This differs from the *usual* expression, which includes the core interaction $\langle \eta_a \phi_b | U_b | \phi_a \eta_b \rangle$ as well. This usual result is obtained if, in the first term of Eq. (10) (which we know to be identically zero), we replace $\chi_a^{(+)}$ by its Born approximation ϕ_a. In this approximation, this term is no longer zero, since η_a and ϕ_a are eigenfunctions of different Hamiltonians. The result is just the extra term found by perturbation techniques. It is clear that this term should *not* be present, and that only the inadequacies of the perturbation approach have led to its appearance. Some of the virtues of an approach in which the exact expression is obtained before approximations are made can be seen from this example.

We may also observe that the so-called "post-prior" discrepancy has evaporated. This discrepancy arises when the ordinary (but incorrect) perturbation result is used, since then either U_a or U_b may enter the expression for $M_R{}^{BA}$. While formally they give identical results, that is $\langle \eta_a \phi_b | U_a | \phi_a \eta_b \rangle = \langle \eta_a \phi_b | U_b | \phi_a \eta_b \rangle$; when approximate bound-state wave functions are introduced into these matrix elements, the equality no longer holds. We now see that this term should not be present at all and that no problem exists.

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Asymptotic Behavior and Subtractions in the Mandelstam Representation*

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It is proved that a two-body reaction amplitude involving scalar particles and satisfying Mandelstam's representation is bounded by expressions of the form $Cs\ln^2 s$ at the forward and backward angles, and $Cs^{3/4}\ln^{3/2}s$ at any other fixed angle in the physical region, C being a constant, s being the total squared c.m. energy. This corresponds to cross sections increasing at most like $\ln^2 s$. These restrictions limit the freedom of choice of the subtraction terms to six arbitrary single spectral functions and one subtraction constant.

I. INTRODUCTION

SINCE the time Mandelstam[1] discovered his representation for two-body reaction amplitudes, there has been in general a little confusion about the question of asymptotic behavior of the different quantities as the energy variables go to infinity.

We shall point out in this paper a number of facts, which, we hope, will help to clarify these questions.

In Sec. II, we derive, from the Mandelstam representation and from a very weakened form of the unitarity condition, an upper bound on the asymptotic behavior of the amplitude in the physical regions.

In Sec. III, we show that these results cannot give us any indication on the behavior of the double spectral function.

In Sec. IV, we write down a general form for the subtracted double dispersion relation, which will prove convenient for the following.

In Sec. V, we investigate the question whether the subtraction constants and the single spectral functions can be determined from the asymptotic conditions which we derived in Sec. II.

II. ASYMPTOTIC PROPERTIES OF THE AMPLITUDE IN THE PHYSICAL REGION

We consider a reaction of the type $a+b \rightarrow c+d$ among scalar particles. We denote by p_1, p_2, $-p_3$, $-p_4$ the momenta of the particles a, b, c, and d, respectively. We introduce the notations $s=(p_1+p_2)^2$; $t=(p_2+p_3)^2$; $u=(p_3+p_1)^2$. Then

$$s+t+u=p_1^2+p_2^2+p_3^2+p_4^2.$$

We shall assume that all masses are equal to the unit of mass as we deal only with asymptotic properties, where the difference between the masses is negligible. Then: $s+t+u=4$. We call channel s the above reaction $a+b \rightarrow c+d$, channel t the reaction $b+\bar{c} \rightarrow \bar{a}+d$ and channel u the reaction $a+\bar{c} \rightarrow \bar{b}+d$. In the channel s, the momentum of one particle in the c.m. system is given by $\mathbf{q}_s^2=(s-4)/4$ and the reaction angle will be defined by $\cos\theta_s = 1+(t/2q_s^2)$. The physical region for channel s will be given by

$$\mathbf{q}_s^2>0, \quad |\cos\theta_s|<1; \quad \text{or} \quad s>4, \quad t\leq 0, \quad u\leq 0.$$

We define the notations in the other channels by a circular permutation among (s,t,u).

