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Nonlinear Electrodynamics in General Relativity*

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General relativistic field equations are derived from a gauge-invariant electromagnetic Lagrangian, which does not involve derivatives of the field, nor any charge density, but otherwise is completely arbitrary. These equations are explicitly solved in the static spherically symmetric case, and it is shown that there are solutions which are everywhere regular and behave, at large distances, like the gravitational and electromagnetic fields of a point charge. Some wave-like solutions are also derived.

1. INTRODUCTION

IT is well known that the equations of vacuum electrodynamics are linear only as a first approximation. General relativity, for instance, implies a gravitational coupling between electromagnetic fields,[1] and thus some nonlinearity. Another (much stronger) nonlinearity is due to vacuum polarization, which is a quantum field effect. In the classical limit of weak fields and large wavelengths, vacuum polarization can be approximated by a suitable modification of the classical electromagnetic Lagrangian.[2] In fact, such modified Lagrangians have been independently introduced by several authors, for various purposes.[3–7]

There have recently been some attempts[8] to remove the classical divergences by introducing general relativistic considerations. (As well known,[9] the quantum field divergences are never worse than the classical ones for fermions, but they may be worse for bosons. Detailed considerations on this problem are, however, beyond the scope of this paper.) The purpose of the present paper is to show that this goal can easily be achieved within the frame of general relativistic nonlinear vacuum electrodynamics, and furthermore, that the concept of charge may arise quite naturally as a first integral of the field equations, and need not be introduced independently.

2. FIELD EQUATIONS[10]

Only two independent algebraic invariants can be formed with an antisymmetric tensor $F_{\mu\nu}$ and a symmetric tensor $g_{\mu\nu}$. They are[11]

$$I=(-g)^{-\frac{1}{2}}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}, \qquad (1)$$

$$J=g^{\alpha\gamma}g^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta}. \qquad (2)$$

Let

$$\mathfrak{L}=(-g)^{\frac{1}{2}}L(I,J), \qquad (3)$$

where L is an arbitrary function of I and J. Further let

$$M=\partial L/\partial I,\quad N=\partial L/\partial J, \qquad (4)$$

and

$$T^{\mu\nu}=(-g)^{-\frac{1}{2}}\partial\mathfrak{L}/\partial g_{\mu\nu}. \qquad (5)$$

It follows that

$$T^{\mu\nu}=2NE^{\mu\nu}+\tfrac{1}{2}g^{\mu\nu}(L-IM-JN), \qquad (6)$$

where

$$E^{\mu\nu}=F^{\mu\nu}F_{\alpha}{}^{\nu}-\tfrac{1}{4}g^{\mu\nu}F^{\beta\alpha}F_{\alpha\beta}, \qquad (7)$$

satisfies[12,13]

$$g_{\mu\nu}E^{\mu\nu}=0, \qquad (8)$$

and

$$E_{\mu}{}^{\lambda}E_{\lambda}{}^{\nu}=\tfrac{1}{4}\delta_{\mu}{}^{\nu}E_{\alpha}{}^{\beta}E_{\beta}{}^{\alpha} \qquad (9)$$

We further define

$$S_{\mu}{}^{\nu}=T_{\mu}{}^{\nu}-\tfrac{1}{4}\delta_{\mu}{}^{\nu}T=2NE_{\mu}{}^{\nu}. \qquad (10)$$

It follows that

$$S_{\mu}{}^{\mu}=0, \qquad (11)$$

and

$$S_{\mu}{}^{\lambda}S_{\lambda}{}^{\nu}=\tfrac{1}{4}\delta_{\mu}{}^{\nu}S_{\alpha}{}^{\beta}S_{\beta}{}^{\alpha}. \qquad (12)$$

These are the well-known algebraic conditions of Rainich.[12,13] It is seen that their validity is quite independent of the choice of the electromagnetic Lagrangian. There are nevertheless two essential differences between linear and nonlinear electrodynamics: the curvature scalar,

$$R=8\pi T=16\pi(L-IM-JN), \qquad (13)$$

may differ from zero, and the generalized electromagnetic field equations are

$$\epsilon^{\alpha\beta\gamma\delta}F_{\beta\gamma,\delta}=0, \qquad (14)$$

$$[M\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}+(-g)^{\frac{1}{2}}NF^{\mu\nu}]_{,\nu}=0. \qquad (15)$$

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[1] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, New York, 1934), pp. 267–285.

[2] W. Heisenberg and H. Euler, Z. Physik **38**, 714 (1936).

[3] M. Born, Proc. Roy. Soc. (London) **A143**, 410 (1934).

[4] M. Born and L. Infeld, Proc. Roy. Soc. (London) **A144**, 425 (1934).

[5] F. Bopp, Ann. Phys. **38**, 345 (1940).

[6] B. Podolsky, Phys. Rev. **62**, 68 (1941).

[7] P. A. M. Dirac, Proc. Roy. Soc. (London) **A257**, 32 (1960).

[8] R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. Letters 4, 375 (1960); Phys. Rev. **120**, 313 (1960).

[9] V. S. Weisskopf, Phys. Rev. **56**, 72 (1939).

[10] Greek indices run from 0 to 3. $\epsilon^{\alpha\beta\gamma\delta}=\pm1$ according to whether $\alpha\beta\gamma\delta$ is an even/odd permutation of 0123, and $\epsilon^{\alpha\beta\gamma\delta}=0$ if any two indices are equal. The signature of the metric $g_{\mu\nu}$ is chosen as $(+---)$ so that its determinant g is always negative. A comma denotes partial differentiation, a semicolon—covariant differentiation. Natural units are used: $c=G=1$.

