Rotational Multiplets in the Spectrum of the Earth*

C. L. Pekeris, Z. Alterman, and H. Jarosch
Department of Applied Mathematics, The Weizmann Institute, Rehovoth, Israel
(Received February 10, 1961)

The doublets in the spectrum of the free oscillations of the earth which have been observed on the gravimetric (UCLA) and strain-meter (Pasadena) records of the great Chilean earthquake of May 22, 1960, are interpreted as multiplets arising from the rotation of the earth. The phenomenon is similar to the Zeeman effect, and is indeed a realization of the mechanical analog from which Larmor deduced the "Larmor precession" in his interpretation of the Zeeman effect. A first-order perturbation calculation yields the result that the degenerate frequency $\sigma_0(n)$ in the absence of rotation is resolved by a slow rotation into $(2n+1)$ lines, $\sigma_n{}^m$ given by

$$\sigma_n{}^m=\sigma_0(n)+m\tau(n)\omega, \quad -n\leqslant m\leqslant n,$$

where ω denotes the angular velocity of rotation of the earth, and m is the azimuthal number of the wave function. $\tau(n)$ is determinable from the zero-order solution in the case of spheroidal oscillations, and is equal to $[n(n+1)]^{-1}$ in the case of purely torsional oscillations. The relative intensities within the quintet $n=2$ and the septet $n=3$ have been determined for an observing station at Los Angeles, on the assumption of an explosive point-source at the earthquake focus in Chile. The strongest lines should be the pair $m=\pm1$ for $n=2$, and the pair $m=\pm2$ for $n=3$. These agree in separation with the pairs observed on the strain-meter and with the gravimetric pair at $n=3$, but less so with the gravimetric pair at $n=2$. There are indications in the strain-meter spectrum for $n=3$ of a weaker line at $m=0$, while the other lines are theoretically of an intensity not exceeding the background noise. The separation in the observed gravimetric doublet for the first overtone of $n=3$ agrees with the interval of the strongest pair $m=\pm2$. The intensities of the lines in the rotational multiplets of the components of displacement for an observing station at Palisades, New York, have also been determined.

1. INTRODUCTION

The first attempts of Zeeman to observe an effect of a magnetic field on spectral lines led to negative results. He resumed the experiment in 1894 when he read in Maxwell's[1] sketch of Faraday's life: "Before we describe this result we may mention that in 1862 he made the relation between magnetism and light the subject of his very last experimental work. He endeavored, but in vain, to detect any change in the lines of the spectrum of a flame when the flame was acted on by a powerful magnet." Zeeman states[2]: "If a Faraday thought of the possibility of the above-mentioned relation, perhaps it might yet be worthwhile to try the experiment again with the excellent auxiliaries of spectroscopy of the present time . . ." The discovery of the Zeeman effect that followed in 1896 came at a time when the basic concepts needed for its interpretation had already been formulated by Lorentz.[3] Zeeman goes on to say,[2] "Professor Lorentz, to whom I communicated these considerations, at once kindly informed me of the manner in which, according to his theory, the motion of an ion in a magnetic field is to be calculated, and pointed out to me that, if the explanation following from his theory be true, the edges of the lines of the spectrum ought to be circularly polarized." The splitting of the original frequency σ_0 by a magnetic field H, which Zeeman derives[2] on the basis of the Lorentz theory, is

$$\sigma=\sigma_0\pm eH/2mc \tag{1}$$

for the case where the lines are viewed parallel to the magnetic field. The two lines given by (1) should be circularly polarized, and this was confirmed already in the first experiment of Zeeman.[2]

The relation (1) had been derived independently[4] by Larmor. When news of Zeeman's discovery reached him, he substituted the value of the mass of hydrogen for m in (1) and concluded that the effect would be inappreciable. He therefore asked Lodge to confirm the experiment. This Lodge succeeded in doing, and on May 20, 1897, he demonstrated the effect at a Royal Society soirée. Lodge later published a disclaimer[5] of "any intention of trespassing on the prerogative of the discoverer." As is well known, Larmor proves that the effect of a uniform magnetic field H on the orbit of a particle of charge e and mass m is to set the whole orbit into a precession around the direction of $\mathbf{H}$ with an angular velocity

$$\omega_L=eH/2mc, \tag{2}$$

provided the centrifugal force can be neglected in comparison with the Coriolis force. ω_L is called the Larmor precession frequency. Larmor's theorem follows directly from equating to zero the sum of the Coriolis force $2m\mathbf{v}\times\boldsymbol{\omega}$ and the Lorentz force $-(e/c)\mathbf{v}\times\mathbf{H}$. Larmor then shows, as Zeeman had done, that the frequency of a harmonic oscillator σ_0 is split by a rotation ω into the two lines given by (1), provided $(\omega/\sigma_0)\ll1$.

While the Zeeman effect furnishes ample experimental verification of the electromagnetic part of Larmor's analogy, an experimental demonstration of the mechanical counterpart has been wanting. A striking case of the splitting of the natural frequency of a purely mechanical system by rotation presented itself

* Supported by the Office of Naval Research and by Project Vela-Uniform of the Advanced Research Projects Agency.

[1] J. C. Maxwell, *Collected Works* (Dover Publications, Inc., New York, 1953), Vol. II, p. 790.

[2] P. Zeeman, Phil. Mag. **43**, 226 (1897) [reprinted in *Verhandelingen van Dr. P. Zeeman over Magneto-Optische Verschijnelsen* (Eduard Ijdo, Leiden, 1921)].

[3] H. A. Lorentz, Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern (Leiden, 1895).

[4] J. Larmor, *Mathematical and Physics Papers* (Cambridge University Press, New York, 1928), Vol. II, p. 140.

[5] O. Lodge, Phil. Mag. 44, 60 (1897).

recently when the records of the great Chilean earthquake of May 22, 1960, were analyzed. This earthquake excited the natural oscillations of the earth. The gravest modes $n=2$ (53.7 min) and $n=3$ (35.5 min) appeared as doublets in the spectra of both the gravimetric records[6] and the strain-meter records.[7] The free oscillations of the earth are governed by gravitational and elastic forces. Taking the distribution of density $\rho(r)$ and of the elastic constants $\lambda(r)$ and $\mu(r)$, as inferred primarily from seismic as well as from other geophysical data, we have determined the spectrum for several proposed models of the earth.[8–11] The investigation was prompted originally by an observation made by Benioff[12] of a 57-min oscillation on the record of the Kamchatka earthquake of 1952. By the time the theory was worked out in detail and the spectrum thoroughly investigated, several years had elapsed; and a question began to emerge with annoying persistence as to why no further natural oscillations had been recorded since 1952, in spite of the continuous improvement in recording facilities. The answer to this question came at the meeting of the International Union of Geodesy and Geophysics held at Helsinki in July, 1960, when Benioff, Press, and Smith, and Ness, Harrison, and Slichter announced that they had identified the free oscillations of the earth in the strain seismograms (52 lines) and the gravimetric records (49 lines), respectively, of the great Chilean earthquake of May 22, 1960. The periods deduced from spectral analysis of the strain-meter records agreed with the gravimetric values to within 1%; and this was also the measure of agreement with the theoretical spectrum[11] for the Gutenberg earth model, and to a lesser extent with the spectrum for the Bullen B model. Free modes ranging from spherical harmonic order $n=2$ up to $n=38$ were identified seismically, and up to $n=41$ gravimetrically.

