Two-Nucleon Interaction from Doublet Splittings

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The splitting of closely spaced doublet levels, of spin $J+\frac{1}{2}$ and $J-\frac{1}{2}$, is investigated in cases which can be approximated by a core of spin J and an s nucleon. Allowance is made for the Thomas shifts of the levels. Using j-j coupling wave functions, the exchange parameters and strength of the effective interaction between nucleons in the nucleus are determined by a least-squares fit to the doublet splittings and give some improvement over results obtained with previously accepted parameters. It is suggested that measurement of the sign of some doublet splittings would clarify the interaction further.

1. INTRODUCTION

THERE is evidence for the existence in several nuclei of closely spaced double levels of spin $J+\frac{1}{2}$ and $J-\frac{1}{2}$ which apparently consist of a core of spin J plus an s nucleon. The small splitting of these s-particle doublets has been taken as an indication of the weak spin dependence of the effective central interaction between nucleons. Such doublets have been studied[1–3] in Al^{28}, P^{32}, Tl^{208}, and B^{11}. Evidence for unresolved doublets in several medium weight nuclei was obtained by Cohen and Price[4] in their (d,p) experiments. Doublets in nuclei with $A=14$ and 16 form part of more general analyses by Unna and Talmi[5] and by Elliott and Flowers.[6]

We have calculated the splittings of several of these doublets on the basis of a simple shell-model description of the nuclear states, in the j-j coupling extreme, and a central interaction between pairs of nucleons. Levels produced by addition of a d nucleon to a spin $\frac{1}{2}$ core have also been included.

The experimental data are presented in Sec. 2. In Sec. 3, the effect of certain factors which can change the splittings is discussed, these are configuration impurities, the Coulomb interaction and Thomas shifts. Formulas for the splittings in terms of two-particle matrix elements are given in Sec. 4 and these matrix elements are evaluated in Sec. 5 for harmonic oscillator wave functions and for a Yukawa-shaped interaction. In Sec. 6, the exchange parameters and strength of the interaction are determined by a least-squares fit to certain doublet splittings in nuclei with $A=14$ and 16. The remaining splittings are calculated and compared with the experimental values, and comparison is also made with splittings calculated for values of the exchange parameters used in previous calculations.

2. EXPERIMENTAL DATA

The data we have used are given in Table I. Configuration assignments in j-j coupling are given for unfilled shells only, and levels are listed by their isobaric spin (T) and total angular momentum (J) values and excitation energies E_{TJ}. Energy values are taken from references.[2–4,7] Levels for $A=14$ and 16 nuclei which are considered to have the least configuration impurity (see Sec. 3) are italicized; the purity of the levels of other nuclei is not known. The first two columns giving differences in energies of levels contain experimental values, those in column (a) being the splittings obtained from the observed excitation energies while in column (b) these have been corrected for the effect of Thomas shifts (Sec. 3). The remaining columns give splittings calculated for the shell model with various exchange parameters.

3. EFFECTS OF CONFIGURATION IMPURITIES, COULOMB INTERACTION AND THOMAS SHIFTS

In Table I, definite j-j coupled wave functions are assigned to each level. This does not mean that we believe j-j coupling to be accurate for all states of all nuclei, but there is experimental and theoretical evidence that certain states are well represented in this way. Thus the calculations of Elliott and Flowers[6] for the negative-parity states of O^{16} and N^{16} indicate that even in the intermediate-coupling region which best fits the observed energy spectra, some of the states, notably the lowest states with $T=1$, belong almost entirely to the configurations $1p_{\frac{1}{2}}^3\,2s_{\frac{1}{2}}$ or $1p_{\frac{1}{2}}^3\,1d_{\frac{3}{2}}$, while the lowest $(TJ)=(01)$, (02), and (03) states contain 75% or less of these configurations. For the lowest (03) state, the presence of other configurations ($1p_{\frac{3}{2}}^7\,1d_{\frac{3}{2}}$ and $1p_{\frac{3}{2}}^7\,1d_{\frac{5}{2}}$) lowers its energy by about 3 Mev. Thus it is important to eliminate from the doublet splittings to be fitted those involving any level in which there is evidence of considerable configuration impurity.

The calculations of Elliott and Flowers[6] suggest that it is safe to use doublet splittings for the $A=16$ nuclei involving only the (00), (10), (11), (12), and (13) levels. The experimental evidence[8,9] on the configuration

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[5] I. Unna and I. Talmi, Phys. Rev. **112**, 452 (1958).
[6] J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **A242**, 57 (1957).
[7] F. Ajzenberg-Selove and T. Lauritsen, Nuclear Phys. **11** 1 (1959).
[8] E. K. Warburton and J. N. McGruer, Phys. Rev. **105**, 639 (1957).
[9] F. B. Hagedorn and J. B. Marion, Phys. Rev. **108**, 1015 (1957).

