Formal Theory of Rearrangement Collisions*

MARVIN H. MITTLEMAN
Lawrence Radiation Laboratory, University of California, Livermore, California
(Received February 13, 1961)

A new formulation of the transition amplitude for the general rearrangement collision is presented. Optical potential ideas are used in the derivation but the final result contains no mention of the optical potential. The result is a matrix element with transitions only between mutually orthogonal states.

I. INTRODUCTION

THE general rearrangement scattering problem has received much attention in connection with nuclear scattering[1] and with atomic scattering.[2] One of the difficulties associated with the general formulation is the lack of orthogonality between the initial and final states of the system. This results in a situation where approximations, which are, in fact, always necessary, may yield nonsensical results. For example, a constant potential will cause a transition in first Born approximation. Another example is the calculation of exchange scattering of an "electron" from "hydrogen" where we neglect the electron-electron interaction. The result should vanish, but in the usual formulation it vanishes only in infinite order of the approximations.[3] The underlying reason for these difficulties is that there is, *in principle*, no interaction which one can treat as being small. The interaction which may act as a small perturbation in the initial (final) state causes binding in the final (initial) state and so is not a small perturbation there.

In this note we shall obtain a result in which the orthogonality difficulties are removed. That is, the transition amplitude will contain transitions only between mutually orthogonal states. The complete symmetrization which is imposed by the Pauli principle is not considered here.

II. FORMAL DEVELOPMENT AND DISCUSSION

We are interested here in a collision in which the final particles are not the same as the initial ones. Symbolically, this is of the form

$$A+B\rightarrow C+D. \tag{1}$$

The total Hamiltonian H may be broken into the Hamiltonian H_i, of the noninteracting particles, A and B, in the initial state, and their interaction V_i. Or it may be broken into the noninteracting Hamiltonian H_f of the final-state particles, C and D, and their interaction V_f. That is,

$$H=H_i+V_i=H_f+V_f. \tag{2}$$

The initial state χ_i of the noninteracting particles, A and B, with a plane wave for their relative motion satisfies

$$(E-H_i)\chi_i=0. \tag{3}$$

Similarly the final noninteracting state satisfies

$$(E-H_f)\chi_f=0, \tag{4}$$

where E is the total energy of the system. We shall now present the usual derivation[4] as a basis for what follows.

The total wave function of system is given by the scattering integral equation,

$$\Psi_i^{(+)}=\chi_i+\frac{1}{a_i}v_i\Psi_i^{(+)}, \tag{5}$$

where

$$a_i=E-H_i+i\eta, \tag{6}$$

and η is a positive infinitesimal which ensures that only outgoing scattered waves are contained in $\Psi_i^{(+)}$. Equation (5) is not in a convenient form for obtaining the rearrangement amplitude since a_i is not diagonal in the final set of states. This can easily be remedied by putting $\Psi_i^{(+)}$ in the form

$$\Psi_i^{(+)}=a_f^{-1}a_i\chi_i+a_f^{-1}V_f\Psi_i^{(+)}, \tag{7}$$

where

$$a_f=E-H_f+i\eta. \tag{8}$$

It is easily seen that the first term in (7) contains no amplitude for outgoing waves in the final state[4] and that the transition matrix obtained from the second term of (7) is

$$T=\langle\chi_f|V_f|\Psi_i^{(+)}\rangle. \tag{9a}$$

The symmetric form for T,

$$T=\langle\Psi_f^{(-)}|V_i|\chi_i\rangle, \tag{9b}$$

where

$$\Psi_f^{(-)}=\chi_f+(a_f^{+})^{-1}V_f\Psi_f^{(-)}, \tag{10}$$

may be obtained from $\Psi_i^{(+)}$ rewritten in the form

$$\Psi_i^{(+)}=\chi_i+a^{-1}V_i\chi_i, \tag{11}$$

where $a=E-H+i\eta$. Here again the first term contains

* Work done under the auspices of the U. S. Atomic Energy Commission.

[1] An extensive bibliography is contained in S. T. Butler and O. H. Hittmair, *Nuclear Stripping Reactions* (John Wiley & Sons, Inc., New York, 1957).

[2] For example, J. D. Jackson and H. Schiff, Phys. Rev. **89**, 359 (1953), and D. R. Bates, Proc. Roy. Soc. (London) **A247**, 294 (1958).

[3] I would like to thank Professor L. Spruch and Professor S. Borowitz for pointing out this example.

[4] B. Lippmann, Phys. Rev. **102**, 264 (1956).

no outgoing waves in the final state and the asymptotic form of the propagator, a^{-1}, yields (9b) for the transition matrix.