The free oscillations fall into two classes: *spheroidal*, with nonvanishing dilatation, and *torsional*. In the latter there is neither dilatation nor vertical displacement, so that they produce no gravity perturbation. Actually, the torsional oscillations were identified only on the seismic records.

Press and Slichter reported that the $n=2$ and $n=3$ lines appear as doublets in the strain-meter and gravimetric spectra. It was then suggested[13] that the splitting is due to the earth's rotation. This conjecture was made on the basis of a recollection of a result in Lamb's treatment[14] of the effect of rotation on the free gravitational oscillation of a circular basin, namely that the wave advancing in the direction of rotation has a longer period than the wave going in the opposite direction. A perturbation calculation,[11] based on Lamb's analysis, showed that in the case of the rotating circular basin the frequency interval $(\sigma_2-\sigma_1)$ in the doublet should be of the order of the angular velocity of rotation ω. Observationally, the quantity $(\sigma_2-\sigma_1)/\omega$ ranged from 0.7 to 1.1, thus lending support to the hypothesis of the rotational origin of the observed doubling of the periods of free oscillations of the earth.

In the following section, we extend our previous analysis of the free oscillations of a nonrotating self-gravitating elastic earth by carrying out a first-order perturbation calculation of the effect of a slow rotation on the *frequency*. Denoting by subscript zero the solution for the case of no rotation, the components of displacement u_0, v_0, w_0 in a spherical system of coordinates (r,θ,ϕ) and the perturbation in gravity ψ_0, are given by

$$u_0=U_0(r)Y_{nm}(\theta,\phi),\quad v_0=V_0(r)\frac{\partial Y_{nm}}{\partial\theta},$$
$$w_0=\frac{V_0(r)}{\sin\theta}\frac{\partial Y_{nm}}{\partial\phi},\qquad \psi_0=P_0(r)Y_{nm}, \tag{3}$$

where

$$Y_{nm}(\theta,\phi)=P_n{}^m(\cos\theta)e^{im\phi}, \tag{4}$$

and a factor $e^{i\sigma t}$ has been omitted. The frequencies $\sigma_0(n)$ are degenerate, and do not depend on the azimuthal number m. It is shown below that the introduction of a slow angular rotation ω ($\ll\sigma_0$) removes the degeneracy, each line $\sigma_0(n)$ being split into a multiplet of $(2n+1)$ lines $\sigma_n{}^m$ given by

$$\sigma_n{}^m=\sigma_0(n)+m\tau(n)\omega,\quad -n\leqslant m\leqslant n, \tag{5}$$

where

$$\tau(n)=\int_0^a\rho r^2(2U_0V_0+V_0{}^2)dr\bigg/\int_0^a\rho r^2[U_0{}^2+n(n+1)V_0{}^2]dr \tag{6}$$

in the case of spheroidal oscillations, and

$$\tau(n)=1/n(n+1) \tag{7}$$

in the case of torsional oscillations.

Taking the earth model Bullen B, for which we have evaluated the functions $U_0(r)$ and $V_0(r)$, we find from (6) that $\tau(2)=0.395$ and $\tau(3)=0.183$. Using these in (5), and the value 7.272×10^{-5} sec^{-1} of the angular velocity of rotation of the earth ω, we get the periods

[6] N. F. Ness, J. C. Harrison, and L. B. Slichter, J. Geophys. Research **66**, 621 (1961); see also L. E. Alsop, G. H. Sutton, and M. Ewing, *ibid.* **66**, 631 (1961).
[7] H. Benioff, F. Press, and S. Smith, J. Geophys. Research **66**, 605 (1961).
[8] C. L. Pekeris and H. Jarosch, *Contributions in Geophysics* (Pergamon Press, New York, 1958), Vol. 1, p. 171.
[9] Z. Alterman, H. Jarosch, and C. L. Pekeris, Proc. Roy. Soc. (London) **A252**, 80 (1959).
[10] Z. Alterman, H. Jarosch, and C. L. Pekeris, Geophys. J. **3**, Jeffreys Jubilee Volume (1961).
[11] C. L. Pekeris, Z. Alterman, and H. Jarosch, Proc. Natl. Acad. Sci. U. S. **47**, 91 (1961).
[12] H. Benioff, Trans. Am. Geophys. Union **35**, 985 (1954).
[13] C. L. Pekeris, Lecture given at the Helsinki meeting of the IUGG (July, 1960).
[14] H. Lamb, *Hydrodynamics* (Cambridge University Press, New York, 1940), 6th ed., p. 320.

TABLE I. The periods T_n^m for model Bullen B.

n	m	T_n^m Theoretical min	Observed Gravimetric min	Observed Seismic min
2	−2	55.33		
			54.98	
2	−1	54.50		54.7
2	0	53.70		
2	1	52.92	52.80	53.1
2	2	52.15		
3	−3	35.99		
3	−2	35.83	35.87	35.9
3	−1	35.67		
3	0	35.50		
3	1	35.34		
3	2	35.18	35.24	35.2
3	3	35.03		

$T_n^m(=2\pi/\sigma_n^m)$ of the quintet for $n=2$ and of the septet for $n=3$ shown in Table I. The question arises as to why only two lines out of a possible five were observed in the case $n=2$, and again why only two out of a possible seven lines in the case $n=3$.

This leads us to an investigation of the relative amplitudes of the lines within a multiplet. The relative amplitudes depend on the nature of the source, its geographical location, and the location of the observing station, as well as on the nature of the quantity that is observed—whether a component of displacement, or of strain, or the perturbation in gravity. We assume an explosive compressional point-source at the earthquake focus in Chile ($\theta_0=128°$), and make use of the results of an earlier investigation[9] of the relative amplitudes of the modes $n=2$ and $n=3$ which such a source excites. If the source were located on either of the poles, there would be, according to our theory, no rotational splitting, because there would be no longitudinal (East-West) component of motion w, and m in (4) and (5) would be zero. Although on the assumption of a point-source there is no longitudinal motion around the axis passing through Chile, there *is* longitudinal motion around the axis of rotation of the earth, and this gives rise to rotational splitting. In the case of the gravimetric measurements, we use the expressions for the amplitudes within the multiplet of the perturbation of gravity, while in the case of the strain-meter we use the appropriate theoretical amplitudes of the longitudinal strain in a horizontally placed long bar.