TABLE I. Doublet splittings. Each j-j coupling configuration in column 2 gives rise to levels T, J, E, and T', J', E'. Column (a) gives the observed energy splittings between these levels, column (b) the splittings corrected for Thomas shifts. Theoretical splittings are given in column (c) using the exchange parameters of Table IV with $A_{10}=-0.5$, in column (d) using Elliott and Flowers' exchange parameters, and in column (e) using Rosenfeld's exchange parameters. Levels for $A=14$ and 16 nuclei which are considered to have the least configuration impurity are indicated by italicized TJ (or $T'J'$) numbers.

						$E_{TJ}-E_{T'J'}$(Mev)				
						Experimental		Calculated		
Nucleus	Configuration	TJ	E_{TJ} (Mev)	$T'J'$	$E_{T'J'}$ (Mev)	(a)	(b)	(c)	(d)	(e)
C^{14}	$1p_{\frac{1}{2}}\,2s_{\frac{1}{2}}$	*10*	6.89	*11*	6.09	0.80	1.33	0.68	1.41	0.97
N^{14}	$1p_{\frac{1}{2}}\,2s_{\frac{1}{2}}$	*10*	8.71	*11*	8.06	0.65	1.08			
N^{14}	$1p_{\frac{1}{2}}\,2s_{\frac{1}{2}}$	*00*	4.91	*01*	5.68	−0.77	−1.06	−1.27	−1.11	−2.57
N^{14}	$1p_{\frac{1}{2}}\,2s_{\frac{1}{2}}$	*10*	8.71	*00*	4.91	3.80	5.85	5.93	5.68	4.36
C^{14}	$1p_{\frac{1}{2}}\,1d_{\frac{5}{2}}$	*12*	7.35	*13*	6.72	0.63	0.78	0.99	1.92	1.32
N^{14}	$1p_{\frac{1}{2}}\,1d_{\frac{5}{2}}$	*12*	9.50	*13*	8.90	0.60	0.76			
N^{14}	$1p_{\frac{1}{2}}\,1d_{\frac{5}{2}}$	02	5.10	*03*	5.83	−0.73	−0.84	−1.47	−0.93	−0.75
N^{14}	$1p_{\frac{1}{2}}\,1d_{\frac{5}{2}}$	*13*	8.90	*03*	5.83	3.07	3.72	3.56	2.40	1.27
N^{16}	$1p_{\frac{1}{2}}{}^3\,2s_{\frac{1}{2}}$	*10*	0.12	*11*	0.39	−0.27	−0.38	−0.29	0.15	−0.80
O^{16}	$1p_{\frac{1}{2}}{}^3\,2s_{\frac{1}{2}}$	*10*	12.78	*11*	13.09	−0.31	−0.51			
O^{16}	$1p_{\frac{1}{2}}{}^3\,2s_{\frac{1}{2}}$	*00*	10.94	01	7.12	3.82	5.14	1.65	2.66	2.75
O^{16}	$1p_{\frac{1}{2}}{}^3\,2s_{\frac{1}{2}}$	*10*	12.78	*00*	10.94	1.84	2.94	3.02	1.91	−0.96
N^{16}	$1p_{\frac{1}{2}}{}^3\,1d_{\frac{5}{2}}$	*12*	0.00	*13*	0.30	−0.30	−0.36	−0.24	0.50	0.29
O^{16}	$1p_{\frac{1}{2}}{}^3\,1d_{\frac{5}{2}}$	*12*	12.96	*13*	13.25	−0.29	−0.35			
O^{16}	$1p_{\frac{1}{2}}{}^3\,1d_{\frac{5}{2}}$	02	8.88	03	6.14	2.74	3.13	2.23	3.35	2.36
O^{16}	$1p_{\frac{1}{2}}{}^3\,1d_{\frac{5}{2}}$	*12*	12.96	02	8.88	4.08	4.84	3.14	1.93	0.93
B^{11}	$1p_{\frac{3}{2}}{}^6\,2s_{\frac{1}{2}}$	$\frac{1}{2}\frac{7}{2}$	9.19	$\frac{1}{2}\frac{5}{2}$	9.28	−0.09		0.67	2.72	0.30
Al^{28}	$1d_{\frac{5}{2}}{}^{11}\,2s_{\frac{1}{2}}$	13	0.00	12	0.03	−0.03		−0.46	0.17	−1.08
P^{32}	$2s_{\frac{1}{2}}{}^3\,1d_{\frac{5}{2}}$	12	0.08	11	0.00	0.08		0.31	−0.11	0.72
Y^{90}	$(2p_{\frac{1}{2}})_p(3s_{\frac{1}{2}})_n$	1		0		±0.06		0.11	−0.04	0.26
Rh^{104}	$(2p_{\frac{1}{2}})_p(3s_{\frac{1}{2}})_n$	1		0		±0.06				
Nb^{94}	$(1g_{9/2})_p(3s_{\frac{1}{2}})_n$	5		4		±0.06		−0.09	0.05	−0.25
In^{114}	$(1g_{9/2}{}^9)_p(3s_{\frac{1}{2}})_n$	5		4		±0.06				
Tl^{208}	$(3s_{\frac{1}{2}})_p(2g_{9/2})_n$	5	0.00	4	0.04	−0.04		−0.09	0.04	−0.22

