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Nucleon Polarizability Correction to High-Energy Electron-Nucleon Scattering*

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The contribution of nucleon polarizability to ultrarelativistic electron-nucleon scattering cross sections is estimated and found to be small for nonforward scattering angles at all energies.

I. INTRODUCTION

THE charge and magnetic moment distributions of nucleons are obtained by analyzing the observed electron-nucleon scattering in Born approximation. Among the higher order terms which are neglected, radiative corrections for the electron scattering can be computed in a straightforward way. Other corrections to the Born approximation involve the exchange of more than one photon between the electron and nucleon. Part of this represents the correction for multiple scattering of the electron by the static electric and magnetic fields of the nucleon. These would be included by explicitly solving the Dirac equation for the electron rather than using the Born approximation. The rest of the many-photon exchange contribution depends upon the dynamical structure of the nucleon charge and moment distribution and cannot be calculated even if the conventional form factors are known. But all of these higher order corrections involve higher powers of $e^2/\hbar c$ and their inclusion would presumably give corrections of order 1% to the nucleon form factors obtained from the Born approximation analyses.

The dynamical contribution to the two-photon exchange depends upon the nucleon polarizability. Because this is anomalously large for photon frequencies near the $(\frac{3}{2},\frac{3}{2})$ pion-nucleon resonance, the effect on electron scattering of nucleon polarizability has been explicitly estimated[1,2] for electron energies below about one Bev to see if it could give an appreciable contribution to the observed cross section. Despite the fact that the polarizability contributed to an exceptionally large cross section for photonucleon scattering near the resonance, it was found to give a negligible ($\sim$1%) correction to the electron scattering. In the latter case, a virtual photon is scattered and it is necessary to average over the frequency and wave number of this virtual photon to calculate its effect on electron scattering. The nucleon polarizability is large and positive for frequencies below resonance and large but negative above; the required average is much smaller than the average magnitude in this region. The approximate magnitude of the polarizability corrections for electron energies $\gg$1 Bev cannot be inferred to be negligible from such calculations of its effect at the lower energies. The nucleon is treated in a nonrelativistic way and in certain approximations the Compton scattering amplitude is extrapolated off the photon energy shell by assuming that real and virtual photons of the same frequency have the same scattering amplitude to terms in e^2.[1] With such approximations the relative contribution of the nucleon polarizability was calculated[1] to become increasingly more significant as the electron energy grows above 1 Bev, but the approximations are clearly inappropriate in the ultrarelativistic region. The assumed dependence of photo-nucleon scattering on frequency alone is equivalent to neglecting the spatial "size" of the nucleon for virtual photon scattering; for large momentum transfers of highly virtual photons this is even qualitatively inadmissible.

In the forward direction the effect of nucleon polarizability on electron-nucleon scattering can be estimated fairly well. The imaginary part of this correction to order e^4 is $(k/4\pi)$ times the total cross section for electron-nucleon inelastic scattering (meson production); k is the electron momentum. This cross section can be estimated in the Weizsäcker-Williams approximation from the measured photomeson production cross section. In this way, it is seen that the imaginary part of the e^4 polarizability correction increases in the forward direction like $k\ln k$. At one Bev it is about 4×10^{-17} cm; at 10^3 Bev it is about 2×10^{-13} cm (center of mass). The real part of the forward polarizability correction is generated from the imaginary part by a dispersion relation. It is also monotonic and comparable in magnitude to the imaginary part. These amplitudes may be compared to the measured *ninety degree* cross section for electron proton scattering of 2×10^{-33} cm²/sr at 900 Mev.[3] Thus at 1 Bev the *forward* polarizability correction is already comparable to the observed amplitude for large momentum transfer. At much higher energies

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[1] S. D. Drell and M. A. Ruderman, Phys. Rev. **106**, 561 (1957).
[2] S. D. Drell and S. Fubini, Phys. Rev. **113**, 741 (1959).
[3] R. Hofstadter, F. Bumiller, and M. Croissiaux, Phys. Rev. Letters **5**, 263 (1960).

it is expected to be much larger than the e^2 ninety degree amplitude. Therefore, to be able to neglect the polarizability with confidence in large-angle ultrahigh-energy electron-nucleon scattering it is necessary to estimate how rapidly it decreases with angle as we move away from the forward direction, but this depends upon the unknown dynamic nucleon polarizability as a function both of the wave number and the frequency separately.

We have tried to make an estimate of the possible contribution of the polarizability to large-angle scattering in two ways. First, a relativistic Weizsäcker-Williams analysis of the electromeson production as a function of impact parameter gives the imaginary part of the electron-proton elastic scattering amplitude for arbitrary incident angular momentum. A one-dimensional dispersion relation yields the real part. These contributions are appropriately summed to give an angular distribution. Such a WKB approximation is applicable only to small-angle scattering. Even here it involves and unknown form factor (photoproduction of mesons for very virtual photons) in a crucial way as soon as the momentum transfer/c becomes larger than a few meson masses. However, it appears that as long as this form factor is not singular for infinite photon 4-momentum, then the polarizability correction for non-forward scattering remains negligible even at ultra-relativistic energies.

Alternatively, the Born approximation and polarizability corrections have also been calculated in relativistic perturbation theory for nucleons coupled to pseudoscalar mesons. The polarizability correction is shown to be negligible and, in this case, a decreasing function of energy for sufficiently high-energy finite angle scattering. Such a model is not expected to give a reliable description of either the polarizability correction or the form factors independently so the generality of the small ratio is unclear. The significance of this and the WKB result lies mainly in confirming that present knowledge and calculations of nucleon structure do not infer that Born approximation analyses of nucleon electromagnetic form factors will be unreliable because of the neglect of polarizability even at ultrahigh energies.

