Spin-Orbit Splitting in Nuclei Due to Tensor Interaction*

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The effect of the tensor force in nuclei with closed shells plus one nucleon has been investigated using second-order perturbation theory. It is found that one can explicitly exhibit the spin-orbit splitting due to the tensor force using some simple identities. The spin-orbit splitting in He^5 is computed, and found to be 3.4 Mev compared with an experimental value of 2.6 Mev.

1. INTRODUCTION

In a previous paper[1] we have formulated a procedure for applying the second-order perturbation method of Bolsterli and Feenberg[2] to doubly magic nuclei, and applied it to O^{16}. We shall now extend this procedure to nuclei with closed shells plus a single nucleon. Particular interest in this calculation will focus on the spin-orbit splitting to be computed for the last nucleon. It has long been recognized that one requires an effective one-body $\mathbf{l}\cdot\mathbf{s}$ interaction[3,4] to interpret nuclear energy levels.

Several authors[5–13] have considered the possibility that the observed spin-orbit splitting is caused by second-order effects of the two-body tensor interaction operator,

$$S_{12}=\boldsymbol{\sigma}_1\cdot\mathbf{n}_{12}\boldsymbol{\sigma}_2\cdot\mathbf{n}_{12}-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2, \tag{1-1}$$

which is known[14] to be present in the nucleon-nucleon interaction. We shall show that the second-order effect of S_{12} may be factored into two parts, the first part displacing $j=l+\frac{1}{2}$ and $j=l-\frac{1}{2}$ states by the same amount, and the second part directly producing the splitting of the two levels. We shall then apply our procedure to obtain the splitting between the $\frac{3}{2}$- and $\frac{1}{2}$-levels of He^5.

Throughout the paper we shall use freely the notation and formulas of reference 1 (hereafter referred to as I).

2. ADDITION OF A SINGLE NUCLEON TO A CLOSED SHELL

In formulating our theory for doubly magic nuclei, we started with a single determinant of particle orbitals for the zero-order wave-function:

$$\psi_0{}^{A+1}=(A!)^{-\frac{1}{2}}|u_1u_2\cdots u_A|. \tag{2-1}$$

We now wish to add one nucleon in a new shell, and label the new orbital $A+1$ so that our wave-function becomes

$$\psi_0=[(A+1)!]^{-\frac{1}{2}}|u_1u_2\cdots u_Au_{A+1}|. \tag{2-2}$$

Now we expand the determinant in Eq. (2-2) by the row of the $(A+1)$ orbital:

$$\psi_0=(A+1)^{-\frac{1}{2}}\sum_{j=1}^{A+1}(-1)^j u_{A+1}{}^{(j)}\psi_0{}^j, \tag{2-3}$$

where the $\psi_0{}^j$ are A-by-A determinants of single-particle orbitals with quantum numbers running from 1 to A and coordinate labels running from 1 to $A+1$ with j excluded.

We now proceed to evaluate matrix elements as in I:

$$(\sum_{i<j}V_{ij})_{00}=\tfrac{1}{2}A(A+1)\int\cdots\int\psi_0{}^*V_{12}\psi_0d\mathbf{r}_1d\mathbf{r}_2\cdots d\mathbf{r}_{A+1} \tag{2-4a}$$

$$=\tfrac{1}{2}A\sum_{i,j=1}^{A+1}(-1)^{i+j}\langle u_{A+1}(i)\psi_0{}^i|V_{12}|u_{A+1}(j)\psi_0{}^j\rangle \tag{2-4b}$$

$$=\tfrac{1}{2}A(A-1)\langle\psi_0{}^{A+1}|V_{12}|\psi_0{}^{A+1}\rangle+\tfrac{1}{2}A\sum_{i,j=1}^{2}(-1)^{i+j}\langle u_{A+1}(i)\psi_0{}^i|V_{12}|u_{A+1}(j)\psi_0{}^j\rangle. \tag{2-4c}$$