These theoretical results are compared with observations in Table III and in Figs. 1 and 2. It is seen that, in general, the doublets observed are those lines of the multiplet which are strongest theoretically, and that the missing lines would not be expected to stand out above the observed noise level.

2. THEORY OF ROTATIONAL MULTIPLETS IN THE SPECTRUM OF THE EARTH

In the absence of rotation, the analysis of the free oscillations of the earth proceeds by first determining the equilibrium static solution, and then superimposing on it a perturbation velocity-field $\mathbf{v}$ which is controlled by the elastic and gravitational restoring forces. The static solution, which we shall designate by the subscript zero, is spherically symmetrical and is governed by the equations

$$dp_0/dr=-g\rho_0, \quad (8)$$

$$g=-d\psi/dr, \quad (9)$$

$$\nabla^2\psi=-4\pi G\rho_0, \quad (10)$$

ψ denoting the gravitational potential and G the gravitational constant. If the earth now rotates with angular velocity ω, Eq. (8) still retains the form[15]

$$\nabla p_0=\rho_0\nabla\Psi, \quad (11)$$

where Ψ is the *geopotential*, comprising the gravitational potential ψ defined in (10) and the centrifugal term,

$$\Psi=\psi+\tfrac{1}{2}\omega^2r^2\sin^2\theta, \quad (12)$$

in a spherical system of coordinates (r,θ,ϕ). The solution of (11) is

$$p_0=p_0(\Psi), \quad \rho_0=dp/d\Psi=\rho_0(\Psi). \quad (13)$$

It can be shown[15] that the surfaces $\Psi=C$ are ellipsoids of revolution whose *maximum* ellipticity is at the sur-

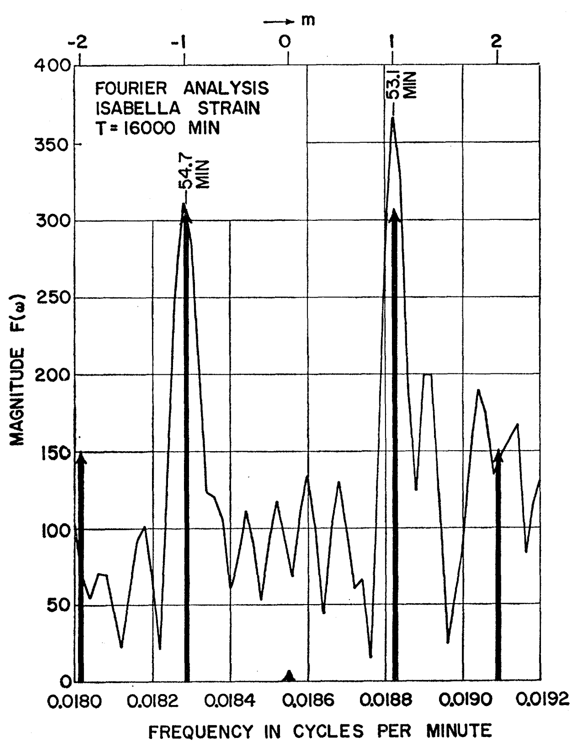

FIG. 1. $n=2$. Spectral intensity of Chilean earthquake observed at Isabella by Benioff, Press, and Smith. Arrows show theoretical positions and amplitudes of multiplet for a compressional point-source.

[15] H. Jeffreys, *The Earth* (Cambridge University Press, New York, 1958), p. 124.

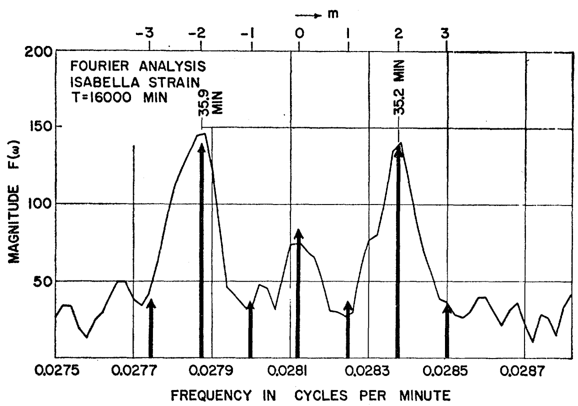

FIG. 2. $n=3$. Spectral intensity of Chilean earthquake observed at Isabella by Benioff, Press, and Smith.[7] Arrows show theoretical positions and amplitudes of multiplet for a compressional point-source.

face $r=a$, where to a close approximation

$$r=a[1+(1/297)(\tfrac{1}{3}-\cos^2\theta)]. \tag{14}$$

The geopotential surfaces are therefore nearly spherical throughout the volume of the earth, to within 1 part in 300. In the following, we shall therefore neglect the ellipticity of the geopotential surfaces and shall assume that p_0, ρ_0, and g_0 are functions of r only. One can also demonstrate the smallness of the effect of the centrifugal force from the relation

$$\nabla^2\Psi=\nabla^2\psi+2\omega^2=-4\pi G\rho+2\omega^2, \tag{15}$$

which, with $\rho\simeq 5$, gives $(2\omega^2/4\pi G\rho)\simeq(1/400)$.

The rotation does affect the motion through the "deflecting force of the earth's rotation," or the so-called Coriolis[16] force $\mathbf{C}$ which, per unit volume, is given by

$$\mathbf{C}=2\rho_0\mathbf{v}\times\boldsymbol{\omega}. \tag{16}$$

Let u, v, w denote the components of displacement in a spherical system of coordinates r, θ, ϕ. The equations of motion and of the perturbation in the gravity field are[8,9]

$$\rho(\partial^2u/\partial t^2)-2\omega\rho\sin\theta(\partial w/\partial t)=R, \tag{17}$$

$$\rho(\partial^2v/\partial t^2)-2\omega\rho\cos\theta(\partial w/\partial t)=S, \tag{18}$$

$$\rho(\partial^2w/\partial t^2)+2\omega\rho\cos\theta(\partial v/\partial t)+2\omega\rho\sin\theta(\partial u/\partial t)=T, \tag{19}$$

$$\Psi=\Psi_0+\psi,$$
$$\nabla^2\psi=4\pi G(\rho\Delta+u\dot{\rho}). \tag{20}$$

Here ρ denotes the unperturbed density ρ_0, Δ is the divergence of the displacement, and a dot denotes differentiation with respect to r. R, S, and T are linear functions of the displacements and of the components of the strain tensor e_{ij}, as shown in the Appendix. The time does not appear explicitly in R, S, and T.