II. WKB APPROXIMATION

The partial wave expansion of the electron-nucleon scattering amplitude is

$$F(\theta)=(2ik)^{-1}\sum_l(2l+1)[\exp(2i\delta_l)-1]P_l(\cos\theta), \quad (1)$$

where δ_l is the complex phase shift. Since each δ_l is small, we have

$$\mathrm{Im}F(\theta)=(k/4\pi)\sum_l\sigma_l{}^{(R)}P_l(\cos\theta). \quad (2)$$

The $\sigma_l{}^{(R)}$ is the reaction cross section for the lth partial wave. In the high-energy limit, we shall consider the reaction cross section associated with a linear electron trajectory of fixed impact parameter $\rho=l/k$. In this WKB approximation,[3a]

$$\mathrm{Im}F(\theta)=(k/4\pi)\int_0^\infty\sigma^{(R)}(\rho)P_{k\rho}(\cos\theta)d\rho. \quad (2')$$

Since $\sigma^{(R)}(\rho)$ is invariant with respect to Lorentz transformations along the trajectory, we may calculate it in the rest system of the struck nucleon and substitute it into Eq. (2′), with all other variables evaluated in the c.m. system. The major part of the partial cross section $\sigma^{(R)}(\rho)$ is the result of meson production by the electron's electromagnetic field; we estimate this in the Weizsäcker-Williams approximation.[4] The canonical form of this approximation gives for the effective flux of real photons incident on the nucleon, caused by an electron with impact parameter ρ and energy of γ rest masses,

$$N(\rho,\omega)\cong\frac{\alpha}{\pi^2\rho^2\omega}\left[\frac{\omega\rho}{\gamma c}K_1\left(\frac{\omega\rho}{\gamma c}\right)\right]^2, \quad (3)$$

where K_1 is the modified Hankel function of first order. Also,

$$\sigma^{(R)}(\rho)=\int_0^{\bar{k}}2\pi\rho N(\rho,\omega)\sigma_{\gamma\pi}(\omega)d\omega, \quad (4)$$

where $\bar{k}$ is the incident electron momentum in the laboratory system and $\sigma_{\gamma\pi}(\omega)$ is the total pion photoproduction cross section for real photons of frequency ω in this system.

The electromagnetic field of an ultrarelativistic electron is not equivalent to that of free photons when the impact parameter is smaller than the nucleon "size." Therefore, for such impact parameters, the combination of Eqs. (3) and (4) must be appropriately altered. Effectively this involves the cross section for pion production by photons which are far off the energy shell, $\omega^2\neq k^2$.

The differential cross section for electromeson production is proportional to

$$|\mathfrak{F}_\mu(\omega,\mathbf{k})(\omega^2-k^2)^{-1}(2p-k)_\mu|^2/4p_4(p-k)_4, \quad (5)$$

the $\mathfrak{F}_\mu(\omega,\mathbf{k})$ is the matrix element for meson production by a photon of 4-momentum $(\omega,\mathbf{k})$ and p is the 4-momentum of the electron. When the high-energy electron, moving in the z direction, is negligibly deviated, $\omega^2-k_z{}^2\sim-\omega^2/\gamma^2$. The cross section (5) can then be somewhat simplified by gauge invariance requirements, converted into a function of the transverse component $\mathbf{k}_\rho$, and Fourier-transformed to exhibit its ρ dependence:

$$\left|\int d^2k_\rho\exp(i\mathbf{k}\cdot\boldsymbol{\rho})[(\omega/\gamma)^2+k_\rho{}^2]^{-1}\omega^{-1}\mathbf{k}_\rho\cdot\mathfrak{F}(\omega,\mathbf{k})\right|^2. \quad (6)$$

[3a] This approach has been previously considered by H. A. Bethe and F. Rohrlich, Phys. Rev. **86**, 10 (1952).

[4] See, e.g., W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, New York, 1954), 3rd ed., appendix 6.

If $\mathfrak{F}(\omega,\mathbf{k})$ is independent of $\mathbf{k}$, then $|\mathfrak{F}(\omega,\omega)|^2 \sim \sigma_{\gamma\pi}(\omega)$ and the expression (6) gives just the result of combining Eqs. (3) and (4), i.e., the usual Weizsäcker-Williams approximation. However, the expression (6) must be integrated over all $\mathbf{k}_\rho$ so that $k_\rho^2+k_z^2-\omega^2 \sim k_\rho^2$ is not on the energy shell. We approximate $\mathfrak{F}(\omega,\mathbf{k})$ by

$$\mathfrak{F}(\omega,\mathbf{k}) \approx \mathfrak{F}(\omega,\omega) f_\omega(\omega^2-k^2). \tag{7}$$

The form factor $f_\omega=1$ when $k^2=\omega^2$, and corrects the photomeson amplitude for virtual photons. It is easily shown that this introduction of f_ω by Eq. (7) is exactly equivalent to writing

$$N(\rho,\omega)=\frac{1}{\hbar\omega}\frac{c}{4\pi^2}\left|\hat{v}\times\int d^2\rho'\, q_\omega(\boldsymbol{\rho}')\mathbf{E}(\boldsymbol{\rho}-\boldsymbol{\rho}';\omega)\right|^2, \tag{8}$$

where $\mathbf{E}$ is the Lienard-Wiechert electric field of an electron moving in the direction $\hat{v}$. If $q(\boldsymbol{\rho}')=\delta^{(2)}(\boldsymbol{\rho}')$, Eq. (8) is the canonical expression for photon density, and leads to Eq. (3); instead here $q(\boldsymbol{\rho}')$ is the two-dimensional Fourier transform of $f_\omega(k_\rho^2)$. We may interpret this, in any frame moving parallel to the z axis, in terms of an effective cylindrical density of the nucleon for photomeson production. The combination of Eqs. (2′), (4), and (8) gives, after straightforward calculation,

$$\operatorname{Im}F(\theta)\cong\frac{\alpha k}{2\pi^2}\int_0^\infty\frac{d\rho}{\rho}P_{k\rho}(\cos\theta)\int_0^{\bar{k}}\frac{d\omega}{\omega}\left[\frac{\omega\rho}{\gamma}K_1\left(\frac{\omega\rho}{\gamma}\right)\right]^2 \times\sigma_{\gamma\pi}(\omega)\left[\int_0^\infty du\, J_1(u)f_\omega(u^2/\rho^2)\right]^2. \tag{9}$$

We explicitly assume that $f_\omega\to 0$ as $\rho\to 0$; otherwise the lower limit of the ρ integration must be cut off at $1/\bar{k}$.

If the form factor f_ω is replaced by unity, the ρ cutoff must be introduced, and in the forward direction we obtain the familiar

$$\operatorname{Im}F(0)\cong\frac{\alpha k}{2\pi^2}\int_0^{\bar{k}}\frac{d\omega}{\omega}\ln\left(\frac{\gamma}{\omega\rho_{\min}}\right)\sigma_{\gamma\pi}(\omega). \tag{10}$$

When this is compared to the Rosenbluth[5] anomalous magnetic moment scattering amplitude in the ultra-relativistic limit,

$$F_B(\theta)\cong(\alpha/2^{\frac{1}{2}}M)KF_2(q^2)\cot(\theta/2), \tag{11}$$

with K the anomalous moment, F_2 is form factor, and $q=(2E/M)\sin(\theta/2)$, it is apparent that in the forward direction, $\operatorname{Im}F(0)$ increases like $k\ln k$, while $F_B(\theta)$ for any fixing nonvanishing θ decreases with increasing k. The neglect of $\operatorname{Im}F(\theta)$ next to $F_B(\theta)$ thus depends critically upon the angular distribution calculated from Eq. (9).