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The first term in (2-4c) represents the mutual interaction of the A particles in the closed shells, which was previously calculated. The next term clearly represents the interaction of the extra nucleon with the A nucleons of the core. Making use of the density matrices,

$$(a|\rho|b)=\sum_{l=1}^{A} u_l^*(a)u_l(b), \tag{2-5}$$

we have:

$$(\sum_{i<j} V_{ij})_{00}=\tfrac{1}{2}A(A-1)\langle\psi_0^{A+1}|V_{12}|\psi_0^{A+1}\rangle+A\langle u_{A+1}(1)\psi_0^1|V_{12}|u_{A+1}(1)\psi_0^1\rangle-A\langle u_{A+1}(1)\psi_0^1|V_{12}|u_{A+1}(2)\psi_0^2\rangle \tag{2-6a}$$

$$=\tfrac{1}{2}\int\cdots\int[(1,2|\rho^2V_{12}|1,2)-(1,2|\rho^2V_{12}|2,1)]d\mathbf{r}_1d\mathbf{r}_2+\int\cdots\int u_{A+1}^*(1)[(2|\rho V_{12}|2)u_{A+1}(1) - (2|\rho V_{12}|1)u_{A+1}(2)]d\mathbf{r}_1d\mathbf{r}_2. \tag{2-6b}$$

One may express the second-order terms in a like manner:

$$\tfrac{1}{2}A(A+1)(V_{12}e^{-\lambda H_0}V_{12})_{00}=\tfrac{1}{2}A(A-1)\langle\psi_0^{A+1}|V_{12}e^{-\lambda H_0}V_{12}|\psi_0^{A+1}\rangle +\tfrac{1}{2}A\sum_{i,j=1}^{2}(-1)^{i+j}\langle u_{A+1}(i)\psi_0^i|V_{12}e^{-\lambda H_0}V_{12}|u_{A+1}(j)\psi_0^j\rangle, \tag{2-7a}$$

$$A(A+1)(A-1)(V_{12}e^{-\lambda H_0}V_{13})_{00}=A(A-1)(A-2)\langle\psi_0^{A+1}|V_{12}e^{-\lambda H_0}V_{13}|\psi_0^{A+1}\rangle +A(A-1)\sum_{i,j=1}^{3}(-1)^{i+j}\langle u_{A+1}(i)\psi_0^i|V_{12}e^{-\lambda H_0}V_{13}|u_{A+1}(j)\psi_0^j\rangle \tag{2-7b}$$

$$\tfrac{1}{4}A(A+1)(A-1)(A-2)(V_{12}e^{-\lambda H_0}V_{34})_{00} =\tfrac{1}{4}A(A-1)(A-2)(A-3)\langle\psi_0^{A+1}|V_{12}e^{-\lambda H_0}V_{34}|\psi_0^{A+1}\rangle +\tfrac{1}{4}A(A-1)(A-2)\sum_{i,j=1}^{4}(-1)^{i+j}\langle u_{A+1}(i)\psi_0^i|V_{12}e^{-\lambda H_0}V_{34}|u_{A+1}(j)\psi_0^j\rangle. \tag{2-7c}$$

Each term in Eqs. (2-7) expands into several parts when expressed by means of the density matrices, for example:

$$A(A-1)\langle u_{A+1}(1)\psi_0^1|V_{12}e^{-\lambda H_0}V_{13}|u_{A+1}(1)\psi_0^1\rangle$$
$$=e^{-\lambda E_0}\int\cdots\int u_{A+1}^*(1)[(2,3|\rho^2\exp(\lambda H_{\rm osc}(1)+\lambda H_{\rm osc}(2))V_{12}\exp(-\lambda H_{\rm osc}(1))V_{13}\exp(-\lambda H_{\rm osc}(2))|2,3) - (2,3|\rho^2\exp(\lambda H_{\rm osc}(1)+\lambda H_{\rm osc}(2))V_{12}\exp(-\lambda H_{\rm osc}(1))V_{13}\exp(-\lambda H_{\rm osc}(2)|3,2)]u_{A+1}(1)d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3. \tag{2-8}$$