[16] The "deflecting force of the earth's rotation" is a dominant term in Laplace's theory of ocean tides. Mém. acad. roy. sci. 88, 75 (1775); Oeuvres Complètes 9, 88, and 187. Laplace does not call it the Coriolis force, for obvious reasons. See the discussion by A. T. Doodson, *Advances in Geophysics* (Academic Press, Inc., New York, 1958), Vol. 5, p. 122.

Having solved[8–11] for the functions $U_0(r)$ and $V_0(r)$ in (3) and for the corresponding $\sigma_0(n)$ for the case of $\omega=0$, we now proceed to carry out a first-order perturbation calculation in the small parameter α defined by

$$\alpha=(\omega/\sigma_0)\ll 1. \tag{21}$$

Let

$$\sigma=\sigma_0+\alpha\sigma_1,\quad u=u_0+\alpha u_1,\quad S=S_0+\alpha S_1,\text{ etc.}; \tag{22}$$

then the zero-order equations are

$$\rho\sigma_0^2u_0+R_0=0,\quad \rho\sigma_0^2v_0+S_0=0, \tag{23}$$

$$\rho\sigma_0^2w_0+T_0=0,\quad \nabla^2\psi_0-4\pi G(\rho\Delta_0+u_0\dot{\rho})=0. \tag{24}$$

The terms proportional to α yield the equations for the determination of u_1, v_1, w_1, and σ_1:

$$\rho\sigma_0^2u_1+R_1=-2\rho\sigma_0\sigma_1u_0-2i\rho\sigma_0^2\sin\theta w_0, \tag{25}$$

$$\rho\sigma_0^2v_1+S_1=-2\rho\sigma_0\sigma_1v_0-2i\rho\sigma_0^2\cos\theta w_0, \tag{26}$$

$$\rho\sigma_0^2w_1+T_1=-2\rho\sigma_0\sigma_1w_0+2i\rho\sigma_0^2\cos\theta v_0+2i\rho\sigma_0^2\sin\theta u_0, \tag{27}$$

$$\nabla^2\psi_1-4\pi G(\rho\Delta_1+u_1\dot{\rho})=0. \tag{28}$$

Let an asterisk signify the complex conjugate. It then follows from Eqs. (23)–(27) that

$$\begin{aligned}(u_1{}^*R_0-u_0{}^*R_1+v_1{}^*S_0-v_0{}^*S_1+w_1{}^*T_0-w_0{}^*T_1)\\ =[2\rho\sigma_0\sigma_1(u_0u_0{}^*+v_0v_0{}^*+w_0w_0{}^*)\\ +2i\rho\sigma_0^2\sin\theta(u_0{}^*w_0-u_0w_0{}^*)\\ +2i\rho\sigma_0^2\cos\theta(v_0{}^*w_0-v_0w_0{}^*)].\end{aligned} \tag{29}$$

It is shown in the Appendix that, by virtue of Eqs. (23)–(28) and the boundary conditions,

$$\int_0^a r^2dr\int_0^\pi \sin\theta d\theta\int_0^{2\pi}d\phi(u_1{}^*R_0-u_0{}^*R_1+v_1{}^*S_0-v_0{}^*S_1+w_1{}^*T_0-w_0{}^*T_1)=0. \tag{30}$$

Hence, by (29), the volume integral over the sphere of the right-hand side of (29) is zero. This furnishes a relation for determining σ_1 in terms of σ_0 and the zero-order solutions defined in (3). The result, derived in the Appendix, is

$$\sigma_1=m\sigma_0\tau(n), \tag{31}$$

with $\tau(n)$ given in Eqs. (6) and (7).

It follows from Eqs. (31) or (5) that each spectral line in (3) of nonvanishing azimuthal number m is shifted either positively or negatively, depending on the sign of m. It is only when the excited mode is symmetrical about the axis of rotation of the earth that the rotation has no effect.

It may be noted that the free oscillations of a self-gravitating liquid Maclaurin ellipsoid, for which the

TABLE II. Relative excitation of modes $n=2$ and $n=3$ of model Bullen B for a compressional point-source at the surface.

n	$U_0(a)$	$V_0(a)$	$\dot{P}_0(a)$
2	1.00	0.0260	0.359
3	1.520	−0.183	2.28

multiplet-separation has been determined[17] for any finite ω, approach the relation (5) in the limit of vanishing ω, with $\tau(n)=1/n$. The separation in the normal Zeeman effect is also given[18] by a relation (5), with $\tau=1$, $m=m_l$, the magnetic quantum number, and $\omega=\omega_L$, the Larmor precession frequency.

3. LINE INTENSITIES IN ROTATIONAL MULTIPLETS

We shall now determine the line intensities in the rotational multiplets of the terrestrial spectrum. In the first instance, we shall treat an explosive compressional point-source at the earthquake focus in Chile. The theory can be generalized to more complicated types of source. For each normal mode, the solution of Eqs. (23) and (24) yields the relative values of $U_0(a)$ and $V_0(a)$ entering in (3), as well as of the factor $\dot{P}(a)$ of Y_{nm} in the perturbation of gravity at the surface $r=a$. The relative excitation of the various modes by a compressional point-source has been determined[9] for model Bullen B, and the relevant amplitudes at the surface are shown in Table II. These were determined for a source situated at the surface, but the relative values do not change by more than about 1% as the depth of focus increases to 200 km.

Let us choose a spherical system of coordinates with the polar axis passing through the earthquake focus in Chile (38°S, 73.5°W), and let θ' denote the polar distance from Chile. Then the components of displacement u', v', w' and the perturbation of gravity at the surface $\dot{P}'$ are given by

$$u'=A(\theta'),\qquad v'=(\partial/\partial\theta')B(\theta'),\quad (32)$$

$$w'=(\sin\theta')^{-1}(\partial/\partial\phi')B(\theta')=0,\quad \dot{P}'=C(\theta'),\quad (33)$$

where

$$A(\theta')=P_2(\cos\theta')e^{i\sigma_2 t}+1.520P_3(\cos\theta')e^{i\sigma_3 t}+\cdots,\quad (34)$$

$$B(\theta')=0.0260P_2(\cos\theta')e^{i\sigma_2 t}-0.183P_3(\cos\theta')e^{i\sigma_3 t}+\cdots,\quad (35)$$

$$C(\theta')=0.359P_2(\cos\theta')e^{i\sigma_2 t}+2.28P_3(\cos\theta')e^{i\sigma_3 t}+\cdots.\quad (36)$$

Here σ_2 and σ_3 denote the frequencies for $n=2$ and $n=3$, respectively.