In order that the double dispersion integrals make sense, we have to require that the double spectral functions be tempered distributions, and similarly, we require that the single spectral functions be also tempered distributions.

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[1] S. Mandelstam, Phys. Rev. 112, 1344 (1958).

Then the amplitude in the s channel is a distribution in s, analytic in t in a domain including the physical region. In order to simplify the language, we adopt the following convention: We shall say that a distribution T behaves at most like a function f at infinity if T/f is bounded in the sense of distributions, or equivalently,[2] if every regularized of T is bounded by some multiple of f in modulus. According to this convention, a tempered distribution is a distribution which behaves at most like some polynomial at infinity.

To get an intuitive idea why the amplitude is bounded in the physical region, let us consider a classical problem: Two particles interact by means of an absorptive Yukawa potential $ge^{-\kappa r}/r$. If a is the impact parameter, the total interaction seen by a particle for large a is likely to be approximately $ge^{-\kappa a}$. If this is small compared to one, there will be practically no scattering. If $|ge^{-\kappa a}|$ is large compared to one, there will be practically complete scattering, so that the cross section will be essentially determined by the value $a=(1/\kappa)\ln|g|$ where $|ge^{-\kappa a}|=1$. It is $\sigma\cong(\pi/\kappa^2)\ln^2|g|$. If we now assume that g is a function of the energy, and increases like a power of the energy, then σ will vary at most like the squared logarithm of the energy.

In the Mandelstam representation the modulus of the spectral function is somehow equivalent to some strength of potential, and it varies at most like some power of the energy. Thus it is natural to expect that this behavior of the total cross section will be also an upper limit for the reactions which satisfy the double dispersion relation. To prove it, let us consider a dispersion relation at fixed s.

$$A(s,\cos\theta_s)=\frac{1}{\pi}(\cos\theta_s)^N\int_{x_1}^{\infty}\frac{\rho(s,x)dx}{x^N(x-\cos\theta_s)}+\frac{1}{\pi}(\cos\theta_s)^N\int_{x_2}^{\infty}\frac{\rho'(s,x)dx}{x^N(x+\cos\theta_s)}+\frac{1}{\pi}\sum_{p=0}^{N-1}\rho_p\cos^p\theta_s.$$

$\rho(s,x)$, $\rho'(s,x)$ are the absorptive parts in the crossed channels, for values of t (or u) $=2q^2(1+x)$. Accordingly $x_1=1+(t_0/2q^2)$, t_0 being the threshold of the absorptive part in the t channel, and $x_2=1+(u_0/2q^2)$, u_0 being the threshold in the u channel. We put $x_0=\min(x_1,x_2)$ and write $x_0=1+(\kappa^2/2q^2)$, where κ^2 is a given constant. This equation has in general to be understood as regularized over a small interval of values of s, as $\rho(s,x)$ and $\rho'(s,x)$ are distributions. Let us compute the partial wave amplitudes:

$$A(s,\cos\theta_s)=\frac{\sqrt{s}}{\pi q_s}\sum_{l=0}^{\infty}a_l(s)(2l+1)P_l(\cos\theta_s),$$

$$a_l(s)=\frac{\pi q}{2\sqrt{s}}\int_{-1}^{+1}A(s,\cos\theta_s)P_l(\cos\theta_s)d(\cos\theta_s),$$

$$a_l(s)=\frac{q}{2\sqrt{s}}\int_{-1}^{+1}d\cos\theta_s\,P_l(\cos\theta_s)\times\left\{\int_{x_0}^{\infty}\frac{\cos^N\theta}{x^N}\left[\frac{\rho(s,x)}{x-\cos\theta}+\frac{q'(s,x)}{x+\cos\theta}\right]dx+\sum_{p=0}^{N-1}\rho_p\cos^p\theta\right\}.$$

If we interchange the order of integration, which is permissible if N is large enough, we find, for $l\geq N$:

$$a_l(s)=\frac{q}{\sqrt{s}}\int_{x_0}^{\infty}Q_l(x)[\rho(s,x)+(-1)^l\rho'(s,x)]dx. \quad (1)$$

We want to find here again the exponential decrease of a_l for large values of l. To do that, we use a little trick in order to get rid of the Legendre function of the second kind Q_l.