[5] M. Rosenbluth, Phys. Rev. **79**, 615 (1950).

For very large k, we replace the Legendre function of Eq. (9) by its asymptotic form $J_0(k\rho\theta)$. This oscillates rapidly for large $k\rho\theta$, and thus cuts off the Hankel function which can be replaced by its limit for small argument. Then, very approximately, we have

$$\operatorname{Im}F(\theta)\cong\frac{\alpha k}{2\pi^2}\int_0^\infty\frac{d\rho}{\rho}J_0(k\rho\theta)\int_0^{\bar{k}}\frac{d\omega}{\omega}\sigma_{\gamma\pi}(\omega) \times\left[\int_0^\infty du\, J_1(u)f_\omega(u^2/\rho^2)\right]^2. \tag{12}$$

The form factor $f_\omega(k^2)$ refers to a four-point vertex, but it might be expected to have a behavior which is not qualitatively dissimilar from the familiar electromagnetic three-point form factor. Therefore we assume $f_\omega(k^2)\sim 1$ for $k^2<\nu^2$, after which there is a rapid decrease, e.g.,

$$f_\omega(k^2)=\sum_{i,n}\nu_i^{2n}(\nu_i^2+k^2)^{-n}C_{ni}, \tag{13}$$

with the ν_i of the order of a few pion masses. Then from Eq. (12), we find that for a typical term in Eq. (13) with $C_n=1$,

$$\operatorname{Im}F(\theta)\cong\frac{\alpha k}{2\pi^2}\int_0^{\bar{k}}\frac{d\omega}{\omega}\sigma_{\gamma\pi}(\omega) \times\int_0^\infty\frac{dx}{x}\left[1-2\left(\frac{x}{2}\right)^n\frac{K_n(x)}{\Gamma(n)}\right]^2 J_0\left(\frac{k\theta}{\nu}x\right). \tag{14}$$

The $\operatorname{Im}F(\theta)$ predicted by Eq. (14) decreases for increasing n or $k\theta$. Even for $n=\frac{1}{2}$, for which the virtual photons are much more effective in photoproduction of mesons than would be estimated from any physical model, the x integral in Eq. (14) gives

$$+2\ln\left\{\frac{\nu}{k\theta}+\left[1+\left(\frac{\nu}{k\theta}\right)^2\right]^{\frac{1}{2}}\right\} -\ln\left\{\frac{2\nu}{k\theta}+\left[1+\left(\frac{2\nu}{k\theta}\right)^2\right]^{\frac{1}{2}}\right\}\xrightarrow[k\theta\to\infty]{}\left(\frac{\nu}{k\theta}\right)^3.$$

For $n=\frac{3}{2}$, the high-momentum transfer limit is $(\nu/k\theta)^4$; when n increases, the limit continues to decrease, but depends upon n less sensitively. Thus in contrast to its contribution to forward scattering, the effect of the polarizability on the large momentum transfer electron scattering is a very rapidly decreasing function of momentum. For $n\gtrsim 1$, $\operatorname{Im}F(\theta)\sim k(\nu/k\theta)^4$. Clearly a nonsingular behavior for the meson photoproduction form factor leads to a polarizability correction which is negligible next to the Born term if the conventional form factors are combinations of Yukawa functions.

We next turn to the evaluation of the contribution of the polarizability to the real part of the scattering amplitude. This is most easily obtained as a dispersion transform of the imaginary part, a relation which in this semiclassical case can be justified on the basis of causality alone. The model of a high-energy electron traversing a straight-line path of fixed impact parameter, such as was considered above, may equivalently be regarded as a wave packet traveling down a long narrow tube, and analyzed into one-dimensional plane waves. If there is an input $e^{i\omega t}$ at one end, then the transmitted wave at the other end is $\exp\{i[\omega t+\eta_\rho(\omega)]\}$, where η_ρ is the complex phase shift. But since the two ends of the tube must be causal with respect to each other, the frequency response function $K_\rho(\omega)=\exp(i\eta_\rho)$ must, by a well-known argument, obey a Kramers-Kronig relation:

$$\mathrm{Im}K_\rho(\omega)=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\mathrm{Re}K_\rho(\omega')\frac{1}{\omega'-\omega}d\omega'.$$

By considering the total cross section to be an integral over impact parameters of the separate cross sections of each tube individually, for small η_ρ the following identification can be made

$$-2\,\mathrm{Re}\{\exp[i\eta_\rho(k)]-1\}=\int_0^{\bar{k}}N(\omega,\rho)\sigma_{\gamma\pi}(\omega)d\omega. \quad (15)$$

In addition, $K_\rho(\omega)=K_\rho^*(-\omega)$ (electrons and positrons produce identical scattering); we then try a no-subtraction dispersion relation for $\mathrm{Re}F(\theta)$:

$$\mathrm{Re}F(\theta)=\frac{2\alpha}{\pi}\int_0^\infty\frac{d\rho}{\rho}P_{k\rho}(\cos\theta)\frac{k}{4\pi}\int_0^\infty\frac{kdk'}{k'^2-k^2}\times\int_0^{\bar{k}'}\frac{d\omega}{\omega}\left[\frac{\omega\rho}{\gamma}K_1\left(\frac{\omega\rho}{\gamma}\right)\right]^2\sigma_{\gamma\pi}(\omega)\times\left[\int_0^\infty du\,J_1(u)f_\omega\left(\frac{u^2}{\rho^2}\right)\right]^2. \quad (16)$$

Again we may use the small argument expansion for K_1 since large ρ is cut off by $P_{k\rho}(\cos\theta)$ for finite θ. Then

$$\frac{\mathrm{Re}F(\theta)}{\mathrm{Im}F(\theta)}=\frac{2}{\pi}P\int_0^\infty\frac{kdk'}{k'^2-k^2}\left[\frac{\int_0^{\bar{k}'}d\omega\,\omega^{-1}\sigma_{\gamma\pi}(\omega)}{\int_0^{\bar{k}}d\omega\,\omega^{-1}\sigma_{\gamma\pi}(\omega)}-1\right]. \quad (17)$$

The real and imaginary parts of the polarizability amplitude F have the same angular dependence in this approximation. Moreover, the bracket is experimentally close to zero when $\bar{k}$ and $\bar{k}'$, the laboratory wave numbers, are both much larger than the nucleon isobar frequency. If most of the $\sigma_{\gamma\pi}(\omega)$ comes from an isobar resonance at k_R, then we have

$$\mathrm{Re}F(\theta)/\mathrm{Im}F(\theta)\sim 2k_R/\pi k,\quad k\gg k_R. \quad (18)$$

Then $\mathrm{Re}F(\theta)$ is much smaller than $\mathrm{Im}F(\theta)$ at high energies and nonforward angles. The indication, then, is that the nucleon polarizability makes a negligible contribution to the scattering amplitude in comparison to the Born term for all energies and scattering angles for which the WKB approximation is qualitatively valid.