purity of these $T=1$ states is based on the relative reduced widths $\mathcal{S}$ for nucleon emission as defined by French.[10,11] The $N^{15}(d,p)N^{16}$ reaction[8] gives $\mathcal{S}\approx 1$ for each of the N^{16} levels.[12] The $N^{15}(p,p)N^{15}$ results,[9] with $\theta_0^2(s)=0.53$, $\theta_0^2(d)=0.29$ as obtained[13] from $C^{12}(p,p)C^{12}$ for the 2.37- and 3.56-Mev levels of N^{13}, give the relative reduced widths $\mathcal{S}=0.42$, 0.38, 0.61, 0.51 for the (10), (11), (12), (13) levels of O^{16}. These low values may indicate considerable configuration impurities, but later in this section it is shown that their smallness may be due to isobaric spin mixing without any need for configuration mixing.

The experimental reduced widths for the C^{14} and N^{14} levels are given by Warburton *et al.*[11] They indicate the extent to which the levels consist of C^{13} ground state plus an s or d nucleon. A shell-model calculation[14] for the negative-parity states of C^{13}, N^{13} shows that many experimental data can be fitted for an intermediate-coupling parameter $\xi/K\approx 4.6$, and then the ground state of C^{13} contains 80% (in intensity) of the configuration $1s_{\frac{1}{2}}{}^4\,1p_{\frac{3}{2}}{}^8\,1p_{\frac{1}{2}}$. Thus for relative reduced widths $\mathcal{S}\approx 1$, about 80% of the listed C^{14}, N^{14} states would be expected to belong to configurations shown (neglecting $1d_{\frac{3}{2}}$ compared with $1d_{\frac{5}{2}}$ nucleons), but presence of other configurations in the C^{13} core would probably not change the calculated splittings appreciably.

The (10), (11), (12), and (13) levels of C^{14} all have $\mathcal{S}=1$ within the experimental uncertainty.[12] The analogous N^{14} levels have $\mathcal{S}\approx 0.9$, 0.6, 0.6, and 0.7, respectively, for proton emission.[11,13] There is little evidence on the $\mathcal{S}$ values for the $T=0$ levels of N^{14}, but apparently[11] $\mathcal{S}\approx 1$ for the (00), (01), and (03) levels and $\mathcal{S}<1$ for the (02) level.

Corresponding $T=1$ levels of C^{14} and N^{14} are expected to have the same configuration impurities, and the difference in their $\mathcal{S}$ values should therefore be attributed to some charge-dependent interaction. The Coulomb interaction, by mixing states of different T values, can have this effect. Consider the nominally (01) and (11) states of the configuration $1p_{\frac{1}{2}}\,2s_{\frac{1}{2}}$ of N^{14} at 5.68 and 8.06 Mev, respectively. The eigenvalues and eigenstates of the Hamiltonian including the Coulomb interaction will be of the form

$$E=E_a:\quad \Phi_a=\alpha_a\Psi(01)+\beta_a\Psi(11),\quad \alpha_a^2+\beta_a^2=1,$$
$$E=E_b:\quad \Phi_b=\alpha_b\Psi(01)+\beta_b\Psi(11),\quad \alpha_b^2+\beta_b^2=1,$$

with

$$\frac{\beta_a}{\alpha_a}=-\frac{\alpha_b}{\beta_b}\approx-\frac{(H^{\mathrm{Coul}})_{01}}{E_b-E_a},$$

where $(H^{\mathrm{Coul}})_{01}$ is the matrix element of the Coulomb

[10] J. B. French, Phys. Rev. **103**, 1391 (1956).
[11] E. K. Warburton, H. J. Rose, and E. N. Hatch, Phys. Rev. **114**, 214 (1959).
[12] M. H. Macfarlane and J. B. French, Revs. Modern Phys. **32**, 567 (1960).
[13] The values of $\theta^2(s)$, $\theta^2(d)$ for $a=4.7\times10^{-13}$ cm given in reference 11 have been combined with values $\mathcal{S}=0.95$, 0.78, respectively, which result from a shell-model calculation of the $A=13$ positive-parity states.
[14] A. M. Lane, Proc. Phys. Soc. (London) **A68**, 197 (1955).

TABLE II. Relative reduced widths $\mathcal{S}$ for nucleon emission from N^{14} states..