We use the generating function of the Q_l's.[3]

$$\sum_{l=0}^{\infty}z^lQ_l(x)=(1-2zx+z^2)^{-\frac{1}{2}}\operatorname{arc\,cosh}\left\{\frac{x-z}{(x^2-1)^{\frac{1}{2}}}\right\}=\int_{x+(x^2-1)^{\frac{1}{2}}}^{\infty}(1-2\zeta x+\zeta^2)^{-\frac{1}{2}}\frac{d\zeta}{\zeta-z}.$$

From this, we can deduce

$$\sum_{l=N}^{\infty}z^lQ_l(x)=z^N\int_{x+(x^2-1)^{\frac{1}{2}}}^{\infty}(1-2\zeta x+\zeta^2)^{-\frac{1}{2}}\frac{d\zeta}{\zeta^N(\zeta-z)}.$$

We have thus, from (1):

$$\sum_{l=N}^{\infty}a_lz^l=\frac{q}{\sqrt{s}}z^N\int_{x_0}^{\infty}dx\int_{x+(x^2-1)^{\frac{1}{2}}}^{\infty}d\zeta\times[\rho(s,x)+(-1)^l\rho'(s,x)](1-2\zeta x+\zeta^2)^{-\frac{1}{2}}\frac{1}{\zeta^N(\zeta-z)}.$$

We can interchange the order of integration, and we get

$$\sum_{l=N}^{\infty}a_lz^l=\frac{q}{\sqrt{s}}z^N\int_{x_0+(x_0^2-1)^{\frac{1}{2}}}^{\infty}\frac{d\zeta[\sigma(s,\zeta)+(-1)^l\sigma'(s,\zeta)]}{(\zeta-z)\zeta^N}. \quad (2)$$

[2] L. Schwartz, *Théorie des distributions* (Hermann & Cie, Paris, 1951), Vol. II, Théorème XXV 2°, p. 57, case $p=\infty$.

[3] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1952), p. 321. The $(z^2-1)^{\frac{1}{2}}$ in this reference has a determination such that $(z^2-1)^{\frac{1}{2}}/z\to-1$ as $z\to\infty$. We have taken the other one as being more natural.

As

$$\sigma(s,\zeta)=\int_{x_0}^{\frac{1}{2}(\zeta+1/\zeta)}\rho(s,x)\frac{dx}{(1-2\zeta x+\zeta^2)^{\frac{1}{2}}},$$

we see that it behaves at most like a polynomial at infinity, just like $\rho(s,x)$.

Now, it is easy to calculate a_l from (2) for $l\geq N$:

$$a_l=\frac{q}{\sqrt{s}}\int_{x_0+(x_0^2-1)^{\frac{1}{2}}}\frac{d\zeta[\sigma(s,\zeta)+(-1)^l\sigma'(s,\zeta)]}{\zeta^{l+1}}. \quad (3)$$

This allows us to put a bound on the behavior of a_l; $\sigma(s,\zeta)$ can be written[2] as

$$\sigma(s,\zeta)=\frac{1}{2}\frac{\partial^M}{\partial\zeta^M}\zeta^{N+M}g(\zeta)B(s),$$

where $g(s,\zeta)$ is a function bounded by unity, and M a sufficiently large fixed integer. $\sigma'(s,\zeta)$ can have a similar expression, and finally:

$$|a_l|\leq\frac{qB(s)}{\sqrt{s}}\int_{x_0+(x_0^2-1)^{\frac{1}{2}}}\zeta^{N+M}|g(\zeta)|\left|\frac{\partial^M}{\partial\zeta^M}\frac{1}{\zeta^{l+1}}\right|d\zeta$$

$$\leq\frac{qB(s)}{\sqrt{s}(l-N)}\left\{\frac{1}{x_0+(x_0^2-1)^{\frac{1}{2}}}\right\}^{l-N}. \quad (4)$$

This is the exponential behavior we are looking for, and $B(s)$ is the analog of the g of the classical model.