In order to determine the rotational splitting of the natural oscillations, we must transform the above quantities to the geographical coordinate system (r,θ,ϕ) referred to the *axis of rotation of the earth*, θ denoting colatitude and ϕ longitude. First we transform $A(\theta')$, $B(\theta')$, and $C(\theta')$ into $A(\theta,\phi)$, $B(\theta,\phi)$, and $C(\theta,\phi)$, respectively, by using the relation

$$P_n(\cos\theta')=\sum_{m=-n}^{m=n}\frac{(n-|m|)!}{(n+|m|)!}P_n^m(\cos\theta)\times P_n^m(\cos\theta_0)e^{im(\phi-\phi_0)},\quad (37)$$

where $P_n^m(\cos\theta)=P_n^{-m}(\cos\theta)$ are the associate Legendre polynomials. Here θ_0 and ϕ_0 denote the colatitude and longitude of the earthquake focus in Chile, and θ and ϕ, the colatitude and longitude of the observing station. The components of displacement u, v, w and of the perturbation in gravity, $\dot{P}$, in the geographical system of coordinates are then given by

$$u=A(\theta,\phi)\qquad v=(\partial/\partial\theta)B(\theta,\phi),\quad (38)$$

$$w=(\sin\theta)^{-1}(\partial/\partial\phi)B(\theta,\phi),\quad \dot{P}=C(\theta,\phi).\quad (39)$$

with v and w now denoting displacements in the directions of South and East, respectively.

Thus we get for the vertical component of displacement u:

$$\begin{aligned}u=A(\theta,\phi)=e^{i\sigma_2 t}\{&0.069P_2(\cos\theta)\\&-0.243P_2^1(\cos\theta)[e^{i(\phi-\phi_0)}+e^{-i(\phi-\phi_0)}]\\&+0.078P_2^2(\cos\theta)[e^{i(2\phi-2\phi_0)}+e^{-i(2\phi-2\phi_0)}]\}\\&+e^{i\sigma_3 t}\{0.517P_3(\cos\theta)+0.134P_3^1(\cos\theta)\\&\times[e^{i(\phi-\phi_0)}+e^{-i(\phi-\phi_0)}]-0.073P_3^2(\cos\theta)\\&\times[e^{i(2\phi-2\phi_0)}+e^{-i(2\phi-2\phi_0)}]+0.019P_3^3(\cos\theta)\\&\times[e^{i(3\phi-3\phi_0)}+e^{-i(3\phi-3\phi_0)}]\}+\cdots.\end{aligned}\quad (40)$$

The rotational separation of the multiplet stems entirely from the dependence on ϕ of $A(\theta,\phi)$, $B(\theta,\phi)$, and $C(\theta,\phi)$.

The Fourier analysis of the multiplets for $n=2$ and $n=3$ was made by Benioff, Press, and Smith[7] from the strain seismograms obtained at Isabella, California. For the interpretation of their results, we must derive an expression for the strain e in a long quartz tube anchored to the ground in a horizontal position, with its axis making an angle β with the East direction. We have[19]

$$e=(\cos^2\beta e_{\phi\phi}+\sin\beta\cos\beta e_{\theta\phi}+\sin^2\beta e_{\theta\theta}),\quad (41)$$

e_{ij} denoting the components of the strain tensor. Using Eq. (3), we get for each partial strain e_n^m corresponding to the spherical harmonic $Y=Y_{nm}(\theta,\phi)$,

$$e_n^m=U_nY+V_n\left[\cos^2\beta\left(\cot\theta\frac{\partial Y}{\partial\theta}-\frac{m^2}{\sin^2\theta}Y\right)+\frac{2im}{\sin\theta}\sin\beta\cos\beta\left(\frac{\partial Y}{\partial\theta}-\cot\theta Y\right)+\sin^2\beta\frac{\partial^2 Y}{\partial\theta^2}\right].\quad (42)$$

[17] G. H. Bryan, Phil. Trans. Roy. Soc. London, Ser. A, **107** (1889). Equation (87) should read: $\lambda=\pm\Lambda+(s/n)\omega$.

[18] M. A. Bethe and E. E. Salpeter, *Quantum Mechanics of Two-Electron Atoms* (Academic Press, Inc., New York, 1958), p. 206.

[19] A. E. H. Love, *The Mathematical Theory of Elasticity* (Dover Publications, Inc., New York, 1944), p. 54.

This leads to

$$ae=\left[A+\cos^2\beta\left(\cot\theta\frac{\partial B}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 B}{\partial\phi^2}\right)\right.$$
$$\left.+\frac{2}{\sin\theta}\sin\beta\cos\beta\frac{\partial}{\partial\phi}\left(\frac{\partial B}{\partial\theta}-\cot\theta B\right)+\sin^2\beta\frac{\partial^2 B}{\partial\theta^2}\right],\quad(43)$$

where a denotes the radius of the earth, $A=A(\theta,\phi)$, and $B=B(\theta,\phi)$.

We note that this expression for the strain along a horizontal axis contains the term $A(\theta,\phi)$ stemming from the vertical component of displacement u. Since the ratio of A/B, as given in (34) and (35), is about 40 for $n=2$, and about 8 for $n=3$, it follows that the first term in (43) predominates, so that in this case the response of the strain-seismograph is not sensitive to the orientation of its axis. Indeed, due to the predominance of the A term, the response of the strain-seismograph in the various lines of the multiplet resembles the relative distribution of amplitude in the vertical component of displacement u and in the gravity perturbation $\dot{P}$.

Using the expressions (38), (39), and (43), we have computed the periods and amplitudes in the rotational multiplets of the Chilean earthquake for observing stations situated at Los Angeles and at Palisades, New York. For Los Angeles, the gravimetric amplitudes were computed from $\dot{P}$ in (39), and the seismic strain amplitudes[20] from Eq. (43). These results are shown in Table III and in Figs. 1 and 2. The values for the components of displacement u, v, and w in Table IV for the Palisades station are based on Eqs. (38) and (39). The periods in the multiplet are based on Eq. (5), with the values of $\tau(n)$ given in Table V.

Before proceeding to a comparison of theory with observations, we shall discuss the implications of our

TABLE III. Theoretical periods T and amplitudes excited by a compressional point-source at Chile ($\theta_0=128°$), and observed at Los Angeles ($\theta=56°$).