J	0			1			2			3		
E_b-E_a(Mev)		3.80			2.38			4.40			3.07	
$(H^{\text{Coul}})_{01}$(Mev)	−0.2	−0.3	−0.4	−0.2	−0.3	−0.4	−0.2	−0.3	−0.4	−0.2	−0.3	−0.4
$\mathcal{S}_a$ calc	1.11	1.16	1.21	1.17	1.25	1.34	1.09	1.14	1.18	1.13	1.20	1.26
$\mathcal{S}_a$ expt		≈1			≈1			<1			≈1	
$\mathcal{S}_b$ calc	0.89	0.84	0.79	0.83	0.75	0.66	0.91	0.86	0.82	0.87	0.80	0.74
$\mathcal{S}_b$ expt		0.86±0.08			0.57±0.02			0.55±0.03			0.70±0.09	

interaction between the states $\Psi(01)$ and $\Psi(11)$. Wilkinson and Bloom[15] have suggested an isobaric spin impurity $|\alpha_b/\beta_b|^2\approx 7\%$ in order to account for observed $E1$ γ-ray intensities in N^{14}, and Warburton *et al.*,[11] note that this amount of impurity could also account for the difference in splittings of the (10) and (11) levels in C^{14} and in N^{14}. Since the reduced widths for nucleon emission contain a Clebsch-Gordan coefficient depending on isobaric spin, the relative reduced widths for proton emission are changed from $\mathcal{S}=1$ for the states $\Psi(01)$ and $\Psi(11)$ to $\mathcal{S}_a=|\alpha_a+\beta_a|^2$, $\mathcal{S}_b=|\alpha_b+\beta_b|^2=|\alpha_a-\beta_a|^2$ for the states Φ_a, Φ_b, respectively.[16] To get $\mathcal{S}_b\approx 0.6$, an isobaric spin impurity of about 4% is required.

For harmonic oscillator wave functions with length parameter $a=1.65\times 10^{-13}$ cm, $(H^{\text{Coul}})_{01}=-0.16$ Mev, giving $|\alpha_b/\beta_b|^2\approx 0.4\%$. It was previously argued[17] that the calculated matrix elements of the Coulomb interaction for configurations of this type were too small to give isobaric spin impurities of the size suggested by Wilkinson and Bloom.[15] These calculations were also based on the use of oscillator wave functions, and it has since been found that the large amount of cancellation of matrix elements which occurs in this case is greatly decreased by the use of other wave functions; for wave functions corresponding to a potential well of finite depth $-(\hbar^2/2Ma^2)(1/\alpha)\exp(-\alpha r^2/a^2)$ with reasonable values of α and a, the Coulomb matrix elements can be increased by factors of order 10 or more, up to magnitudes of about 0.5 Mev. These values are probably more realistic.

Similar isobaric spin mixing is expected in the N^{14} levels with other J values. The calculated value of the Coulomb matrix element is the same for $J=0$ as for $J=1$, and for $J=2$ or 3 is $(H^{\text{Coul}})_{01}=-0.32$ Mev again for oscillator wave functions. In Table II, values of $\mathcal{S}$ are given for various values of the Coulomb matrix element and compared with the experimental values; for $\mathcal{S}_b$ these include errors due to uncertainties in the measured widths of levels. For $(H^{\text{Coul}})_{01}\approx -0.4$ Mev, the only clear discrepancies between calculated and experimental values are for the $J=2$ levels. The low experimental value of $\mathcal{S}_b$ for $J=2$ could be obtained if a (02) level of N^{14} containing an appreciable amount of the configuration $1p_{\frac{1}{2}}\,1d_{\frac{5}{2}}$ were situated near the (12) level at 9.50 Mev, and this is not improbable. Thus the only level of C^{14} and N^{14} for which there is clear evidence of configuration impurity is the (02) N^{14} level at 5.10 Mev.

The splitting of the (10) and (11) levels in N^{14} is calculated to be 0.03 Mev less than the corresponding splitting in C^{14}, for $(H^{\text{Coul}})_{01}=-0.4$ Mev. Thus this effect does not account for much of the observed difference of 0.15 Mev. The isobaric spin impurity of the (11) level is about 3%. Energy shifts due to admixture of states of different isobaric spin are small compared with those expected for comparable admixtures of states of other configurations, because the off-diagonal matrix elements of the Coulomb interaction are small compared with the expected matrix elements of the interaction coupling states of different configurations.

The effect of isobaric spin mixing on the $\mathcal{S}$ values for O^{16} states should also be considered. For $(H^{\text{Coul}})_{01}=-0.4$ Mev, and for the states in Table I, $\mathcal{S}_b=0.56$, 0.87, 0.80, 0.89 for the nominally (10), (11), (12), (13) states compared with the experimental values 0.42, 0.38, 0.61, 0.51, respectively. For $J=1$, 2, 3, however, there is evidence[7] for $T=0$ states closer to the $T=1$ states than those we have chosen, and as for the (12) state of N^{14} these may cause the $\mathcal{S}_b$ values to be decreased to near the experimental values for the (11), (12), and (13) states.