Now everything becomes straightforward: To calculate an upper bound on the forward amplitude, we write

$$A(s,1)=\frac{\sqrt{s}}{\pi q_s}\sum_{l=0}^{\infty}a_l(s)(2l+1).$$

Let us call L a value of l for which the upper bound (4) is less than unity. Then, for $l<L$, $|a_l|$ is bounded by one, a_l being an element of a unitary matrix, and for $l\geq L$, we can use the upper bound (4). Consequently

$$|A(s,1)|\leq\frac{\sqrt{s}}{\pi q_s}\sum_{l=0}^{L-1}(2l+1)$$

$$+\frac{\sqrt{s}}{\pi q_s}\sum_{l=L}^{\infty}\frac{qB(s)}{\sqrt{s}(l-N)}\left\{\frac{1}{x_0+(x_0^2-1)^{\frac{1}{2}}}\right\}^{l-N}.$$

The last series is bounded term by term by

$$\sum_{l=L}^{\infty}\left\{\frac{1}{x_0+(x_0^2-1)^{\frac{1}{2}}}\right\}^{l-L}=\frac{1}{1-x_0+(x_0^2-1)^{\frac{1}{2}}};$$

then

$$|A(s,1)|<\frac{\sqrt{s}}{\pi q_s}\left\{L^2+\frac{1}{1-x_0+(x_0^2-1)^{\frac{1}{2}}}\right\}.$$

L can be chosen as $\{\ln B(s)/\ln[x_0+(x_0^2-1)^{\frac{1}{2}}]\}+N$.

$$x_0+(x_0^2-1)^{\frac{1}{2}}=1+\frac{\kappa^2}{2q^2}+\left[\left(1+\frac{\kappa^2}{2q^2}\right)^2-1\right]^{\frac{1}{2}}$$

$$=1+\frac{\kappa}{q}+O\left(\frac{1}{q^2}\right),$$

so that we finally get

$$|A(s,1)|<(q^2/\kappa^2)\ln^2B(s).$$

$B(s)$ behaves at most like a polynomial in s, because all functions involved do so, and we get the following result: At forward or backward angles, the modulus of the amplitude behaves at most like $s\ln^2 s$, as s goes to infinity. We can use the optical theorem to derive that the total cross sections behave at most like $\ln^2 s$, as s goes to infinity. At nonforward angles, we may proceed along similar lines, but, in this case, the Legendre polynomial $P_l(\cos\theta)$ behaves like $f(\theta)/\sqrt{l}$. Using our upper bound, we find that at angles different from 0 or π, the amplitude behaves at most like $s^{\frac{3}{4}}\ln^{\frac{3}{2}}s$, as s goes to infinity.

III. ASYMPTOTIC BEHAVIOR OF THE DOUBLE SPECTRAL FUNCTIONS

We want to emphasize in this section that there is little hope that the preceding results could give any hint on the asymptotic behavior of the double spectral functions. Let a function $f(z)$ of one complex variable z be analytic in a plane cut from 0 to $+\infty$. Is there a relationship between the asymptotic behavior of the jump over the cut (the spectral function) and the asymptotic behavior of the function on the negative real axis? A simple counterexample will show that it is not the case:

Consider the function $z^N\exp[-(-z)^{\frac{1}{2}}]$. This function goes to zero faster than any power of $1/z$, in any direction not parallel to the positive real axis. Nevertheless the dispersion relation requires N subtractions, as the spectral function is $x^N\sin(\sqrt{x})$. Another example will show that in general, the asymptotic behavior in different directions of the complex plane may be different. Consider the function $f=\exp(i\ln^2 z)$. If we go to infinity along a line $z=\rho e^{i\alpha}$, α fixed, the modulus of f is $\rho^{-2\alpha}$; the asymptotic behavior depends upon the direction α.