			Amplitude		Observed periods	
n	m	T (min)	Gravi-metric	Strain	Gravi-metric[a]	Strain[b]
2	−2	55.33	0.115	0.302	54.98	
2	−1	54.50	0.242	0.615		54.7
2	0	53.70	0.0015	0.0023		
2	1	52.92	0.242	0.615	52.80	53.1
2	2	52.15	0.115	0.302		
3	−3	35.99	0.396	0.401		
3	−2	35.83	1.257	1.46	35.87	35.9
3	−1	35.67	0.282	0.377		
3	0	35.50	0.624	0.873		
3	1	35.34	0.282	0.377		
3	2	35.18	1.257	1.46	35.24	35.2
3	3	35.03	0.396	0.401		

[a] See reference 6.
[b] See reference 7.

[20] The over-all sensitivity of the Isabella strain-meter varies by less than 10% in the period range of 17 to 54 min, so that its response is proportional to the strain within the above limit.

TABLE IV. Theoretical periods T and amplitudes of the displacements u, v, and w of multiplets in the spectrum of the earth excited by a point-source at Chile ($\theta_0=128°$), and observed at Palisades, New York ($\theta=49°$).

n	m	T (min)	u Vertical	v South	w East
2	−2	55.33	0.265	0.0120	0.0182
2	−1	54.50	0.721	0.0053	0.0248
2	0	53.70	0.020	0.0053	0
2	1	52.92	0.721	0.0053	0.0248
2	2	52.15	0.265	0.0120	0.0182
3	−3	35.99	0.200	0.063	0.096
3	−2	35.83	0.814	0.058	0.260
3	−1	35.67	0.350	0.144	0.056
3	0	35.50	0.288	0.162	0
3	1	35.34	0.350	0.144	0.056
3	2	35.18	0.814	0.058	0.260
3	3	35.03	0.200	0.063	0.096

assumption of an impulsive compressional point-source. To begin with, the SH torsional content of the source has excited torsional free modes which have been observed seismically[7] and which lie in a different part of the spectrum. Secondly, we have to consider the evidence put forward by Benioff, Press, and Smith[7] to the effect that the source was a progressive rupture proceeding from the epicenter southward for about 1000 km with a velocity of about 3 km/sec. As far as the purely compressional content is concerned, such a disturbance could be represented by a point source traveling over a distance of about 10° of latitude and lasting some 5 min, with most of the energy probably released in the initial burst near the epicenter. The finite time extension, as against an assumed δ-function pulse, may affect the relative excitation of the entire modes $n=2$ and $n=3$, but not the distribution of intensity within the multiplet of each mode. The spatial spreading of the source over 10° of latitude will change the coefficients for the relative excitation of the components of a multiplet, such as is given in (40), to a degree by which the functions $P_n^m(\cos\theta_0)$ vary as θ_0 goes from 128° to 138°. For $n=2$ and $n=3$ this variation is small and is of the same order of magnitude as the α terms in (22), which we have neglected. Since $\alpha=(\omega/\sigma)$ is around 1/24, we may expect errors of that order of magnitude from the latter source in the amplitudes given in Tables III and IV.

There remains to be considered the effect of torques (SV) whose axis has horizontal components. An SV torque will excite spheroidal oscillations which are not symmetrical about an axis passing through the source. These will have a different distribution of intensity in the rotational multiplets from that of a compressional point-source.

4. COMPARISON OF THEORETICAL WITH OBSERVED MULTIPLETS IN THE SPECTRUM OF THE EARTH

The theory presented above predicts that every frequency of spheroidal oscillations of the earth of order n

TABLE V. Values of $\tau(n)$ in Eq. (6).

		Bullen B model[a]			Gutenberg model		
n	Overtone	$\sigma_0\times10^3$ (sec^{-1})	T_0 (min)	$\tau(n)$	$\sigma_0\times10^3$ (sec^{-1})	T_0 (min)	$\tau(n)$
2	0	1.951	53.67	0.395	1.957	53.52	0.395
2	I	4.235	24.73	0.239	4.306	24.32	0.230
2	II	6.764	15.48	0.117	6.912	15.15	0.120
3	0	2.953	35.46	0.183	2.964	35.33	0.182
3	I	5.844	17.92	0.213	5.940	17.63	0.207
4	0	4.076	25.69	0.099	4.100	25.54	0.098
4	I	7.289	14.37	0.198	7.420	14.11	0.192

[a] The periods given here are taken from reference (9), where the compressional velocity C_p and the shear velocity C_s in the top 33 km were taken as 7.65 km/sec and 4.30 km/sec, respectively. In reference (11), these values were changed to 6.10 km/sec and 3.54 km/sec, resulting in differences up to 0.04 min in the periods.

is split by rotation into a multiplet of $(2n+1)$ lines whose frequencies are given by Eq. (5), m denoting the azimuthal number appearing in (3) and (4). Values of $\tau(n)$ are shown in Table V. Using these, and the previously determined values[11] of the central frequencies $\sigma_0(n)$, we give in Table III the periods of the multiplet lines for the fundamental modes of spheroidal oscillations of order $n=2$ and $n=3$, in the case of model Bullen B. Also shown are the theoretical amplitudes of the lines for an assumed compressional point-source at Chile, and the observed periods.

As for the periods, we note that the average of each of the two observed doublets in the case $n=2$ is 53.9 min, as against the theoretical value of 53.7 for model Bullen B. In the case $n=3$, the average period of the observed strain doublet is 35.7 and of the gravimetric doublet 35.56, as compared with the theoretical value of 35.5 for model Bullen B. These central frequencies depend on the model assumed, being even smaller by 0.18 min each in the case of the Gutenberg model. The discrepancies with the observed values disappear for the Gutenberg model for $n>5$. It is not the purpose of the present discussion to deal with the question of the modifications which would be required in the structure of the model in order to eliminate the 0.2-min discrepancy with the observed central periods. We shall rather assume that the whole doublet is shifted so as to bring the central line into coincidence with the observed value, and we shall compare the observed *intervals* between the lines in a multiplet with the theoretical values for the intervals.

A. Gravimetric

Referring to Table III, in the case $n=2$, the central line $m=0$ should be very weak and was actually not observed. The theoretical interval between the strongest $m=-1$ and $m=1$ lines is 1.6 min, as against the observed interval of 2.2 min. Since, however, the $m=\pm2$ lines are theoretically half as strong as the $m=\pm1$ lines, the observed lines may represent the unresolved pairs $m=1$ and $m=2$. The average periods of the pairs, weighted according to their amplitudes, are 54.77 and 52.67 min, giving an interval of 2.1 min, which is close to the observed interval of 2.2 min.

In the case $n=3$, the strongest lines should be the pair $m=\pm2$, which are very close to the observed two lines. The $m=0$ line should be half as strong, and should have been observed if the noise level were low and the resolution adequate.