We may modify the above development to include a distorted wave for the interparticle motion in the initial state. This is accomplished by including the distorting potential, $U_i^{(+)}$, in the initial state propagator, a_i^{-1}. Thus (11) can be rewritten

$$\Psi_i^{(+)}=\Psi_{ci}^{(+)}+a^{-1}(V_i-U_i^{(+)})\Psi_{ci}^{(+)}, \tag{11a}$$

where

$$\Psi_{ci}^{(+)}=\chi_i+a_i^{-1}U_i^{(+)}\Psi_{ci}^{(+)}. \tag{12}$$

The requirement that $\Psi_{ci}^{(+)}$ differ from χ_i only by replacing the plane wave relative motion of A and B by a scattered wave with outgoing wave boundary conditions implies only that $U_i^{(+)}$ be diagonal in the states of A and B.

The T matrix is obtained from (11a) as

$$T=\langle\Psi_f^{(-)}|\,V_i-U_i^{(+)}|\Psi_{ci}^{(+)}\rangle, \tag{13}$$

in the same way as (9b) is obtained from (11).

The crucial point of the derivation is now to indentify $\Psi_{ci}^{(+)}$ as the true elastic scattering wave function in the incident channel. This implies that $U_i^{(+)}$ is an optical potential for the elastic scattering in the incident channel. Let us define an operator Π_i which projects onto the initial state of the system $A+B$. Then we have

$$\Pi_i\chi_i=\chi_i, \tag{14}$$

and

$$\Pi_i\Psi_i^{(+)}=\Psi_{ci}^{(+)}. \tag{15}$$

Then if we multiply (5) by Π_i, we obtain

$$\Psi_{ci}^{(+)}=\chi_i+a_i^{-1}\Pi_iV_i\Psi_i^{(+)}, \tag{16}$$

since Π_i commutes with a_i. Now comparison of (16) and (12) yields

$$U_i^{(+)}\Psi_{ci}^{(+)}=\Pi_iV_i\Psi_i^{(+)}. \tag{17}$$

We now use (15) and (17) in (13) to obtain

$$T=\langle\Psi_f^{(-)}|\,[V_i,\Pi_i]\,|\Psi_i^{(+)}\rangle, \tag{18}$$

which is our final result.[5] A similar derivation which includes a distorted wave for the final state will yield the symmetric form

$$T=\langle\Psi_f^{(-)}|\,[\Pi_f,V_f]\,|\Psi_i^{(+)}\rangle, \tag{18a}$$

where Π_f projects onto the final state of the system $C+D$. It should be noted that the results contained in (18) and (18a) are independent of the distorted wave treatment. That is: the wave functions Ψ_i and Ψ_f may be approximated in any of the usual ways.

The form (18) may be rewritten as

$$T=\langle\Psi_f^{(-)}|\,(1-\Pi_i)V_i\Pi_i|\Psi_i^{(+)}\rangle - \langle\Psi_f^{(-)}|\,\Pi_iV_i(1-\Pi_i)\,|\Psi_i^{(+)}\rangle.$$

In this form T has an interesting interpretation. The first term is a transition from the elastic channel by V_i to that portion of the final state orthogonal to the elastic channel. The second term is a transition from the inelastic initial channels to that part of the final state parallel to the elastic channel. In this form the transition only between mutually orthogonal states is evident. In the commutator form of (18) the difficulties mentioned in the introduction do not arise since the commutator will vanish. Only that part of the interaction which changes the state will contribute.

We note that even in lowest order the Born approximation is changed from the usual result. One form is

$$T_B=\langle\chi_f|\,V_f(1-\Pi_f)\,|\chi_i\rangle.$$

In a recent paper[6] this matrix element was evaluated in the impact parameter formalism for the problem of charge exchange of protons in hydrogen. There it was seen that for high energies the second term behaved only like $E^{-\frac{1}{2}}$ relative to the first, and that the correction to the cross section was a factor of 2 at 300 kev.

ACKNOWLEDGMENT

I would like to thank Dr. Warren Heckrotte for critical discussions.

[5] The only property of the optical potential used here is contained in (17). There are many equivalent optical potentials but all of them satisfy this equation, so we need specify $U_i^{(+)}$ no further. For completeness we should demonstrate the existence of $U_i^{(+)}$ as defined above. This is omitted since it has been done many times before. For example, F. Coester and H. Kummel, Nuclear Phys. **9**, 225 (1958) and M. H. Mittleman and K. M. Watson, Phys. Rev. **113**, 198 (1959).

[6] Marvin H. Mittleman, Phys. Rev. **122**, 499 (1961).