Let us remark that this phenomenon can only occur if the spectral function undergoes an infinite number of oscillations,[4] as in both examples above.

In the case of several variables, examples of this kind are even easier to find, e.g., $(t_0-t)^{\alpha(s)}$, where $\alpha(s)$ is a function analytic in the s plane cut from s_0 to infinity. If $\alpha(s)<0$ for values of $-\infty\leq s<0$, and takes a finite range of values, this function satisfies a Mandelstam representation with a number of subtractions determined by the value of max Reα, although it stays

[4] This was proved by S. Weinberg (private communication).

bounded at infinity in every physical region. Of course it again oscillates an infinite number of times. This example seems to have some connection with the problem of potential scattering, and this may indicate the plausibility of such a phenomenon even in the relativistic problem.

IV. GENERAL SUBTRACTED FORM

We want here to write down an explicit formula for the subtracted double dispersion relation, allowing any number of subtractions, in a way suitable for the analysis of the next section.

Given an amplitude $A(s,t,u)$, behaving at most like some polynomial in $|s|$ and $|t|$ at infinity, its double spectral functions are determined by

$$\begin{aligned}\rho(s',t')=-\tfrac{1}{4}\{&A(s'+i\epsilon,\,t'+i\epsilon,\,u'-2i\epsilon)\\&+A'(s'-i\epsilon,\,t'-i\epsilon,\,u'+2i\epsilon)\\&-A(s'+i\epsilon,\,t'-i\epsilon,\,u')-A(s'-i\epsilon,\,t'+i\epsilon,\,u')\}\end{aligned}$$

in the limit $\epsilon\to 0$, and by the circular permutations among (s,t,u).

Once the double spectral functions are known, we choose N large enough to ensure the convergence of

$$f(s,t,u)=\frac{s^Nt^N}{\pi^2}\int\frac{\rho(s',t')ds'dt'}{s'^Nt'^N(s'-s)(t'-t)}+P_{stu}.$$

P_{stu} denotes the two terms which are deduced from the first by two circular permutations among s, t, and u. Then $A(s,t,u)-f(s,t,u)$ has a vanishing double spectral function. The jump over the s cut, say, is an entire function in t, and is thus a polynomial, due to the limitations to the increase of A at infinity.

We can then write down explicitly the single spectral functions, which are the coefficients of this polynomial, in terms of the jump of $A-f$. We subtract the single dispersion terms from $A-f$, and we get an entire function in s and t which is again a polynomial; so that we finally get

$$\begin{aligned}A(s,t,u)=&\frac{1}{\pi^2}s^Nt^N\int\int\frac{\rho(s',t')ds'dt'}{s'^Nt'^N(s'-s)(t'-t)}+P_{stu}\\&+\sum_{p=0}^{M}t^p\frac{1}{\pi}s^M\int\frac{\rho_{p,s}(s')ds'}{(s'-s)s'^M}+P_{stu}\\&+\sum_{p,q=0}^{L}t^ps^q\rho_{p,q}.\qquad(5)\end{aligned}$$

Remember that P_{stu} means the two terms deduced by a *circular* permutation among (s,t,u), from the term standing just before it. It is clear that any amplitude satisfying Mandelstam's conditions can be represented in this form by taking N,M,L sufficiently large, and even a finite number of amplitudes can have the same (N,M,L) set, provided that each N, M, and L is sufficiently large.