[11] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951), p. 69.

[12] G. Y. Rainich, Trans. Am. Math. Soc. **27**, 106 (1925).

[13] C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957).

The last relation is obtained from $\delta\mathfrak{L}/\delta A_\mu=0$, where A_μ is defined by $F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu}$.

3. STATIC SPHERICALLY SYMMETRIC SOLUTIONS

We now take the metric[14]

$$ds^2=e^\nu dt^2-e^\lambda dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2), \quad (16)$$

where λ and ν are functions of r only.

Only the diagonal elements of $S_\mu{}^\nu$ do not vanish, and it follows from (11), (12) and from the spherical symmetry that

$$S_0{}^0=S_1{}^1=-S_2{}^2=-S_3{}^3. \quad (17)$$

Only the first of the above equalities is not trivial, and it readily leads, together with the Einstein equations

$$R_\mu{}^\nu-\tfrac{1}{4}\delta_\mu{}^\nu R=-8\pi S_\mu{}^\nu, \quad (18)$$

to[14]

$$\lambda+\nu=0. \quad (19)$$

Furthermore, the only nonvanishing component of $F_{\mu\nu}$ is $F_{01}=E(r)$, say, and one has

$$I=0, \quad J=-2E^2. \quad (20)$$

It then follows from (15) that

$$E=q/2Nr^2, \quad (21)$$

where q is an integration constant.

Furthermore, one has from (6), (7), and (13)

$$JN=T_2{}^2-T_0{}^0, \quad (22)$$

$$L-JN=T_2{}^2+T_0{}^0, \quad (23)$$

whence

$$L=2T_2{}^2, \quad (24)$$

and, by virtue of (20) and (21)

$$JN^2=-q^2/2r^4. \quad (25)$$

From (22) and (25) it is possible to find J and N as functions of the metric, and one can easily verify that the relation

$$N=\partial L/\partial J=L'/J' \quad (26)$$

(where the prime denotes the derivative with respect to r) follows identically from $T_\mu{}^\nu{}_{;\nu}=0$.

The solution of this problem can now be completed. If one gives arbitrarily $\nu=\nu(r)$, then one obtains $L=L(r)$ from (24) and $J=J(r)$ from (22) and (25). One thus obtains $L=L(0,J)$. Notice that the dependence of L on I remains arbitrary.

On the other hand, if $L=L(J)$ is given, then one knows $N=N(J)$, and by virtue of (25), $J=J(r)$. Now, from (22) and (23) one has[14]

$$T_0{}^0=-JN+\tfrac{1}{2}L=(r-re^\nu)'/8\pi r^2, \quad (27)$$

which is easily solved for ν.

[14] R. C. Tolman, see reference 1, pp. 241–242.

4. PHYSICAL INTERPRETATION

We henceforth suppose that

$$L=\tfrac{1}{2}J+(\text{small coefficient}) \times(\text{higher powers of } I \text{ and } J). \quad (28)$$

Thus, for weak fields, our equations differ very little from the Maxwell-Einstein equations, and we get the usual interaction laws between charges. However, the notion of "charge" is only a makeshift hiding the real nonlinear structure of the field. Actually, it is possible to obtain solutions that are everywhere regular, and behave asymptotically, at large distances, like point charges. For instance, one can take

$$g_{00}=-(g_{rr})^{-1}=1-(2mr^{-1}-e^2r^{-2})\exp(-e^4/m^2r^2). \quad (29)$$

For large r, this goes over into the familiar solution for a point charge,[15] and one can show, by standard methods,[7,16] that one obtains the usual equations of motion as long as the distance between the "sources" is much larger than e^2/m.

It may be objected, however, that the electric field E becomes infinite at $r=0$, by virtue of (21). In fact, this singularity is only a mathematical, but not a physical, one. Indeed, the usual definition of the electric field is formulated in terms of force per unit charge, while charge has really no meaning in the present theory (except as a parameter in asymptotic expansions). Therefore even the electromagnetic field $F_{\mu\nu}$ has no intrinsic meaning, except as a complicated function of the metric field $g_{\mu\nu}$. The above-mentioned difficulty therefore does not appear at all if we adopt the point of view of geometrodynamics,[13] according to which the only basic field is the metric one.

Finally, it is likely that the stability of the static spherically symmetric solution can be proved by a method similar to the one used by Regge and Wheeler[17] in the case of the Schwarzschild singularity, although an explicit proof seems rather difficult.

5. NONLINEAR ELECTROMAGNETIC WAVES[18]

Null electromagnetic fields are defined by $I=J=0$. It then follows from (13) and (28) that

$$M=T=0 \quad \text{and} \quad N=\tfrac{1}{2}. \quad (30)$$

Since any solution for null electromagnetic fields in general relativity[19] must satisfy (12) and (30), then it follows, by virtue of (10), (14), and (15), that it must also be an admissible solution in the present more general nonlinear theory.

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[15] R. C. Tolman, see reference 1, p. 266.
[16] L. Infeld and P. R. Wallace, Phys. Rev. **57**, 797 (1940).
[17] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[18] M. Lutzky and J. S. Toll, Phys. Rev. **113**, 1649 (1949).
[19] A. Peres, Phys. Rev. **118**, 1105 (1960).