III. FIELD-THEORETIC MODEL

The previous results, obtained through semiclassical arguments, may be qualitatively confirmed for a particular model in relativistic perturbation theory. In an expansion of the S matrix in powers of the electromagnetic coupling, e,

$$S=1+i\sum_n e^nT^{(n)};$$

the nucleon polarizability effect is associated with the term $T^{(4)}$ involving two-photon exchange.[2] The unitarity of S, along with time reversal invariance implies a relation between $T^{(4)}$ and $T^{(2)}$:

$$\mathrm{Im}\langle f|T^{(4)}|i\rangle=\tfrac{1}{2}\sum_\beta\langle f|T^{(2)}|\beta\rangle\langle\beta|T^{(2)}|i\rangle, \quad (19)$$

where the states f and i are real two-particle states containing one electron and one physical nucleon, and the states β contain the electron and any combination of particles which couple strongly to the nucleon.

The intermediate state β of just one nucleon and one electron, without any mesons, can be disposed of quickly. It would occur (and, in fact, be the only term) in a calculation of electron-nucleon scattering, even in the absence of a meson field; the presence of the meson coupling merely modifies the static nucleon charge and moment by insertion of form factors dependent on the momentum transfer. This type of correction term can easily be calculated, and with inclusion of extrapolations of experimentally measured form factors, will presumably be small for all energies and scattering angles relative to the one photon exchange terms.

The remaining terms, in which the β contain one or more mesons and/or baryon pairs, and which represent the polarizability effect to be investigated, cannot be completely evaluated with available analytic techniques. One approximation, however, would be a simple one-meson model, calculated in lowest order perturbation theory; it would be natural, then, to compare the resulting expression for $T_{fi}^{(4)}$ with the Born approximation, $T_{fi}^{(2)}$, in which the unknown form factors are also obtained for the same perturbation theory model. Although this calculation is certainly not reliable in either application, it is possible that the ratio $T_{fi}^{(4)}$ to $T_{fi}^{(2)}$, each estimated in this way, may be indicative of the true momentum transfer dependence; the calculated ratio *is* independent of the meson-nucleon coupling constant.

In the particular case of the neutron, the relevant Feynman diagrams for application of the model to $\langle\beta|T^{(2)}|i\rangle$ are given in Fig. 1; they involve electromeson production from a bare neutron. Their contribution,

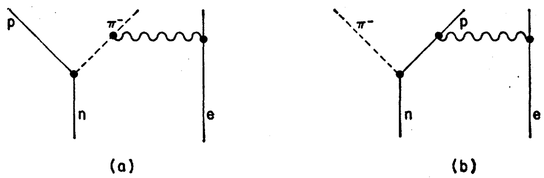

FIG. 1. Diagrams used in model taken to represent amplitude $\langle ne\pi | T^{(2)} | ne \rangle$.

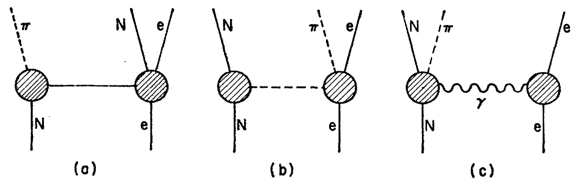

FIG. 2. Diagrams representing pole contributions to amplitude $\langle ne\pi | T^{(2)} | ne \rangle$; (a), (b), and (c) give nucleon, pion, and photon poles, respectively.

according to the usual rules and Eq. (19), is[6]

$$\sum_{\beta} \langle f | T^{(2)} | \beta \rangle \langle \beta | T^{(2)} | i \rangle$$

$$= \frac{\alpha^2 g^2 M^2 m^2}{4\pi^4 (E_1 E_2 E_1' E_2')^{\frac{1}{2}}} \delta^4(p_1' + p_2' - p_1 - p_2)$$

$$\times \int \frac{d^3 p_3}{E_3} \frac{d^3 p_4}{E_4} \frac{d^3 p_5}{E_5} \frac{\delta^4(p_3 + p_4 + p_5 - p_1 - p_2)}{(p_2 - p_4)^2 (p_2' - p_4)^2}$$

$$\times \bar{w}(p_1') \left[\frac{p_5 \gamma_\mu}{(p_1' - p_5)^2 - M^2} + \frac{(p_1' - p_3 + p_5)_\mu}{(p_1' - p_3)^2 - \mu^2} \right]$$

$$\times \Lambda_-(p_3) \left[\frac{\gamma_\nu p_5}{(p_1 - p_5)^2 - M^2} + \frac{(p_1 - p_3 + p_5)_\nu}{(p_1 - p_3)^2 - \mu^2} \right] w(p_1)$$

$$\times \bar{w}(p_2') \gamma^\mu \Lambda_+(p_4) \gamma^\nu w(p_2). \quad (20)$$

Even in this simplified model, expression (20) is too ponderous to integrate directly. However, since an order of magnitude estimate at the high-momentum-transfer limit is all that is required, a crude approximation technique is employed. The basis of this scheme lies in noting that most of the dependence of the integrand on the integration variables is in the propagator denominators; these propagators become many orders larger at certain points in the range of the variables, for very high energies. Furthermore, the half-widths of these spikes, although very narrow, are sufficient that most of the contribution to an integration of them against a slowly varying function comes from these neighborhoods. The numerator does display dips at the points in question, but not severe enough to offset the peaking of the denominator. Hence a possible approximation is to evaluate the numerator at the dominant points, and take it outside the integration.

An alternative explanation of the procedure is in terms of the singularities of the S matrix. The unitarity condition has related $\langle f | T^{(4)} | i \rangle$ to a bilinear sum over states β of $\langle f | T^{(2)} | \beta \rangle$ and $\langle \beta | T^{(2)} | i \rangle$. The first simplification of the model chosen was to truncate this sum by restricting β to contain only one pion besides the nucleon and electron. Next, of the singularities of the physical amplitude $\langle ne\pi | T^{(2)} | ne \rangle$ considered as a function of complex energy and momentum transfer variables, only poles were retained; this corresponds to calculating only to lowest order in renormalized meson-nucleon coupling constant. There are three poles (Fig. 2): two, in the unphysical region, represent the possibility of a single nucleon or meson intermediate state, while the third is a photon pole on the edge of the physical region. In the high-energy limit, furthermore, all the singularities which lie outside move in towards the edges and have an increasingly significant influence on the amplitude inside the region.[7] The mathematical approximation to Eq. (20), described above, corresponds to replacing the second order amplitude by the most singular terms in its Laurent expansion about the poles.