A different way in which the Coulomb interaction might contribute to the splittings of doublet levels is that it might have different expectation values for the two levels. Actually for the $A=14$ and $A=16$ levels considered here, the only nonzero contributions to the splittings occur for O^{16} levels of the same T and different J, but these contributions are of order 0.01 Mev and so can be neglected.

The effect on the splittings of Thomas shifts[18] has still to be considered. These shifts are due to the differences, for each channel c, in the values of the logarithmic derivatives evaluated at the interaction radius a_c of the shell-model single particle wave function $(-\bar{b}_c)$ and of the single-particle wave function needed to fit on to the wave function in the external region $[g_c^{\text{Re}}(\epsilon_c)]$. The contribution to the shift from channel c is proportional to the reduced width γ_c^2 of the level

[15] D. H. Wilkinson and S. D. Bloom, Phil. Mag. **2**, 63 (1957).
[16] F. C. Barker and A. K. Mann, Phil. Mag. **2**, 5 (1957).
[17] F. C. Barker, Phil. Mag. **2**, 286 (1957).
[18] R. G. Thomas, Phys. Rev. **88**, 1109 (1952).

for breaking up into that channel. The Thomas shifts for the levels of the $A=14$, 16 nuclei given in Table I have been calculated, assuming pure configurations and including only the channels for neutron emission from C^{14} and N^{16} to the ground states of C^{13} and N^{15}, and for neutron and proton emission from N^{14} and O^{16} to the ground states of N^{13}, C^{13} and O^{15}, N^{15}. Isobaric spin mixing is not expected to affect the shifts appreciably, because for the N^{14} and O^{16} levels S(neutrons) $+S$(protons)$=2$, irrespective of the amount of mixing. Then the difference in the shifts for two levels of a doublet does not depend on the value taken for $\bar{b}_c$, as this is the same for all states of the same configuration. In calculating the shifts, we have used $a_c=4.8\times10^{-13}$ cm, $\theta_0^2(s)=0.53$, $\theta_0^2(d)=0.29$, and for convenience taken Coulomb wave functions appropriate to proton emission from oxygen. In Table I the contributions to the splittings calculated to be due to Thomas shifts have been subtracted from the measured splittings to give the values in the column headed (b).

For the remaining nuclei in Table I, it is not known how well the levels belong to the configurations shown. In the case of the nuclei investigated by Cohen and Price,[4] the "levels" observed may actually represent averages over several levels containing parts of the given configurations, and the resultant splittings may therefore be just the quantities calculated; the experimental splittings are order of magnitude estimates and the signs are not known. Correction for Thomas shifts should increase the magnitude of the splittings for all these nuclei to some extent.

4. CALCULATION OF SPLITTINGS IN TERMS OF TWO-PARTICLE MATRIX ELEMENTS

The splittings are assumed to be due to a central interaction between pairs of nucleons[6]:

$$V^c=\sum_{i<j}[W+MP_{ij}^x-HP_{ij}^\tau+BP_{ij}^\sigma]v(r_{ij}).$$

A one-body spin-orbit interaction gives no contribution to the splittings calculated here, and two-body spin-orbit, tensor and many-body interactions are neglected. The splittings can be calculated by standard techniques[19,20] with j-j fractional parentage coefficients[21]; in all cases except for the B^{11} levels, the fractional parentage coefficients have simple forms.

Formulas for the splittings of several of the s-particle doublets have been given previously.[3,5,22] Some of the levels are also included in the calculation of Elliott and Flowers.[6] The splittings can be expressed in terms of two-particle matrix elements

$$\langle nln'l'[\lambda]L\|v(r)\|nln'l'[\lambda]L\rangle$$

of the potential $v(r)$; the two particles in the single particle states nl and $n'l'$ are combined to form a state of orbital angular momentum L with orbital symmetry or antisymmetry for $[\lambda]=[2]$ or $[11]$, respectively. The coefficients of these matrix elements can be written in terms of the exchange parameters A_{TS}, where

$$A_{01}=W+M+H+B=-1,\quad A_{10}=W+M-H-B,$$
$$A_{00}=W-M+H-B,\quad A_{11}=W-M-H+B,$$

corresponding to the two particles in a state of isobaric spin T and ordinary spin S. $[\lambda]$ is $[2]$ or $[11]$ for $(-)^{T+S+1}=1$ or -1, respectively.