As we did in I, we may perform the spin summations, isobaric spin summations, and space integrations independently. There is, however, one important difference. We are now summing effectively over incompleted shells. To illustrate this, consider the sum on isobaric spin. Let $\eta_{\frac{1}{2}}$ represent the proton state and $\eta_{-\frac{1}{2}}$ the neutron state (later we use $\chi_{\frac{1}{2}}$ for spin up and $\chi_{-\frac{1}{2}}$ for spin down). For closed shells we have isobaric spin sums of the form:

$$(\tau_1,\tau_2|\rho^2|\tau_1,\tau_2)=\sum_{i,j=-\frac{1}{2}}^{+\frac{1}{2}}\eta_i^*(1)\eta_j^*(2)\eta_i(1)\eta_j(2), \tag{2-9}$$

but now if the $A+1$ orbital represents a proton we obtain, for the interaction of this orbital with the core, terms of the form

$$\eta_{\frac{1}{2}}^*(1)(\tau_2|\rho|\tau_2)\eta_{\frac{1}{2}}(1). \tag{2-10}$$

Since

$$(\tau_1,\tau_2|\rho^2|\tau_1,\tau_2)=\eta_{\frac{1}{2}}^*(1)(\tau_2|\rho|\tau_2)\eta_{\frac{1}{2}}(1) +\eta_{\frac{1}{2}}^*(1)(\tau_2|\rho|\tau_2)\eta_{-\frac{1}{2}}(1), \tag{2-11}$$

we effectively are dealing with "half-filled" shells in isobaric spin space in Eq. (2-10). A similar argument holds for the spin except that here the $A+1$ orbital may be a mixture of $\chi_{\frac{1}{2}}$ and $\chi_{-\frac{1}{2}}$ states and hence terms of the sort:

$$\chi_{\frac{1}{2}}^*(1)(\sigma_2|\rho|\sigma_2)\chi_{-\frac{1}{2}}(1) \tag{2-12}$$

may appear. In the next section we shall consider the spin sums relevant to the tensor interaction, and display a simple theorem which directly exhibits the spin-orbit splitting due to these terms.

3. THE SPIN-ORBIT SPLITTING

The spin sums over the second-order terms involving the tensor operator ($S_{12}=\boldsymbol{\sigma}_1\cdot\mathbf{n}_{12}\boldsymbol{\sigma}_2\cdot\mathbf{n}_{12}-\frac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$) consist of three major types:

$$\chi_m^*(a)(\sigma_b|\rho S_{12}S_{1'2'}|\sigma_\beta)\chi_{m'}(\alpha)=E_{ab,\alpha\beta}{}^{mm'}[(\mathbf{n}_{12}\cdot\mathbf{n}_{1'2'})^2-\tfrac{1}{3}]$$
$$+A_{ab,\alpha\beta}{}^{mm'}(\mathbf{n}_{12}\cdot\mathbf{n}_{1'2'})(\mathbf{n}_{12}\times\mathbf{n}_{1'2'})_{m'-m},\quad a,b=1,2;\quad \alpha,\beta=1,2 \tag{3-1}$$

$$\chi_m^*(a)(\sigma_b,\sigma_c|\rho^2 S_{12}S_{1'3}|\sigma_\beta,\sigma_\gamma)\chi_{m'}(\alpha)=E_{abc,\alpha\beta\gamma}{}^{mm'}[(\mathbf{n}_{12}\cdot\mathbf{n}_{1'3})^2-\tfrac{1}{3}]$$
$$+A_{abc,\alpha\beta\gamma}{}^{mm'}(\mathbf{n}_{12}\cdot\mathbf{n}_{1'3})(\mathbf{n}_{12}\times\mathbf{n}_{1'3})_{m'-m},\quad a,b,c=1,2,3;\quad \alpha,\beta,\gamma=1,2,3 \tag{3-2}$$