In addition to the doublets in the fundamental $n=2$ and $n=3$ modes, the first overtone of the $n=3$ mode was also reported as a gravitational doublet: $T=17.88$ and 17.68 min. Theoretically, the relative amplitudes within a multiplet for the overtones should be the same as for the fundamental, as given in Table III. The strongest lines in all the overtones of $n=3$ should therefore again be the pair $m=\pm2$. The frequency interval is $4\omega\tau_{\rm I}(3)=6.20\times10^{-5}$ sec^{-1}, using the value of 0.213 for $\tau_{\rm I}(3)$ given in Table V. This leads to a period-interval for the pair of 0.19 min, which agrees with the observed value of 0.20 min.

B. Seismic

Two doublets were reported by Benioff, Press, and Smith[7] from a Fourier analysis of the strain seismogram at Isabella, California. In Fig. 1 is reproduced their spectral intensity curve in the vicinity of the $n=2$ pair of 54.7 and 53.1 min. In the same figure are shown by arrows the theoretical separations and amplitudes of the five lines making up the $n=2$ quintet, in accordance with the results given in Table III. The central $m=0$ line should be extremely weak, and is indeed below the background noise. The $m=2$ line is not far from the observed intensity level. On the other hand, the $m=-2$ line is above the observed intensity in its vicinity. One should expect a rise in intensity immediately to the left of this line, which is outside the limit of the figure.

In Fig. 2 is shown the observed spectral curve for $n=3$, and the theoretical positions and amplitudes of the $n=3$ septet. There is an indication of the central $m=0$ line, and its theoretical intensity is close to the observed value. The $m=\pm1$ and $m=\pm3$ pairs are close to the general noise level in their respective neighborhoods.

The answer to the question posed previously, as to why only two lines out of the quintet for $n=2$ have been observed at Los Angeles, is that the observed doublets are the theoretically strongest lines, that the central line should be weak, and that the outermost lines would not be expected to stand out above the prevailing noise level. In the case of the septet for $n=3$, the observed doublets are again theoretically the strongest; there is actually an indication of a third line at $m=0$ in the strain record, while the missing two pairs $m=\pm1$ and $m=\pm3$ are theoretically again just within the background noise.

It would be of interest to make a more refined Fourier analysis of the observed spectral curves in the multiplets, and also to determine the theoretical multiplet intensities for an SV torque-source. Further test of the theory could be made by observations at other latitudes, where the intensity distribution within the multiplet would be different. We are at the threshold of a new science: terrestrial spectroscopy.

ACKNOWLEDGMENT

I am indebted to Professor H. Benioff for sending me the response curve of the Isabella strain-meter.

APPENDIX

We shall first prove that, by virtue of Eqs. (23)–(28) and the boundary conditions,

$$I=\int_0^a r^2dr\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi(u_1{}^*R_0-u_0{}^*R_1+v_1{}^*S_0-v_0{}^*S_1+w_1{}^*T_0-w_0{}^*T_1)=0. \quad \text{(A.1)}$$

In the absence of rotation, the components of displacement u_0, v_0, and w_0 and the perturbation in the gravitational potential ψ_0 are given by

$$u_0=U_0(r)Y_{nm}(\theta,\phi),\quad v_0=V_0(r)\frac{\partial}{\partial\theta}Y_{nm}, \quad \text{(A.2)}$$

$$w_0=\frac{V_0(r)}{\sin\theta}\frac{\partial}{\partial\phi}Y_{nm},\quad \psi_0=P_0(r)Y_{nm}, \quad \text{(A.3)}$$

with

$$Y_{nm}(\theta,\phi)=P_n{}^m(\cos\theta)e^{im\phi}. \quad \text{(A.4)}$$

We have[8,9]

$$R_i=\left\{\rho g\Delta+\rho\frac{\partial\psi}{\partial r}-\rho\frac{\partial}{\partial r}(gu)+\frac{\partial}{\partial r}\left(\lambda\Delta+2\mu\frac{\partial u}{\partial r}\right)+\frac{\mu}{r}\frac{\partial e_{r\theta}}{\partial\theta}+\frac{\mu}{r\sin\theta}\frac{\partial e_{r\phi}}{\partial\phi}+\frac{\mu}{r}(4e_{rr}-2e_{\theta\theta}-2e_{\phi\phi}+\cot\theta e_{r\theta})\right\}_i, \quad \text{(A.5)}$$

$$S_i=\left\{\frac{\rho}{r}\frac{\partial\psi}{\partial\theta}+\frac{\partial}{\partial r}(\mu e_{r\theta})+\frac{1}{r}\frac{\partial}{\partial\theta}(-g\rho u+\lambda\Delta+2\mu e_{\theta\theta})+\frac{\mu}{r\sin\theta}\frac{\partial e_{\theta\phi}}{\partial\phi}+\frac{\mu}{r}\left[2\cot\theta\left(\frac{1}{r}\frac{\partial v}{\partial\theta}-\frac{v}{r}\cot\theta-\frac{1}{r\sin\theta}\frac{\partial w}{2\phi}\right)+3e_{r\theta}\right]\right\}_i, \quad \text{(A.6)}$$

$$T_i=\left\{\frac{\rho}{r\sin\theta}\frac{\partial\psi}{\partial\phi}+\frac{\partial}{\partial r}(\mu e_{r\phi})+\frac{\mu}{r}\frac{\partial e_{\theta\phi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-g\rho u+\lambda\Delta+2\mu e_{\phi\phi})+\frac{3\mu}{r}e_{r\phi}+\frac{2\mu}{r}\cot\theta e_{\theta\phi}\right\}_i. \quad \text{(A.7)}$$

Here

$$e_{rr}=\frac{\partial u}{\partial r},\qquad e_{\theta\theta}=\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}, \quad \text{(A.8)}$$

$$e_{\phi\phi}=\frac{1}{r\sin\theta}\frac{\partial w}{\partial\phi}+\frac{v}{r}\cot\theta+\frac{u}{r},\quad e_{r\theta}=\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}, \quad \text{(A.9)}$$

$$e_{r\phi}=\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}+\frac{\partial w}{\partial r}-\frac{w}{r}, \quad \text{(A.10)}$$

$$e_{\theta\phi}=\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{w}{r}\cot\theta+\frac{1}{r\sin\theta}\frac{\partial v}{\partial\phi}, \quad \text{(A.11)}$$

$$\Delta=\frac{\partial u}{\partial r}+\frac{2u}{r}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial w}{\partial\phi}. \quad \text{(A.12)}$$

We now substitute from (A.5), (A.6), and (A.7) into (A.1) and carry out the indicated integration. Every term which involves derivatives of λ, μ, Δ and of the components of the strain-tensor e_{ij} we integrate out partially. The integrated quantities vanish because of (1) the boundary conditions at the surface $r=a$:

$$\lambda\Delta+2\mu(\partial u/\partial r)=e_{r\theta}=e_{r\phi}=0, \quad \text{(A.13)}$$