Formulas for all the splittings listed in Table I are given in Appendix 1, using the abbreviation

$$\langle nln'l'[\lambda]L\|v(r)\|nln'l'[\lambda]L\rangle=X_{\pm L}$$

for levels of the configuration $nl_j^m n'l'_{j'}$, the $+$ or $-$ sign referring to $[\lambda]=[2]$ or $[11]$. It is to be noted that if the two-particle matrix elements are assumed not to change from nucleus to nucleus, the splittings are not all independent, but those for $A=11$, 14, and 16 are related, and those for $A=28$ and 32 are related.

5. EVALUATION OF TWO-PARTICLE MATRIX ELEMENTS

The matrix elements

$$\langle nln'l'[\lambda]L\|v(r)\|nln'l'[\lambda]L\rangle$$

can be evaluated when the single-particle radial wave functions $u_{nl}(r_1)$ and the potential function $v(r_{12})$ are prescribed. For a harmonic oscillator potential well with length parameter a,

$$u_{nl}(r)=\left(\frac{2}{\Gamma(n+l+\frac{1}{2})\Gamma(n)}\right)^{\frac{1}{2}}(r/a)^l\exp(-r^2/2a^2)$$
$$\times\sum_{s=0}^{n-1}(-)^s(r/a)^{2s}\frac{\Gamma(n+l+\frac{1}{2})\Gamma(n)}{\Gamma(s+l+\frac{3}{2})\Gamma(s+1)\Gamma(n-s)},$$

normalized by

$$\int_0^\infty u_{nl}^2(r)r^2dr=a^3.$$

The potential $v(r)$ is assumed to have a Yukawa shape:

$$v(r)=V_0e^{-\chi r}/\chi r,$$

with range χ^{-1} and depth V_0. The two-particle matrix elements can then be calculated by standard methods.[23] They are proportional to V_0 and depend on χ and a only through the product χa, being roughly proportional to $(\chi a)^{-2}$. We take χ^{-1} equal to the π-meson Compton wavelength $\chi^{-1}=\hbar/m_\pi c=1.41\times10^{-13}$ cm; the value of V_0 required to fit the deuteron binding energy is then 50.0 Mev. For nuclei with $A\geqslant28$, the length parameter a is chosen for each nucleus to fit the measured root-

[19] J. P. Elliott, Proc. Roy. Soc. (London) **A218**, 345 (1953).
[20] J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **A229**, 536 (1955).
[21] A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) **A214**, 515 (1952).
[22] F. C. Barker, Nuclear Phys. **19**, 110 (1960).
[23] H. A. Jahn (unpublished notes).

TABLE III. Radius parameter a of harmonic oscillator potential well and χa, where χ is the inverse pion Compton wavelength, for various mass numbers A.

A	11	14	16	28	32	90	94	104	114	208
$a(10^{-13}$ cm)	1.55	1.64	1.66	1.69	1.74	2.09	2.11	2.14	2.15	2.41
χa	1.10	1.16	1.18	1.20	1.23	1.48	1.50	1.52	1.52	1.71

mean-square radius, taken as $(\frac{3}{5})^{\frac{1}{2}}r_0A^{\frac{1}{3}}$ with $r_0=1.2\times10^{-13}$ cm. For the lighter nuclei we use values of a determined from differences of Coulomb energies for analogous states of neighboring isobars. This leads to the values of a and χa in Table III. For simplicity we take $\chi a=1.2$ for $A\leqslant32$, $\chi a=1.6$ for $A\geqslant90$ in the calculations. Changes in χa give splittings proportional to $(\chi a)^{-2}$ for fixed V_0. The two-particle matrix elements then have the values given in Appendix 2.

6. DETERMINATION OF EXCHANGE PARAMETERS A_{TS} AND POTENTIAL DEPTH V_0

We now choose the parameters A_{TS} and the potential depth V_0 to make the calculated splittings agree as well as possible with the experimental splittings obtained after subtraction of Thomas shifts, for those doublets in nuclei with $A=14$ and 16 which involve only levels of pure configurations (italicized in Table I). There are more data than unknowns, so we use a least-squares fit. This is similar to the method of Unna and Talmi[5] except that they included the two-particle matrix elements in the unknown parameters, and also they did not consider only separations of levels of similar type. A wide range of parameters gives near the minimum root-mean-square deviation, defined as in reference 5; for given values of A_{10} in the range $-0.2\geqslant A_{10}\geqslant-0.8$, which is of most interest from other considerations, the parameters A_{00}, A_{11}, and V_0 have been varied to give the best fit and the resulting values are given in Table IV.