Returning to Eq. (20), it is seen that factors in the denominator nearly vanish for the four cases

$$(p_4 \cong p_2,\ p_5 \cong p_1), \quad (p_4 \cong p_2,\ p_3 \cong p_1), \quad (21a)$$

and

$$(p_4 \cong p_2',\ p_5 \cong p_5'), \quad (p_4 \cong p_2',\ p_3 \cong p_1'). \quad (21b)$$

In the center-of-mass frame, the first pair leads to the substitutions into the numerator

$$\begin{aligned} p_5 &\rightarrow p_{51} \equiv [E_1 - \tfrac{1}{2}M,\ (p_1 - \tfrac{1}{2}M)\hat{p}_1], \\ p_3 &\rightarrow p_{31} \equiv [M, 0], \\ p_4 &\rightarrow p_{41} \equiv [E_2 - \tfrac{1}{2}M,\ (p_2 - \tfrac{1}{2}M)\hat{p}_2], \end{aligned} \quad (22)$$

where the factor $(p_1 - p_5)^2 - M^2$ occurs in the denominator, and

$$\begin{aligned} &p_5 \rightarrow p_{52} \equiv [\mu, 0], \quad p_3 \rightarrow p_{32} \equiv [E_1 - \tfrac{1}{2}\mu,\ (p_1 - \tfrac{1}{2}\mu)\hat{p}_1], \\ &p_4 \rightarrow p_{42} \equiv [E_2 - \tfrac{1}{2}\mu,\ (p_2 - \tfrac{1}{2}\mu)\hat{p}_2], \end{aligned} \quad (23)$$

where $(p_1 - p_3)^2 - \mu^2$ occurs in the denominator.

Furthermore, if the net momentum transfer in the scattering is large, so that p_1 is not approximately equal to p_1', then the peaks from the pair of points (21a) are widely separated from, and nearly nonoverlapping with, those of the pair (21b). Thus it is also legitimate to treat the noneffective denominator factors in any particular region as slowly varying, too, just like the numerator. The extent of validity of this procedure can be illustrated by the following simplified typical example, which can be analyzed in detail.

[6] The notation follows the conventions of S. Schweber, H. Bethe, and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, Illinois, 1956), Vol. I.

[7] The prominence of the poles at high energies has been used recently in a different context by S. D. Drell, Phys. Rev. Letters **5**, 278, 342 (1960).

Consider the integral

$$I\equiv\int d\Omega_q(a-\hat{p}\cdot\hat{q})^{-1}(a-\hat{p}'\cdot\hat{q})^{-1}, \quad (24)$$

over all directions of $\hat{q}$, with $a>1$. An exact evaluation yields

$$I=\frac{4\pi}{2\sin(\frac{1}{2}\theta)[a^2-\cos^2(\frac{1}{2}\theta)]^{\frac{1}{2}}}\times\ln\left|\frac{\sin(\frac{1}{2}\theta)+[a^2-\cos^2(\frac{1}{2}\theta)]^{\frac{1}{2}}}{-\sin(\frac{1}{2}\theta)+[a^2-\cos^2(\frac{1}{2}\theta)]^{\frac{1}{2}}}\right|, \quad (25)$$

where $\cos\theta\equiv\hat{p}\cdot\hat{p}'$. If $a-1\ll 1$, then the integrand of I is very similar to the one at hand, Eq. (20). If in addition, $a^2-1\ll\sin^2(\frac{1}{2}\theta)$, then Eq. (25) reduces to

$$I\cong\frac{4\pi}{1-\cos\theta}\ln\frac{1-\cos\theta}{a-1}. \quad (26)$$

But the further condition just states that the two peaks of the integrand (when $\hat{q}\|\hat{p}$, and $\hat{q}\|\hat{p}'$) are well separated in comparison to their half-widths. Thus if the peaks are assumed to be strictly nonoverlapping, and the nondominant factor to be slowly varying, then

$$I\cong\frac{2}{a-\hat{p}\cdot\hat{p}'}\int\frac{d\Omega_q}{a-\hat{p}\cdot\hat{q}}\cong\frac{4\pi}{1-\cos\theta}\ln\frac{2}{a-1}. \quad (27)$$

The two results are sufficiently similar, so that the approximation procedure is pertinent; the ratio of the arguments of the logarithm, $\sin^2(\frac{1}{2}\theta)$, will be of importance later, however.

Applying these techniques, Eq. (20) becomes

$$\sum_\beta\langle f|T^{(2)}|\beta\rangle\langle\beta|T^{(2)}|i\rangle$$
$$=\frac{2\alpha^2g^2M^2m^2}{4\pi^4E_1E_2}\frac{\delta^4(p_1'+p_2'-p_1-p_2)}{(p_2'-p_2)^2}$$
$$\times\int\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}\frac{d^3p_5}{E_5}\frac{\delta^4(p_3+p_4+p_5-p_1-p_2)}{(p_2-p_4)^2}\bar{w}(p_1')$$
$$\times\left[\frac{p_1\gamma_\mu\Lambda_-(p_{31})\gamma_\nu p_1}{[(p_1'-p_1)^2-M^2][(p_1-p_5)^2-M^2]}+\text{cross terms}\right.$$
$$\left.+\frac{(p_1'-p_{32}+p_{52})_\mu\Lambda_-(p_{32})(p_1-p_{32}+p_{52})_\nu}{[(p_1'-p_1)^2-\mu^2][(p_1-p_3)^2-\mu^2]}\right]$$
$$\times w(p_1)\bar{w}(p_2')\gamma^\mu\Lambda_+(p_2)\gamma^\nu w(p_2). \quad (28)$$