$$\chi_m^*(a)(\sigma_b,\sigma_c,\sigma_d|\rho^3 S_{12}S_{34}|\sigma_\beta,\sigma_\gamma,\sigma_\delta)\chi_{m'}(\alpha)$$
$$=E_{abcd,\alpha\beta\gamma\delta}{}^{mm'}[(\mathbf{n}_{12}\cdot\mathbf{n}_{34})^2-\tfrac{1}{3}]+A_{abcd,\alpha\beta\gamma\delta}{}^{mm'}(\mathbf{n}_{12}\cdot\mathbf{n}_{34})(\mathbf{n}_{12}\times\mathbf{n}_{34})_{m'-m},$$
$$a,b,c,d=1,2,3,4;\quad \alpha,\beta,\gamma,\delta=1,2,3,4, \tag{3-3}$$

where we use the spherical components for vectors $n_0=n_z$, $n_{\pm1}=n_x\pm in_y$. The terms multiplying $E^{mm'}$ do not depend on the orientation ($E^{mm'}=\delta_{mm'}E^{mm}$) of the spin with respect to the orbital angular momentum but the terms in $A^{mm'}$ do, and therefore provide a spin-orbit splitting. The corresponding spin sums over closed shells are of the form[1]:

$$(\sigma_1,\sigma_2|\rho^2 S_{12}S_{1'2'}|\sigma_\alpha,\sigma_\beta)=E_{\alpha\beta}[(\mathbf{n}_{12}\cdot\mathbf{n}_{1'2'})^2-\tfrac{1}{3}] \tag{3-4}$$

$$(\sigma_1,\sigma_2,\sigma_3|\rho^3 S_{12}S_{1'3}|\sigma_\alpha,\sigma_\beta,\sigma_\gamma)$$
$$=E_{\alpha\beta\gamma}[(\mathbf{n}_{12}\cdot\mathbf{n}_{1'3})^2-\tfrac{1}{3}] \tag{3-5}$$

$$(\sigma_1,\sigma_2,\sigma_3,\sigma_4|\rho^4 S_{12}S_{34}|\sigma_\alpha,\sigma_\beta,\sigma_\gamma,\sigma_\delta)$$
$$=E_{\alpha\beta\gamma\delta}[(\mathbf{n}_{12}\cdot\mathbf{n}_{34})^2-\tfrac{1}{3}]. \tag{3-6}$$

We see that the $A^{mm'}$ terms disappear when we sum over a complete spin shell. From Eqs. (3-1) to (3-6), one may deduce

$$E^{\frac{1}{2}\frac{1}{2}}=E^{-\frac{1}{2}-\frac{1}{2}},\quad A^{\frac{1}{2}\frac{1}{2}}=-A^{-\frac{1}{2}-\frac{1}{2}}$$
and
$$E^{\frac{1}{2}-\frac{1}{2}}=E^{-\frac{1}{2}\frac{1}{2}}=0. \tag{3-7}$$

Needed $E^{\frac{1}{2}\frac{1}{2}}$ and $A^{\frac{1}{2}\frac{1}{2}}$ are displayed in Table I.

TABLE I. The $E^{mm'}$ and $A^{mm'}$ coefficients of two-, three-, and four-particle terms. Coefficients for absent permutations vanish.

		a	b	α	β			$E_{ab,\alpha\beta}{}^{\frac{1}{2}\frac{1}{2}}$	$A_{ab,\alpha\beta}{}^{\frac{1}{2}\frac{1}{2}}$
		1	2	1	2			2	$2i$
		1	2	2	1			2	$2i$
	a	b	c	α	β	γ		$E_{abc,\alpha\beta\gamma}{}^{\frac{1}{2}\frac{1}{2}}$	$A_{abc,\alpha\beta\gamma}{}^{\frac{1}{2}\frac{1}{2}}$
	1	2	3	1	3	2		2	$2i$
	3	1	2	2	1	3		2	$2i$
	2	1	3	3	1	2		2	$2i$
	1	2	3	3	1	2		2	$2i$
	2	1	3	1	3	2		2	$2i$
	3	1	2	2	3	1		2	$2i$
a	b	c	d	α	β	γ	δ	$E_{abcd,\alpha\beta\gamma\delta}{}^{\frac{1}{2}\frac{1}{2}}$	$A_{abcd,\alpha\beta\gamma\delta}{}^{\frac{1}{2}\frac{1}{2}}$
1	2	3	4	4	3	2	1	2	$2i$
2	1	3	4	3	4	2	1	2	$2i$
3	1	2	4	2	4	3	1	2	$2i$
4	1	2	3	1	4	3	2	2	$2i$
1	2	3	4	4	3	1	2	2	$2i$
2	1	3	4	3	4	1	2	2	$2i$
3	1	2	4	1	4	3	2	2	$2i$
4	1	2	3	2	4	3	1	2	$2i$

