity of $M_k{}^l(q)$. However, it may be used in an approximate way as we will see in the next section.

[11] It is only necessary to use the more complicated presentation derived in (1) for higher angular momenta.

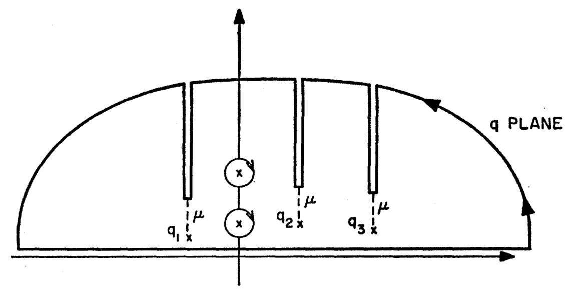

FIG. 1. Path for application of Cauchy theorem.

V. RESCATTERING CORRECTIONS TO THE BORN APPROXIMATION

The results obtained so far may be useful for practical purposes. Indeed, we have learned that if singularities of the matrix element exist, they consist of the points $q_\nu{}^\pm = \pm i[\chi-\tau_\nu(k)]$ and of the lines $q_\nu{}^\pm \pm i\xi$ with ξ running from μ to ∞ where μ, we recall, is the inverse of the range of the force which generates the scattering. In many problems such a range is very short, and so the lines of singularities start far from the singular points. In these cases we are naturally led to the approximation of neglecting such lines of singularities as far apart from the physical region of interest. In such an approximation Eq. (29) becomes an integral equation for the matrix element. Indeed, the discontinuity across the positive real axis can be evaluated using the property stated by Eq. (6). One finds:

$$\mathrm{Im}M_k{}^l(q)=\rho^l(q,k)\sin\delta_l = M_k{}^l(q)h_l{}^*(q), \quad (30)$$

where

$$h_l(q)=\exp[i\delta_l(q)]\sin\delta_l(q),$$

and therefore one obtains for $M_k{}^l(q)$ the singular integral equation (we limit ourselves to l even and no final bound state; obvious changes are needed if l is odd; if bound states are present, one has to add the pole terms to the Born term).

$$M_k{}^l(q)=B_k{}^l(q)+\frac{1}{\pi}\int_0^\infty \frac{M_k{}^l(q')h^*(q')dq'^2}{q'^2-q^2-i\epsilon}, \quad (31)$$

whose solution reads[12]:

$$\begin{aligned} M_k{}^l(E) = \Bigg\{ B_k{}^l(E)\cos\delta_l(E) + \frac{1}{\pi}e^{\rho(E)}P \\ \times\int_0^\infty \frac{B_k{}^l(E')\sin\delta_l(E')e^{-\rho(E')}dE'}{E'-E}\Bigg\} \\ \times\exp[i\delta_l(E)], \quad (32)\end{aligned}$$

[12] R. Omnes, Nuovo cimento 8, 316 (1958).

where $E=q^2$,

$$\rho(E)=\frac{1}{\pi}P\int_0^\infty \frac{\delta_l(E')}{E'-E}dE'.$$

The meaning of this solution is evident. It takes into account the rescattering of the particles emerging in the disintegration. The validity of such a formula is of course limited to those processes for which the unphysical cuts are not very important.

VI. CONCLUSIONS

We have studied the analytic properties of the transition matrix elements between a bound state and the continuum for a general operator, and from these analytic properties we have derived an expression for the rescattering corrections to Born approximations. In order to derive such an expression we were obliged to neglect certain cuts on which we were unable to determine the discontinuity of the matrix element. Fortunately, we found that the cuts we neglect start at a distance μ from the nearest singularities, μ being the inverse of the range of the force which generates the scattering, and so in the problems involving nuclear forces they start quite apart from the physical region. We wish to point out, however, that also in the case in which one cannot neglect the unphysical cuts, Eq. (29) may be very useful in suggesting other kinds of approximations: For example, one can introduce some phenomenological constants in the problem, representing the cuts by means of one or more poles. Finally, it is worthwhile to illustrate what kinds of phenomena the unphysical cuts represent. It is clear that such a specific aspect can only be reached through specific examples. The particular model of a scalar two-body photodisintegration was studied in some detail from such a point of view.[1] The result is that the unphysical cuts are connected with the anomalous thresholds. Since the argument used in this investigation can be applied to any operator representing a physical interaction, we can conclude that the connection between unphysical cuts in the transition matrix element and anomalous thresholds is quite general.

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APPENDIX A

We will give here the proof that in the case the operator $O(k,r)$ is invariant under time reversal, the function $R_l(k,q)$ defined by Eq. (7) is a real function. In fact, recalling that by definition (8) φ_l is only a function of q^2, we have

$$R_l^*(k,q^2)=q^l\int_0^\infty rdr\,\psi_B^*(r)O^*(k,r)\varphi_l^*(q^2,r).$$

Now if U is a unitary transformation such that[13]:

$$UO^*(k,r)U^{-1}=O(k,r),$$

then

$$R_l^*(k,q)=R_l(k,q),$$

since

$$U\varphi_l^*(q^2,r)=\varphi_l^{\text{rev}}(q^2,r)=\varphi_l(q^2,r),$$
$$U\psi_B^*(r)=\psi_B^{\text{rev}}(r)=\psi_B(r).$$

APPENDIX B

We wish to outline briefly here how to express the circular integrals around poles due to bound states of the final particles as expectation values of the operator $O(k,r)$ between the initial and final bound-state wave functions.

Let us recall that

$$M_k^l(q)=\int\psi_B(r),O(k,r)\psi_l^{\text{out}}(q,r)rdr=R_l(k,q)/f_l(-q). \tag{B.1}$$

Since $R_l(k,q)$ is regular in the point $q=i\chi_\rho$ with χ_ρ real and positive in which a bound state exists, we have:

$$\frac{1}{\pi i}\oint\frac{M_k^l(q')dq'}{q'-q}=-\frac{2R_l(k,i\chi_\rho)}{i\chi_\rho-q}\frac{1}{\dot{f}_l(-i\chi_\rho)}, \tag{B.2}$$

in which use has been made of the relation

$$f_l(-i\chi_\rho)=0, \tag{B.3}$$

and of the notation:

$$\dot{f}_l(-i\chi_\rho)=\left[\frac{df_l(q)}{dq}\right]_{q=-i\chi_\rho}.$$

We use now the well-known relation[14]:

$$f_l(i\chi_\rho)/\dot{f}_l(-i\chi_\rho)=iN_{l,\rho}^2, \tag{B.4}$$

where

$$N_{l,\rho}^{-2}=\int_0^\infty f_l^2(-i\chi_\rho,r)dr.$$

From Eqs. (7), (8), and (B.3) it follows that

$$\frac{R_l(k,i\chi_\rho)}{\dot{f}_l(-i\chi_\rho)}=\tfrac{1}{2}(-)^lN_{l,\rho}\int\psi_B(r),\,O(k,r)\psi_{\chi_\rho}{}^l(r)rdr.$$

[13] See, for example, J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, New York, 1952), p. 525.

[14] This relation can be obtained, for example, using Eqs. (4.20) and (4.21′) of reference 3.