Using the commutation relations for the gamma matrices and the fact that $\bar{w}(p)$ is an eigenspinor of the free-particle Dirac equation, the numerator can be reduced:

$$\bar{w}(p_1')p_1\gamma_\mu\Lambda_-(p_{31})\gamma_\nu p_1w(p_1)\bar{w}(p_2')\gamma^\mu\Lambda_+(p_2)\gamma^\nu w(p_2)$$
$$=\frac{M}{2m}\bar{w}(p_1')\{(m-i\sigma_{\mu\nu}p_2^\nu)[(1+\gamma_4)M-2E_1]+2p_{1\mu}(1+\gamma_4)p_2-2E_2\gamma_\mu M\}\times w(p_1)\bar{w}(p_2')\gamma^\mu w(p_2), \quad (29)$$
$$\cong\frac{M}{2m}\bar{w}(p_1')[2iE_1\sigma_{\mu\nu}p_2^\nu+2p_{1\mu}(1+\gamma_4)p_2]\times w(p_1)\bar{w}(p_2')\gamma^\mu w(p_2), \quad (30)$$

when masses are neglected compared to energies. In a similar manner, one obtains for the other term,

$$\bar{w}(p_1')(p_1'-p_{32}+p_{52})_\mu\Lambda_-(p_{32})(p_1-p_{32}+p_5)_\nu w(p_1)\times\bar{w}(p_2')\gamma^\mu\Lambda_+(p_2)\gamma^\nu w(p_2)$$
$$\cong-\frac{3\mu^3E_2}{8Mm}\bar{w}(p_1')(\gamma_4+\boldsymbol{\gamma}\cdot\hat{p}_1)w(p_1)\bar{w}(p_2')\gamma_4w(p_2). \quad (31)$$

A further reduction can be achieved by using the formulas for spinor matrix elements, referring to ultra-relativistic particles in the center-of-mass frame:

$$\begin{aligned}m\bar{w}(p',\zeta')w(p,\zeta)&=\bar{\epsilon}E\sin(\tfrac{1}{2}\theta),\\ m\bar{w}(p',\zeta')\gamma_4w(p,\zeta)&=\epsilon E\cos(\tfrac{1}{2}\theta),\\ m\bar{w}(p',\zeta')\boldsymbol{\gamma}w(p,\zeta)&=-\epsilon E(\hat{p}+\hat{p}'-i\zeta\hat{p}\times\hat{p}')\times(1+\hat{p}\cdot\hat{p}')^{-1}\cos(\tfrac{1}{2}\theta),\\ m\bar{w}(p',\zeta')\gamma_4\boldsymbol{\gamma}w(p,\zeta)&=\bar{\epsilon}E(\hat{p}-\hat{p}'+i\zeta\hat{p}\times\hat{p}')\times(1-\hat{p}\cdot\hat{p}')^{-1}\sin(\tfrac{1}{2}\theta),\\ m\bar{w}(p',\zeta')\boldsymbol{\sigma}w(p,\zeta)&=\bar{\epsilon}\zeta E(\hat{p}-\hat{p}'+i\zeta\hat{p}\times\hat{p}')\times(1-\hat{p}\cdot\hat{p}')^{-1}\sin(\tfrac{1}{2}\theta).\end{aligned} \quad (32)$$

Here $\cos\theta\equiv\hat{p}\cdot\hat{p}'$, while ζ is a spin polarization index assuming values ±1 such that $\zeta=\bar{w}(p,\zeta)\boldsymbol{\sigma}\cdot\hat{p}w(p,\zeta)$. It further proves convenient to define $\epsilon\equiv\frac{1}{2}(1+\zeta\zeta')$, $\bar{\epsilon}\equiv\frac{1}{2}(1-\zeta\zeta')$. Substituting formulas (32) into Eq. (30) yields

$$2E_1^2E_2^2m^{-2}[2\epsilon_1\cos(\tfrac{1}{2}\theta)+\bar{\epsilon}_1\sin(\tfrac{1}{2}\theta)]\epsilon_2\cos(\tfrac{1}{2}\theta); \quad (33)$$

the same substitution into Eq. (31) yields

$$-3\mu^3E_2^2(8Mm^2)^{-1}\epsilon_2\bar{\epsilon}_1\cos(\tfrac{1}{2}\theta)\sin(\tfrac{1}{2}\theta). \quad (34)$$

It can now be seen that for high energies, the second of these two expressions is much smaller than the first. Since they are coefficients of terms in Eq. (28), the second term can be neglected; it will be shown later that the integrals of the denominators of these two terms do not differ significantly in magnitude. The cross-terms may similarly be neglected. These approximations correspond to retaining only Feynman diagrams in which the photon is attached to the bare nucleon, not the meson; or equivalently to discarding the meson pole of Fig. 2.

After these reductions, Eq. (28) reads

$$\sum_\beta \langle f|T^{(2)}|\beta\rangle\langle\beta|T^{(2)}|i\rangle$$

$$\cong \alpha^2 g^2 \pi^{-4} M^2 E_1 E_2 [2\epsilon_1 \cos(\tfrac{1}{2}\theta)+\bar{\epsilon}_1 \sin(\tfrac{1}{2}\theta)]$$

$$\times \epsilon_2 \cos(\tfrac{1}{2}\theta)\frac{\delta^4(p_1'+p_2'-p_1-p_2)}{(p_1'-p_1)^2-M^2}\frac{1}{(p_2'-p_2)^2}$$

$$\times \int \frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}\frac{d^3p_5}{E_5}\frac{\delta^4(p_3+p_4+p_5-p_1-p_2)}{(p_2-p_4)^2[(p_1-p_5)^2-M^2]}. \quad (35)$$

The integration over momenta has now been reduced to a manageable form. The delta-function is used to integrate the four variables $\mathbf{p}_3$ and θ_{45}, where the latter is determined by conservation of energy-momentum:

$$2\mathbf{p}_4\cdot\mathbf{p}_5 \equiv 2p_4p_5\cos\theta_{45} = (E_1+E_2)^2-2(E_1+E_2)(E_4+E_5)+2E_4E_5+\mu^2+m^2-M^2. \quad (36)$$

Thus with spherical coordinates chosen in the center of mass,

$$I \equiv \int \frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}\frac{d^3p_5}{E_5}\frac{\delta^4(p_3+p_4+p_5-p_1-p_2)}{(p_2-p_4)^2[(p_1-p_5)^2-M^2]}$$

$$=\frac{1}{4}\int \frac{dE_4dE_5d\Omega_4d\phi_{45}}{(E_2E_4-m^2-\mathbf{p}\cdot\mathbf{p}_4)(E_1E_5-\frac{1}{2}\mu^2-\mathbf{p}\cdot\mathbf{p}_5)}. \quad (37)$$