It is now possible to exhibit the spin-orbit splitting in a simplified manner. Let us abbreviate

$$\mathcal{E}=(\mathbf{n}_{ij}\cdot\mathbf{n}_{kl})^2-\tfrac{1}{3}, \tag{3-8a}$$

$$\mathcal{A}_M=(\mathbf{n}_{ij}\cdot\mathbf{n}_{kl})(\mathbf{n}_{ij}\times\mathbf{n}_{kl})_M, \tag{3-8b}$$

and consider the $A+1$ orbital (angular part only) to be $u_{j,m}$. We need

$$u_{l+\frac{1}{2},l+\frac{1}{2}}=\chi_{\frac{1}{2}}Y_{ll}, \tag{3-9a}$$

$$u_{l+\frac{1}{2},l-\frac{1}{2}}=(2l+1)^{-\frac{1}{2}}[\chi_{-\frac{1}{2}}Y_{ll}+(2l)^{\frac{1}{2}}\chi_{\frac{1}{2}}Y_{ll-1}], \tag{3-9b}$$

$$u_{l-\frac{1}{2},l-\frac{1}{2}}=(2l+1)^{-\frac{1}{2}}[(2l)^{\frac{1}{2}}\chi_{-\frac{1}{2}}Y_{ll}-\chi_{\frac{1}{2}}Y_{ll-1}], \tag{3-9c}$$

where the Y_{lm} are spherical harmonics. Now consider the energy shift due to a typical term in the $j=l+\frac{1}{2}$, $m=l+\frac{1}{2}$, state:

$$\Delta E_{l+\frac{1}{2}}{}^{\alpha\beta}=\langle\mathcal{Y}_{l+\frac{1}{2},l+\frac{1}{2}}{}^{\alpha}|S_{ij}S_{kl}|\mathcal{Y}_{l+\frac{1}{2},l+\frac{1}{2}}{}^{\beta}\rangle$$
$$=E^{\frac{1}{2}\frac{1}{2}}\langle\mathcal{Y}_{ll}{}^{\alpha}|\mathcal{E}|\mathcal{Y}_{ll}{}^{\beta}\rangle+A^{\frac{1}{2}\frac{1}{2}}\langle\mathcal{Y}_{ll}{}^{\alpha}|\mathcal{A}_0|\mathcal{Y}_{ll}{}^{\beta}\rangle. \tag{3-10}$$

The $\mathcal{Y}_{JM}$ are complex functions of several $(ijkl)$ particles which carry angular momentum J with z-component M. Their explicit evaluation is not pertinent to this derivation, and would only obscure matters. Obviously one can calculate the same energy shift in the $j=l+\frac{1}{2}$, $m=l-\frac{1}{2}$ state:

$$\Delta E_{l+\frac{1}{2}}{}^{\alpha\beta}=E^{\frac{1}{2}\frac{1}{2}}\langle\mathcal{Y}_{ll}{}^{\alpha}|\mathcal{E}|\mathcal{Y}_{ll}{}^{\beta}\rangle+(2l+1)^{-1}(2l-3)$$
$$\times\langle\mathcal{Y}_{ll}{}^{\alpha}|\mathcal{A}_0|\mathcal{Y}_{ll}{}^{\beta}\rangle A^{\frac{1}{2}\frac{1}{2}}+C, \tag{3-11}$$