The splittings calculated with these parameters for $A_{10}=-0.5$ are given in column (c) of Table I. It is seen that, for the splittings fitted, the greatest discrepancy with the values of column (b) occurs for the $E_{10}-E_{11}$ splittings of C^{14} and N^{14}; since this splitting is calculated to be about 0.7 times the $E_{12}-E_{13}$ splitting in C^{14} and N^{14} more or less independent of the A_{TS} and V_0 values it is obvious that some discrepancy must occur as the experimental ratio is about 1.5. An intermediate-coupling calculation similar to that of Elliott and Flowers[6] for the parameters of Table IV with $A_{10}=-0.5$ has been carried out to estimate the effect of configuration mixing on the splittings involving the negative-parity states of the $A=16$ nuclei. The energies of the lowest states of given (TJ), calculated for spin-orbit strengths $\xi_{1p}=-4.22$ Mev, $\xi_{1d}=-2.03$ Mev, and with $2s$ and $1d$ single-particle energies adjusted to fit the relative energy of the lowest two states of O^{17}, are compared in Table V both with the experimental energies (as observed and after subtraction of Thomas shifts calculated for $a_c\bar{b}_c=3$) and with those calculated in the j-j coupling extreme. It is seen that with these exchange parameters, these states of the $A=16$ nuclei are more or less pure j-j coupled states, except for the (03) state. Then for the $A=16$ splittings the greatest remaining discrepancy is in the $E_{00}-E_{01}$ values.

The difficulty of getting the lowest (01) level of O^{16} at a low enough energy was noted by Elliott and Flowers.[6] As there are more (01) levels of O^{16} in the region below about 13 Mev than can be explained using the configurations assumed by Elliott and Flowers, and in the above calculation, it seems that other configurations must be giving appreciable contributions and so depressing the energy of the lowest (01) level.

Also given in Table I are splittings calculated for the Elliott and Flowers[6] values [column (d)] and the Rosenfeld[24] values [column (e)] of the A_{TS}, with V_0 adjusted to give the best fit to the $A=14$, 16 splittings involving italicized levels. It is seen that the values in column (c) agree substantially better than those in columns (d) or (e) with the experimental values in column (b). As far as the splittings for other nuclei are concerned, a determination of the sign of the splittings observed by Cohen and Price[4] would be of interest.

TABLE IV. Sets of exchange parameters A_{TS} and well depth V_0 which fit doublet splittings for nuclei with $A=14$ and $A=16$, together with Elliott and Flowers' values [column (d)] and Rosenfeld's values [column (e)].

							(d)	(e)
A_{01}	−1	−1	−1	−1	−1	−1	−1	−1
A_{10}	−0.2	−0.4	−0.5	−0.6	−0.7	−0.8	−0.7	−0.6
A_{00}	−0.343	−0.652	−0.806	−0.959	−1.111	−1.262	−0.5	1.8
A_{11}	0.323	−0.145	−0.376	−0.602	−0.820	−1.018	0.26	0.333
V_0 (Mev)	48.6	64.7	77.3	95.4	122.9	164.8	48.3	35.6
Rms deviation (Mev)	0.33	0.31	0.31	0.32	0.37	0.52	0.84	1.58

[24] L. Rosenfeld, *Nuclear Forces* (North-Holland Publishing Company, Amsterdam 1948).

TABLE V. Relative energies (in Mev) of the lowest negative-parity states of the $A=16$ nuclei. Line 2 gives the experimental values. Line 3 gives the experimental values corrected for Thomas shifts. Lines 4, 5 are theoretical values.

TJ	00	01	02	03	10	11	12	13
Experimental	10.94	7.12	8.88	6.14	12.78	13.09	12.96	13.25
Experimental (corrected for Thomas shifts)	10.94	5.80	7.15	4.02	13.88	14.39	11.99	12.34
jj coupling extreme	10.94	9.29	8.43	6.20	13.96	14.25	11.57	11.81
Intermediate coupling	10.94	9.13	8.31	5.04	13.83	13.94	11.03	11.49

The effective central interaction between nucleons obtained from fitting these doublet splittings is more or less spin-independent, for $A_{10}\approx-0.6$, and so differs markedly from the interaction used to fit p-shell data, in agreement with the conclusion of Bilaniuk and French.[3]

APPENDIX 1. FORMULAS FOR DOUBLET SPLITTINGS

$A=14$ $1p_{1/2}\,2s_{1/2}$

$$E_{10}-E_{11}=A_{10}(-1/3X_{+1})+A_{11}(1/3X_{-1})$$
$$E_{00}-E_{01}=A_{01}(1/3X_{+1})+A_{00}(-1/3X_{-1})$$
$$E_{10}-E_{00}=A_{01}(-X_{+1})+A_{11}(X_{-1})$$

$1p_{1/2}\,1d_{5/2}$

$$E_{12}-E_{13}=A_{10}(2/15X_{+2}-1/3X_{+3})+A_{11}(12/25X_{-1}+2/15X_{-2}-31/75X_{-3})$$
$$E_{02}-E_{03}=A_{01}(108/225X_{+1}+2/15X_{+2}-31/75X_{+3})+A_{00}(2/15X_{-2}-1/3X_{-3})$$
$$E_{13}-E_{03}=A_{01}(-2/9X_{+2}-4/9X_{+3})+A_{10}(1/3X_{+3})+A_{00}(-1/3X_{-3})+A_{11}(2/9X_{-2}+4/9X_{-3})$$