The angular integrations can be performed exactly, with the result

$$I=\pi^2\int dE_4dE_5\,\Delta^{-1}\ln\left\{\frac{E_2E_4-m^2-pp_4}{E_2E_4-m^2+pp_4}\,\frac{(E_2E_4-m^2+pp_4)\Gamma+\Delta-pp_4(E_1E_5-\frac{1}{2}\mu^2-pp_5\cos\theta_{45})}{(E_2E_4-m^2-pp_4)\Gamma+\Delta-pp_4(E_1E_5-\frac{1}{2}\mu^2+pp_5\cos\theta_{45})}\right\},$$

where

$$\Delta^2=p^2p_4^2[(E_1E_5-\tfrac{1}{2}\mu^2)^2-pp_5\sin^2\theta_{45}]+p^2p_5^2(E_2E_4-m^2)^2+2(E_1E_5-\tfrac{1}{2}\mu^2)p^2p_4p_5\cos\theta_{45}(E_2E_4-m^2), \quad (38)$$

and

$$-\Gamma\Delta=(E_1E_5-\tfrac{1}{2}\mu^2)p^2p_4p_5\cos\theta_{45}-p^2p_5^2(E_2E_4-m^2).$$

The upper and lower limits on the E_5 integration for fixed E_4, which correspond to the situations $\mathbf{p}_5$ antiparallel to both $\mathbf{p}_3$ and $\mathbf{p}_4$, and $\mathbf{p}_3$ antiparallel to both $\mathbf{p}_4$ and $\mathbf{p}_5$, respectively, are given by

$$E_5^{\pm}=\{(E_1+E_2-E_4)[(E_1+E_2-E_4)^2-p_4^2+\mu^2-M^2]\pm p_4[((E_1+E_2-E_4)^2-p_4^2+\mu^2-M^2)^2-4\mu^2((E_1+E_2-E_4)^2-p_4^2)]^{\frac{1}{2}}\}/2[(E_1+E_2-E_4)^2-p_4^2]. \quad (39)$$

The lower limit on the E_4 integration is $E_4^-=m$; the upper limit is

$$E_4^+=[(E_1+E_2)^2+m^2-(M+\mu)^2]/2(E_1+E_2), \quad (40)$$

corresponding to the situation $\mathbf{p}_4$ antiparallel to both $\mathbf{p}_3$ and $\mathbf{p}_5$.

In order to make further progress, the high-energy limit of I must be taken; that is, p_1 and p_2 are to be the four-momenta of ultrarelativistic particles. Setting

$$\mathcal{E}_4^2\equiv(E_2E_4-m^2-pp_4)/pp_4,$$

and

$$\mathcal{E}_5^2\equiv(E_1E_5-\tfrac{1}{2}\mu^2-pp_5)/pp_5, \quad (41)$$

the $\mathcal{E}$'s are seen to be $\ll 1$ in the high-energy limit for almost all E_4 and E_5. Thus keeping only terms of lowest order in the $\mathcal{E}$'s,

$$I\cong 2\pi^2p^{-2}\int dE_4dE_5[p_4p_5(1+\cos\theta_{45}+\mathcal{E}_4^2+\mathcal{E}_5^2)]^{-1}\times\ln(1+\cos\theta_{45}+\mathcal{E}_4^2+\mathcal{E}_5^2)/\mathcal{E}_4\mathcal{E}_5. \quad (42)$$

Furthermore, in this limit

$$p_4p_5(1+\cos\theta_{45})\cong 2(E-E_4)(E-E_5);$$
$$E_5^+\cong E,\quad E_5^-\cong E-E_4,\quad E_4^+\cong E,\quad E_4^-\cong 0;$$
$$\mathcal{E}_4^2\cong m^2(E-E_4)^2/2E^2E_4^2,$$
$$\mathcal{E}_5^2\cong[M^2E_5^2+\mu^2E(E-E_5)]/2E^2E_5^2; \quad (43)$$

and where $E\equiv(E_1+E_2)/2$.

Inside the region of integration, the denominator factor becomes smaller with increasing E_4 and E_5, but does not vanish entirely; at the corner $E_4=E_5=E$, it has a minimum value $M^2/2$. The logarithm factor does not vary dramatically in the region of integration, except near the minimum corner of the denominator, where it becomes infinite as $\ln(E-E_4)$. Thus the dominant contribution to the integral comes from the region E_4 and $E_5\sim E$, and a number of alterations based on this behavior may be made to allow evaluations in closed form; these are consistent with the previous treatment of the numerator of Eq. (20).

$$p_4p_5(1+\cos\theta_{45}+\mathcal{E}_4^2+\mathcal{E}_5^2)\to 2(E-E_4)(E-E_5)+\tfrac{1}{2}M^2,$$
$$p_4p_5\mathcal{E}_4\mathcal{E}_5\to mM(E-E_4)/2E,$$
$$\int_{E-E_4}^{E}dE_5\to\int_0^E dE_5. \quad (44)$$

After these compromises,

$$I\cong\frac{2\pi^2}{E^2}\int_0^E dE_4\int_0^E dE_5\frac{1}{2(E-E_4)(E-E_5)+\frac{1}{2}M^2}\times\ln\frac{2(E-E_4)(E-E_5)+\frac{1}{2}M^2}{mM(E-E_4)/2E}. \quad (45)$$

The change of variable

$$x=M^2/[4(E-E_4)(E-E_5)+M^2],\qquad y=M^2/[4(E-E_4)^2+M^2], \tag{46}$$

transforms the integral to

$$I=\frac{\pi^2}{E^2}\int_{(M/2E)^2}^{1}\frac{dy}{y(1-y)}\int_{y}^{1}\frac{dx}{x}\ln\left[\frac{4E^2y}{mMx(1-y)}\right], \tag{47}$$

and after performing the x integration and some rearrangement,

$$I=\frac{\pi^2}{E^2}\int_{(M/2E)^2}^{1}dy\ln\frac{1}{y}\left\{\frac{1}{y}\ln\frac{4E^2y^{\frac{1}{2}}}{mM}+\frac{1}{1-y}\left[\frac{1}{y}\ln\frac{1}{1-y}+\ln\frac{4E^2y^{\frac{1}{2}}}{mM}\right]\right\}. \tag{48}$$

The terms in square brackets would lead to a finite integral even if the lower limit of integration were extended to zero, whereas the remaining term would diverge. Thus in the high-energy limit, the latter term is the dominant contribution to I:

$$I\cong\frac{\pi^2}{E^2}\int_{(M/2E)^2}^{1}\frac{dy}{y}\ln\frac{1}{y}\ln\frac{4E^2y^{\frac{1}{2}}}{mM}=\frac{\pi^2}{E^2}\frac{2}{3}\left(\ln\frac{2E}{M}\right)^2\ln\left[\left(\frac{2E}{m}\right)^3\left(\frac{2E}{M}\right)\right]. \tag{49}$$

By inserting Eq. (49) into Eq. (35), the absorptive part of the fourth-order amplitude in the high energy limit can now be exhibited explicitly. Before comparison can be made with the second-order amplitude, Eq. (9), however, the magnetic form factor must be evaluated in the same perturbation theoretic model just used for the fourth-order amplitude. This calculation has already been investigated for small momentum transfers by Rosenbluth[5] (proton) and Fried[8] (neutron). The relevant Feynman diagrams in the case of the neutron are given in Fig. 3; application of the usual rules for these diagrams leads to the parametric expression[8]

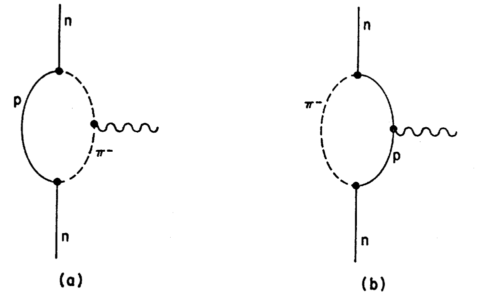

FIG. 3. Diagrams used in one-meson model to represent neutron form factors.