where C is the cross term. We have used the fact that $\mathcal{E}$ is a scalar and $\mathcal{A}$ a vector in relating the matrix elements:

$$\langle\mathcal{Y}_{ll}{}^{\alpha}|\mathcal{E}|\mathcal{Y}_{ll}{}^{\beta}\rangle=\langle\mathcal{Y}_{ll-1}{}^{\alpha}|\mathcal{E}|\mathcal{Y}_{ll-1}{}^{\beta}\rangle, \tag{3-12a}$$

$$\langle\mathcal{Y}_{ll}{}^{\alpha}|\mathcal{A}_0|\mathcal{Y}_{ll}{}^{\beta}\rangle=\frac{l}{l-1}\langle\mathcal{Y}_{ll-1}{}^{\alpha}|\mathcal{A}_0|\mathcal{Y}_{ll-1}{}^{\beta}\rangle, \tag{3-12b}$$

by the Wigner-Eckart theorem. Comparing (3-10) and (3-11), we find

$$C=\frac{4}{2l+1}A^{\frac{1}{2}\frac{1}{2}}\langle\mathcal{Y}_{ll}{}^{\alpha}|\mathcal{A}_0|\mathcal{Y}_{ll}{}^{\beta}\rangle. \tag{3-13}$$

The energy shift in the $j=l-\frac{1}{2}$ state is then

$$\Delta E_{l-\frac{1}{2}}^{\alpha\beta}=E^{\frac{1}{2}\frac{1}{2}}\langle \mathcal{Y}_{ll}^{\alpha}|\mathcal{E}|\mathcal{Y}_{ll}^{\beta}\rangle+A^{\frac{1}{2}\frac{1}{2}}(2l+1)^{-1}\left[\frac{l-1}{l}-2l\right]\times\langle \mathcal{Y}_{ll}^{\alpha}|\mathcal{A}_0|\mathcal{Y}_{ll}^{\beta}\rangle-C, \quad (3\text{-}14)$$

where we have precisely the same cross term C, but with a negative coefficient. Eliminating C from (3-14), we obtain

$$\Delta E_{l-\frac{1}{2}}^{\alpha\beta}=E^{\frac{1}{2}\frac{1}{2}}\langle \mathcal{Y}_{ll}^{\alpha}|\mathcal{E}|\mathcal{Y}_{ll}^{\beta}\rangle-\langle \mathcal{Y}_{ll}^{\alpha}|\mathcal{A}_0|\mathcal{Y}_{ll}^{\beta}\rangle\left(\frac{l+1}{l}\right)A^{\frac{1}{2}\frac{1}{2}}. \quad (3\text{-}15)$$

Comparing (3-15) with (3-10), we see that

$$\Delta E_{l+\frac{1}{2}}^{\alpha\beta}-\Delta E_{l-\frac{1}{2}}^{\alpha\beta}=\left(\frac{2l+1}{l}\right)A^{\frac{1}{2}\frac{1}{2}}\langle \mathcal{Y}_{ll}^{\alpha}|\mathcal{A}_0|\mathcal{Y}_{ll}^{\beta}\rangle, \quad (3\text{-}16)$$

which explicitly exhibits the spin-orbit splitting due to a typical term in the tensor interaction to second-order.

The splitting due to a conventional one-body $\mathbf{l}\cdot\boldsymbol{\sigma}$ term:

$$V_{\text{s.o.}}=V(r)\mathbf{l}\cdot\boldsymbol{\sigma}, \quad (3\text{-}17\text{a})$$

is customarily expressed by

$$\Delta E_{l+\frac{1}{2}}-\Delta E_{l-\frac{1}{2}}=(2l+1)\langle R_{nl}(r)|V(r)|R_{nl}(r)\rangle. \quad (3\text{-}17\text{b})$$

One must note that in both (3-16) and (3-17b) there is a "hidden" dependence on l in the radial part of the matrix element. This hidden dependence is illustrated in Appendix I by a simple example.