V. LIMITATIONS ON THE NUMBER OF ARBITRARY SUBTRACTION TERMS

We will assume that the double spectral functions are given. Suppose that one amplitude derived from them satisfies the asymptotic requirements found in Sec. II. We want to look for the possible changes in the subtraction terms which do not contradict these asymptotic requirements. That is to say that we have two different amplitudes $A^{(1)}$ and $A^{(2)}$, both satisfying our asymptotic conditions and admitting the same double spectral functions. Then, the difference $A^{(1)}-A^{(2)}$ satisfies also the asymptotic conditions, and by virtue of the representation (5), we can write

$$\begin{aligned}A^{(1)}-A^{(2)}\equiv\Delta A=&\sum_{p=0}^{M}t^p\frac{1}{\pi}s^M\int\frac{\Delta\rho_{p,s}(s')ds'}{(s'-s)s'^M}\\&+P_{stu}+\sum_{p,q=0}^{L}t^ps^q\Delta\rho_{p,q}.\qquad(6)\end{aligned}$$

Assume, as is usually done, that all spectral functions are real. Then, in the physical region of the s channel:

$$\mathrm{Im}\Delta A=\sum_{p=0}^{M}t^p\Delta\rho_{p,s}(s).$$

Consider $M+1$ fixed angles θ_i all different, put $\cos\theta_i=1-2\lambda_i$, then $t=(4-s)\lambda_i$,

$$\mathrm{Im}\Delta A(\theta_i)=\sum_{p=0}^{M}(4-s)^p\lambda_i{}^p\Delta\rho_{p,s}(s).\qquad(7)$$

One can solve for each term, as the determinant of this system is

$$\textstyle\prod_{i<j}(\lambda_i-\lambda_j)\neq 0$$

and we find that $(4-s)^p\Delta\rho_{p,s}(s)$ behaves at most like $\mathrm{Im}\Delta A(\theta_i)$, that is, at most like $s^{\frac{3}{2}}\ln^{\frac{3}{2}}s$, according to the results of Sec. 2. This can be repeated in each channel, and allows us to undo the subtractions which are present in the expression (6), except for $p=0$, by dropping the factors of s^M/s'^M and changing the values of the coefficients $\Delta\rho_{p,q}$ of the residual polynomial. One can still go farther than that in order to exhibit more clearly the asymptotic behavior of the terms in the expression (6); namely, for $p\geq 2$,

$$\begin{aligned}\frac{1}{\pi}\int\frac{\Delta\rho_{p,s}(s')ds'}{(s'-s)}&=-\sum_{q=0}^{p-2}\frac{1}{\pi}\int\frac{\Delta\rho_{p,s}(s')s'^qds'}{s^{q+1}}+\frac{1}{\pi}\int\frac{\Delta\rho_{p,s}(s')s'^{p-1}ds'}{s^{p-1}(s'-s)}\\&=\sum_{q=0}^{p-2}\frac{A_{p,s,q}}{s^{q+1}}+\frac{1}{\pi}\frac{1}{s^{p-1}}\int\frac{\Delta\rho_{p,s}(s')s'^{p-1}ds'}{s'-s}.\end{aligned}$$

It is easily deduced from Eq. (7) that this last term behaves at most like $s^{\frac{3}{2}-p}\ln^{\frac{3}{2}}s$ at infinity. We make the same manipulation on each term of (6) and get

$$\Delta A=\sum_{p=1}^{M}\sum_{q=0}^{p-2}\frac{t^p}{s^{q+1}}A_{p,s,q}+P_{stu}+\sum_{p,q=0}^{L}\Delta\rho_{p,q}s^pt^q+\text{other terms}.$$

Consider again this expression at a number of fixed angles θ_i, $t=(4-s)\lambda_i$, $u=(4-s)(1-\lambda_i)$. We first notice that all the terms which we did not write behave at most like $s^{\frac{3}{2}}\ln^{\frac{3}{2}}s$, so that we have written explicitly all the leading terms in the asymptotic expansion, down to terms linear in s. Proceeding downwards from $2L$ or M, whichever is greater, we will prove that all terms of a given degree m vanish, by taking a number of values of λ and solving, just as above. We have only to make sure that, at each step, the determinant is nonzero. The coefficients for the terms of degree m are λ_i^p, $(1-\lambda_i)^p/\lambda_i^{q+1}$, and $1/(1-\lambda_i)^{q+1}$, where $m+1\leq p\leq M$, $p=m+q+1$, for the $A_{p,q}$ terms, and λ_i^q, $0\leq q\leq m$ for the $\Delta\rho_{p,q}$ terms, so that we have in general $3M-2m+1$ terms and $3M-2m+1$ values of λ. The determinant never vanishes, being equal to

$$\Delta=\prod_{i<j}(\lambda_i-\lambda_j)/\prod_i[\lambda_i(1-\lambda_i)]^{M-m}.$$