$A=16$ $1p_{1/2}{}^3\,2s_{1/2}$

$$E_{10}-E_{11}=A_{01}(1/6X_{+1})+A_{10}(-1/6X_{+1})+A_{00}(-1/6X_{-1})+A_{11}(1/6X_{-1})$$
$$E_{00}-E_{01}=A_{01}(-1/6X_{+1})+A_{10}(-1/2X_{+1})+A_{00}(1/6X_{-1})+A_{11}(1/2X_{-1})$$
$$E_{10}-E_{00}=A_{01}(-1/2X_{+1})+A_{10}(1/2X_{+1})+A_{00}(-1/2X_{-1})+A_{11}(1/2X_{-1})$$

$1p_{1/2}{}^3\,1d_{5/2}$

$$E_{12}-E_{13}=A_{01}(6/25X_{+1}+1/15X_{+2}-31/150X_{+3})+A_{10}(1/15X_{+2}-1/6X_{+3})+A_{00}(1/15X_{-2}-1/6X_{-3})+A_{11}(6/25X_{-1}+1/15X_{-2}-31/150X_{-3})$$
$$E_{02}-E_{03}=A_{01}(-6/25X_{+1}-1/15X_{+2}+31/150X_{+3})+A_{10}(1/5X_{+2}-1/2X_{+3})+A_{00}(-1/15X_{-2}+1/6X_{-3})+A_{11}(18/25X_{-1}+1/5X_{-2}-31/50X_{-3})$$
$$E_{12}-E_{02}=A_{01}(2/25X_{+1}-1/5X_{+2}-77/150X_{+3})+A_{10}(-1/45X_{+2}+7/18X_{+3})+A_{00}(1/45X_{-2}-7/18X_{-3})+A_{11}(-2/25X_{-1}+1/5X_{-2}+77/150X_{-3})$$

$A=11$ $1p_{3/2}{}^6\,2s_{1/2}$

$$E_{1/2\,7/2}-E_{1/2\,5/2}=A_{01}(7/24X_{+1})+A_{10}(-7/8X_{+1})+A_{00}(-7/24X_{-1})+A_{11}(7/8X_{-1})$$

$A=28$ $1d_{5/2}{}^{11}\,2s_{1/2}$

$$E_{13}-E_{12}=A_{01}(3/10X_{+2})+A_{10}(-3/10X_{+2})+A_{00}(-3/10X_{-2})+A_{11}(3/10X_{-2})$$

$A=32$ $2s_{1/2}{}^3\,1d_{3/2}$

$$E_{12}-E_{11}=A_{01}(-1/5X_{+2})+A_{10}(1/5X_{+2})+A_{00}(1/5X_{-2})+A_{11}(-1/5X_{-2})$$

$A=90$ 104 $(2p_{1/2})_p(3s_{1/2})_n$

$$E_1-E_0=A_{01}(-1/6X_{+1})+A_{10}(1/6X_{+1})+A_{00}(1/6X_{-1})+A_{11}(-1/6X_{-1})$$

$A=94$ 114 $(1g_{9/2}{}^{1,9})_p(3s_{1/2})_n$

$$E_5-E_4=A_{01}(5/18X_{+4})+A_{10}(-5/18X_{+4})+A_{00}(-5/18X_{-4})+A_{11}(5/18X_{-4})$$

$A=208$ $(3s_{1/2})_p(2g_{9/2})_n$

$$E_5-E_4=A_{01}(5/18X_{+4})+A_{10}(-5/18X_{+4})+A_{00}(-5/18X_{-4})+A_{11}(5/18X_{-4})$$

APPENDIX 2. VALUES OF TWO-PARTICLE MATRIX ELEMENTS

$\chi a=1.2$	$1p\ 2s$	$X_{+1}=0.10091V_0$	$X_{-1}=0.06433V_0$
	$1p\ 1d$	$X_{+1}=0.16308V_0$	$X_{-1}=0.06399V_0$
		$X_{+2}=0.02895V_0$	$X_{-2}=0.06267V_0$
		$X_{+3}=0.15130V_0$	$X_{-3}=0.02739V_0$
	$1d\ 2s$	$X_{+2}=0.07987V_0$	$X_{-2}=0.04680V_0$
$\chi a=1.6$	$2p\ 3s$	$X_{+1}=0.03435V_0$	$X_{-1}=0.02069V_0$
	$1g\ 3s$	$X_{+4}=0.01825V_0$	$X_{-4}=0.01195V_0$
	$2g\ 3s$	$X_{+4}=0.01764V_0$	$X_{-4}=0.01040V_0$