4. APPLICATION TO He^5

We shall consider here an application of the preceding discussion to He^5. We use a force presented in I, which is a Serber mixture with a repulsive core fitted to the properties of H^2, H^3, He^3, and He^4:

$$\begin{aligned} V_{12}=J_R&\exp(-r_{12}^2/R^2)\\ &+(1/16)J_C[(1-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(3+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\\ &+(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(1-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)]\exp(-r_{12}^2/r_0^2)\\ &+\tfrac{1}{4}(1-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)J_S(r_{12}/r_0)^2S_{12}\\ &\times\exp(-r_{12}^2/r_0^2), \end{aligned} \quad (4\text{-}1)$$

where:

$$J_C=-58.65\text{ Mev},\quad J_S=-107.29\text{ Mev},$$
$$J_R=+189.75\text{ Mev},\quad r_0=1.54(10^{-13})\text{ cm},$$

and

$$R=r_0/\sqrt{8}. \quad (4\text{-}1\text{a})$$

We compute, for the fifth nucleon in He^5, only the spin-orbit splitting between the $p_{\frac{3}{2}}$ and $p_{\frac{1}{2}}$ states from Eqs. (2-7). This is the simplest possible example. Unfortunately it has the drawback that the fifth nucleon is not bound and consequently the size parameter is uncertain. We obtain our rough estimate by fitting the Coulomb energy difference between He^5 and Li^5 (0.95 Mev in the $p_{\frac{3}{2}}$ state, and 0.85 Mev in the $p_{\frac{1}{2}}$ state[15]). We compute a splitting of 3.4 Mev compared with an experimental value of 2.6 Mev.

The fact that our estimate is about 30% too high may be attributed either to the problem of determining the size parameter, the fact that the splitting is certainly sensitive to the force used, or more likely a combination of both. One should note here that the wave function and the potential both have Gaussian radial dependence and overlap exceedingly well. (The oscillator well parameter $\hbar\omega$ used was 19 Mev.) This would appear to strengthen the view that the overestimation of the doublet P splitting is due to the radial functions used.

Feingold[7] has examined the doublet P splitting in He^5, and found that for a tensor force of the Serber type the splitting vanished. We too found considerable cancellation among terms with a Serber mixture, in fact the two-particle terms in Eq. (2-7a), which are of the type that produce splitting, cancel completely. We are left with three-particle terms from Eq. (2-7b) however, which would cancel were it not for the rigorous treatment of the operator $(E-H_0)^{-1}$ by the Bolsterli-Feenberg method.

Since the two-particle terms in Eq. (3-16) cancel in He^5, one does not mix in any states where two particles are excited out of the ground state. The three-particle terms,

$$\langle\psi_0|V_{12}|\psi_n\rangle\langle\psi_n|V_{13}|\psi_0\rangle, \quad (4\text{-}2)$$

involve excitations of only one particle. We emphasize that this cancellation occurs only when *one* nucleon interacts with a closed 1*s-shell*. As one adds on more particles, the two-body terms may build up. This effect may be partially responsible for the building up of the doublet splitting in the first p shell.

Teresawa[11] has computed the doublet splitting in He^5 and attributes the effect to the fact that the nucleon outside the closed shell suppresses configuration interaction of the core nucleons due to the Pauli principle more effectively in the $j=l-\frac{1}{2}$ state than in the $j=l+\frac{1}{2}$ state. Although this explanation appears quite different from the derivation presented here, where the doublet splitting is directly obtained from a spin-orbit term in the expansion of the tensor force in second order, there is a correlation between the two. Three-particle terms like that in Eq. (4-2) represent the interaction between particles 2 and 3 due to their mutual interaction with particle 1. Part of this interaction must be related to the fact that particles 2 and 3 cannot be excited to an orbital occupied by particle 1. The fact that terms of this type contribute strongly to the doublet splitting verifies the effect found by Teresawa.