This was the case for $L\geq m$, $M>m$. For any other case, the determinant is a subdeterminant of this one, and thus one can pick up a set of λ_i for which it does not vanish. We thus find that all $A_{p,q}$ vanish, as well as all terms of the residual polynomial, except $\Delta\rho_{00}$. We can write (6) as

$$\Delta A=\frac{1}{\pi}\sum_{p=0}^{M}\frac{t^p}{s^{p-1}}\int\frac{\Delta\rho_{p,s}(s')s'^{p-1}ds'}{s'-s}+P_{stu}+\Delta\rho_{00}.$$

If we keep s fixed at some negative value, and let t go to infinity, θ_t goes to π and the only terms which will violate the conditions of Sec. II at backward angle are

$$\sum_{p=2}^{M}\frac{t^p}{s^{p-1}}\int\frac{\Delta\rho_{p,s}(s')s'^{p-1}ds'}{s'-s}.$$

Thus for each negative value of s this polynomial in t vanishes identically, and

$$\frac{1}{s^{p-1}}\int\frac{\Delta\rho_{p,s}(s')s'^{p-1}ds'}{s'-s}=0\quad\text{for}\quad p\geq 2.$$

This analytic function vanishes for an interval of values of s, therefore it vanishes everywhere and $\Delta\rho_{p,s}(s')=0$ for $p\geq 2$. We get our final result, which is

$$\Delta A=\frac{s}{\pi}\int\frac{\Delta\rho_{0,s}(s')ds'}{s'(s'-s)}+P_{stu}+\frac{t}{\pi}\int\frac{\Delta\rho_{1,s}(s')ds'}{s'-s}+P_{stu}+\Delta\rho_{00}. \quad (8)$$

The single spectral functions, moreover, are bound to behave at most like $s^{\frac{3}{2}}\ln^{\frac{3}{2}}s$ for $\Delta\rho_{s,0}$ and $s^{-\frac{1}{2}}\ln^{\frac{3}{2}}s$ for $\Delta\rho_{s,1}$, and similarly by circular permutation.

One could summarize the preceding results by saying that, given a double spectral function, the s- and p-wave subtractions only are free.

VI. CONCLUSIONS

The net result of this work is the obtaining of a limit upon the growth at infinity of an amplitude satisfying Mandelstam's hypothesis, and the consequence that a relatively small number of subtractions are free. However, we would like to stress a number of points which may be interesting for applications. First, if one believes that there is no reason why there should be such a large forward or backward peak in inelastic reactions, and if one arbitrarily sets a limit like $s^{1-\epsilon}$ on the forward behavior of the amplitude, it is then possible to reduce further the number of subtractions, by suppressing the freedom of choice of some p-wave subtractions.

Second, we would like to emphasize that the problem of finding an amplitude having the right asymptotic behavior from a given double spectral function is not a simple one. In general it has no solution, but if it has one, it would be possible to find it by following the lines of thought of our proof. However, this involves at a point an analytic continuation, which appears in our proof when we deduce that $\Delta\rho_{p,s}=0$ from the fact that $\int\Delta\rho_{p,s}(s')ds'/(s'-s)=0$; it would be very difficult in general.

Finally, we would like to remark that, although we only considered here the case of scalar particles, it does not seem that any essential difficulty could arise in applying our method to the case of particles with spin. However, a general study for particles with arbitrary spins seems to be difficult, as, to the knowledge of the author, there does not exist any systematical way of specifying the invariant amplitudes which satisfy the Mandelstam representation.

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