[8] B. D. Fried, Phys. Rev. **88**, 1142 (1952).

$$KF_2(q^2)=-\frac{g^2M^2}{16\pi^2}\int_0^1 2xdx\int_0^1 dy\left[\frac{x^2}{\Delta_{\rm I}}+\frac{(1-x)^2}{\Delta_{\rm II}}\right], \tag{50}$$

where

$$\Delta_{\rm I}=-\mu^2(1-x)-M^2x^2+q^2x^2y(1-y),$$
$$\Delta_{\rm II}=-\mu^2x-M^2(1-x)^2+q^2x^2y(1-y).$$

For $q^2\gg M^2$, the neighborhood of $x=0$ is strongly emphasized, and a valid approximation to this integral is

$$KF_2(q^2)\cong-\frac{g^2M^2}{16\pi^2}\int_0^1 2xdx\int_0^1\frac{dy}{-M^2+q^2x^2y(1-y)}. \tag{51}$$

The y integration is easily performed; for large momentum transfers, the resulting x integration may be approximated in a manner similar to that used above for the fourth-order amplitude. The result is

$$KF_2(q^2)\cong\frac{g^2}{16\pi^2}\frac{M^2}{q^2}\left[\ln\frac{q^2}{M^2}\right]^2,\quad q^2\gg M^2. \tag{52}$$

Note that in this model, F_2 falls off with increasing q^2, but quite slowly.

Now if the lesson of the simple integration example, Eqs. (26) and (27), is taken seriously, an additional factor $\sin^2\frac{1}{2}\theta$ should be supplied in the argument of the squared logarithm in Eq. (49); when this is done, the expression (52) for the magnetic form factor can be recognized in the resulting fourth-order amplitude. After this substitution is made,

$$\begin{aligned}\mathrm{Im}\langle f|T^{(4)}|i\rangle&\cong4\alpha^2KF_2(q^2)[2\epsilon_1\cos(\tfrac{1}{2}\theta)+\bar{\epsilon}_1\sin(\tfrac{1}{2}\theta)]\epsilon_2\cos(\tfrac{1}{2}\theta)\\&\times\delta^4(p_1'+p_2'-p_1-p_2)(q^2-M^2)^{-1}\\&\times\ln\left[\frac{2E}{m}\left(\frac{2E}{M}\right)^{\frac{1}{3}}\right].\end{aligned} \tag{53}$$

Employing this result, the ratio of the imaginary part of the fourth order amplitude to the second order amplitude is seen to vary roughly as $\alpha(M/E)\ln(E/M)$, a decreasing function of energy. For an electron center-of-mass energy of 10 Bev, the ratio is about 5%; it is below 1% for energies in excess of 50 Bev.

For estimating the real part of the fourth order amplitude from the imaginary part, two lines of argument are possible. The first is to exploit the similarity with the semiclassical imaginary part in order simply to claim that the real parts should correspond as well. The other line of argument is to proceed boldly with an application of a conventional relativistic energy dispersion relation, despite the difficulty of the substantial unphysical integration region, arising from the requirement of large momentum transfers. A fixed-momentum-transfer, no-subtraction dispersion relation for the fourth-order amplitude Eq. (53) (excluding kinematic factors) predicts

$$\mathrm{Re}\langle f|T^{(4)}|i\rangle = \mathrm{Im}\langle f|T^{(4)}|i\rangle \frac{1}{\pi} P\int^{\infty} dE'\left(\frac{1}{E'-E}-\frac{1}{E'+E}\right) \times[\ln(2E'/m^*)]/[\ln(2E/m^*)], \quad (54)$$

where $(m^*)^4 \equiv m^3M$. If the integration is carried out by just discarding the unphysical region, that is, setting the lower limit at $E_{\min}'=E\sin(\frac{1}{2}\theta)$, then

$$\frac{\mathrm{Re}\langle f|T^{(4)}|i\rangle}{\mathrm{Im}\langle f|T^{(4)}|i\rangle} \cong \frac{8}{3\pi}\sin(\tfrac{1}{2}\theta)\frac{\ln[2E\sin(\frac{1}{2}\theta)]/m^*}{\ln(2E/m^*)}; \quad (55)$$

if the analytic expression (53) is used even in the unphysical region where strictly speaking it is not valid, that is, setting $E_{\min}'=M$, then the ratio is

$$\frac{\mathrm{Re}\langle f|T^{(4)}|i\rangle}{\mathrm{Im}\langle f|T^{(4)}|i\rangle} \cong \frac{2\pi}{3\ln(2E/m^*)}. \quad (56)$$

In neither case does the real part decrease with energy less rapidly than the imaginary part.

If the effect of highly virtual photons is examined in more detail we see that the relation between the two may be expressed by Eqs. (17) and (13) with $n=1$. In the WKB approximation the imaginary part of $F(\theta)$ is then expected to fall off with a rate between $(k\theta)^{-3}$ and $(k\theta)^{-4}$. From Eqs. (52) and (53) we see that, for small finite angles and $k\to\infty$, $\mathrm{Im}\langle f|T^{(4)}|i\rangle \sim (k\theta)^{-4}\ln^2(k\theta)\ln k$, in qualitative agreement with the previous estimate. The ratio of real to imaginary parts as given by Eq. (56), approximately $1/\ln k$, is in agreement with Eq. (17) in that it has no angular dependence and is a decreasing function of k. Since the field-theoretic model does not reproduce the dominance of the resonance at low k in $\sigma_{\gamma\pi}$, the ratio is not expected to fall off as rapidly as the estimate of Eq. (18).

The explicit model calculations thus confirm the more general estimates of the WKB approximation and again raise no hint that the polarizability corrections will be significant for finite-angle electron scattering at very high energies.