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We would emphasize that this is an effect of the doublet splitting, however, and not the cause.

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APPENDIX I. THE l DEPENDENCE OF THE SPIN-ORBIT SPLITTING CONTAINED IN THE RADIAL INTEGRALS

The radial integrals arising from the interaction of one nucleon in a state of orbital angular momentum l with the first s shell are all of the form:

$$\Gamma_t=\frac{1}{\pi^6}\int\cdots\int(r_{1-}r_{3+})^l(r_{12+}r_{34-}-r_{12-}r_{34+})(\mathbf{r}_{12}\cdot\mathbf{r}_{34})\exp\Big[-\sum_{i,j=1}^{4}\gamma_{ij}\mathbf{r}_i\cdot\mathbf{r}_j\Big]d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3d\mathbf{r}_4. \tag{A-1}$$

This integral is easily performed by the method of I, obtaining

$$\Gamma_t=\tfrac{1}{2}l(l!)[(\gamma_{22}-\gamma_{12})(\gamma_{44}-\gamma_{34})-(\gamma_{24}+\gamma_{14})(\gamma_{24}+\gamma_{23})]\{D^{-(l+\frac{5}{2})}(2l+3)\Gamma_a^{l-1}\Gamma_c+(l-1)D^{-(l+\frac{3}{2})}\Gamma_a^{l-2}\Gamma_b\}, \tag{A-2}$$

where

$$D=\det|\gamma_{ij}|, \tag{A-2a}$$

$$\Gamma_a=\gamma_{13}(\gamma_{22}\gamma_{44}-\gamma_{24}^2)+\gamma_{22}\gamma_{14}\gamma_{34}+\gamma_{44}\gamma_{23}\gamma_{12}+\gamma_{24}(\gamma_{12}\gamma_{34}+\gamma_{23}\gamma_{14}), \tag{A-2b}$$

$$\Gamma_b=(\gamma_{22}-\gamma_{12})(\gamma_{44}-\gamma_{34})-(\gamma_{24}+\gamma_{23})(\gamma_{24}+\gamma_{14})-2\gamma_{24}\gamma_{13}+2\gamma_{14}\gamma_{23}, \tag{A-2c}$$

and

$$\Gamma_c=\gamma_{13}(\gamma_{22}-\gamma_{12})(\gamma_{44}-\gamma_{34})+\gamma_{24}(\gamma_{11}-\gamma_{12})(\gamma_{33}-\gamma_{34})-\gamma_{14}(\gamma_{22}-\gamma_{12})(\gamma_{33}-\gamma_{34})-\gamma_{23}(\gamma_{11}-\gamma_{12})(\gamma_{44}-\gamma_{34})$$
$$-(\gamma_{13}\gamma_{24}-\gamma_{23}\gamma_{14})(\gamma_{13}+\gamma_{24}+\gamma_{14}+\gamma_{23}). \tag{A-2d}$$

Setting $l=1$ in Eq. (A-2) yields the needed space integrations for He^5.

We may compare our Γ_t with the analogous radial integral arising from the $\mathbf{l}\cdot\boldsymbol{\sigma}$ term in Eq. (3-17) if $V(r)$ has the dependence $\exp(-ar^2)$:

$$\Gamma_{s1}=\pi^{-\frac{3}{2}}\int\int\int(r_-r_+)^l\exp[-(a+k)r^2]d\mathbf{r} \tag{A-3}$$

$$=(-1)^l(a+k)^{-\frac{1}{2}}\frac{d^l}{da^l}\left(\frac{1}{a+k}\right) \tag{A-3a}$$

$$=l!(a+k)^{-(l+\frac{3}{2})}, \tag{A-3b}$$

where k is the size parameter for our oscillator wave function.

Comparing (A-3) and (A-2) with (3-16) and (3-17), we see that in this simple case the l dependence of the spin-orbit splitting is proportional to $(2l+1)$ for both the tensor and the $\mathbf{l}\cdot\boldsymbol{\sigma}$ interaction, insofar as they are comparable.