(2) the regularity of the e_{ij} and of Δ at the poles, and (3) the periodicity condition in ϕ. It is found that the terms in the integrand multiplying λ, μ, and $g\rho$ each cancel out, leading to

$$I=\int d\tau\,\rho\left[u_1{}^*\frac{\partial\psi_0}{\partial r}-u_0{}^*\frac{\partial\psi_1}{\partial r}+\frac{1}{r}\left(v_1{}^*\frac{\partial\psi_0}{\partial\theta}-v_0{}^*\frac{\partial\psi_1}{\partial\theta}\right)+\frac{1}{r\sin\theta}\left(w_1{}^*\frac{\partial\psi_0}{\partial\phi}-w_0{}^*\frac{\partial\psi_1}{\partial\phi}\right)\right], \quad \text{(A.14)}$$

where $\int d\tau$ denotes the volume integral in (A.1).

For the proof of the vanishing of (A.14), we make use of the equations for the gravity potential:

$$\nabla^2\psi_0{}^*-4\pi G(\rho\Delta_0{}^*+u_0{}^*\dot{\rho})=0, \quad \text{(A.15)}$$

$$\nabla^2\psi_1{}^*-4\pi G(\rho\Delta_1{}^*+u_1{}^*\dot{\rho})=0. \quad \text{(A.16)}$$

Multiplying (A.15) by ψ_1 and (A.16) by $-\psi_0$, adding, and integrating over the volume of the sphere, we get

$$\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi\, a^2\times\left[\psi_1\left(\frac{\partial\psi_0{}^*}{\partial r}-4\pi G\rho u_0{}^*\right)-\psi_0\left(\frac{\partial\psi_1{}^*}{\partial r}-4\pi G\rho u_1{}^*\right)\right]_{r=a}$$
$$=4\pi G\int d\tau\left[u_1{}^*\frac{\partial\psi_0}{\partial r}-u_0{}^*\frac{\partial\psi_1}{\partial r}+\frac{1}{r}\left(v_1{}^*\frac{\partial\psi_0}{\partial\theta}-v_0{}^*\frac{\partial\psi_1}{\partial\theta}\right)+\frac{1}{r\sin\theta}\left(w_1{}^*\frac{\partial\psi_0}{\partial\phi}-w_0{}^*\frac{\partial\psi_1}{\partial\phi}\right)\right]. \quad \text{(A.17)}$$

Now, while u_0 and ψ_0 in (A.2) and (A.3) are each represented by a single spherical harmonic of order n, the solutions for u_1 and ψ_1 from equations (25) and (28), respectively, are of more general type:

$$u_1=\sum_k u_1^{(k)}, \quad u_i^{(k)}=U_i^{(k)}(r)Y_{km}(\theta,\phi). \quad (A.18)$$

$$\psi_1=\sum_k \psi_1^{(k)}, \quad \psi_i^{(k)}=P_i^{(k)}(r)Y_{km}(\theta,\phi), \quad (A.19)$$

$$u_0=u_0^{(n)}, \qquad \psi_0=\psi_0^{(n)}. \quad (A.20)$$

The boundary condition at the surface for the gravitational potential is[8,9]

$$(\partial\psi^{(k)}/\partial r)-4\pi G\rho u^{(k)}=-[(k+1)/a]\psi^{(k)};\ r=a, \quad (A.21)$$

for each k. Hence

$$\left[\psi_1\left(\frac{\partial\psi_0^*}{\partial r}-4\pi G\rho u_0^*\right)-\psi_0\left(\frac{\partial\psi_1^*}{\partial r}-4\pi G\rho u_1^*\right)\right]$$

$$=\frac{1}{a}\sum_k [(k+1)\psi_0^{(n)}\psi_1^{*(k)}-(n+1)\psi_0^{*(n)}\psi_1^{(k)}]. \quad (A.22)$$

The surface integral of (A.22) vanishes because of the orthogonality of the spherical harmonics when $n\neq k$, and because of identical vanishing for $n=k$. Hence, the volume integral in (A.17) vanishes, and with it I, as given by (A.14).

The above derivation holds for both spheroidal and torsional oscillations, since all the boundary conditions are satisfied in both cases. In the case of the torsional oscillations, some of the boundary conditions are satisfied identically because for them

$$u_0=\Delta_0=T_0=\psi_0=0. \quad (A.23)$$

Evaluation of $\tau(n)$

In the case of spheroidal oscillations, u_0, v_0, and w_0 are given by (A.2) and (A.3), and the volume integral of the right-hand side of (29) leads to

$$2\sigma_0\sigma_1\int d\tau\,\rho\left\{U_0^2p^2+V_0^2\left[\left(\frac{dp}{d\theta}\right)^2+\frac{m^2p^2}{\sin^2\theta}\right]\right\}$$

$$=4m\sigma_0^2\int d\tau\rho\left\{U_0V_0p^2+\cot\theta p\frac{dp}{d\theta}V_0^2\right\}, \quad (A.24)$$

where

$$p(\theta)\equiv P_n^m(\cos\theta). \quad (A.25)$$

We have

$$A=\int_0^\pi \sin\theta d\theta\, p^2=\frac{2}{(2n+1)}\frac{(n+m)!}{(n-m)!}, \quad (A.26)$$

$$B=\int_0^\pi \sin\theta\left(\frac{dp}{d\theta}\right)^2 d\theta=n(n+1)A-m^2C, \quad (A.27)$$

$$C=\int_0^\pi \frac{d\theta}{\sin\theta}p^2=\frac{1}{m}\frac{(n+m)!}{(n-m)!}, \quad (A.28)$$

$$m\int_0^\pi \cos\theta p\frac{dp}{d\theta}d\theta=\tfrac{1}{2}mA. \quad (A.29)$$

Using these in (A.24), we are led to Eqs. (3) and (4).

In the case of torsional oscillations, we have[8,9]

$$u_0=0, \quad v_0=(V_0(r)/\sin\theta)(\partial Y_{nm}/\partial\phi),$$
$$w_0=-V_0(r)(\partial Y_{nm}/\partial\theta), \quad (A.30)$$

and the volume integral of the right-hand side of (29) leads to

$$2\sigma_0\sigma_1\int d\tau\,\rho V_0^2[(dp/d\theta)^2+(m^2/\sin^2\theta)p^2]$$

$$=4m\sigma_0^2\int d\tau\,\rho V_0^2\cot\theta p(dp/d\theta). \quad (A.31)$$

From this, relation (7) follows.