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Plasma Density Fluctuations in a Magnetic Field*

E. E. SALPETER†

Laboratory of Nuclear Studies and Center for Radiophysics and Space Research, Cornell University, Ithaca, New York and School of Physics, Sydney University, Sydney, N.S.W., Australia

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Sinusoidal electron charge density fluctuations with propagation vector **k** are considered for a fully ionized gas in complete thermodynamic equilibrium in a constant magnetic field. Let α and ϵ be the ratio of the Debye length and of an electron gyroradius, respectively, to the wavelength k^{-1}. A general formula is derived for the frequency spectrum of these fluctuations for arbitrary values of α, ϵ, and of the angle $(\phi-\frac{1}{2}\pi)$ between **k** and the magnetic field. The dispersion relation implied by this expression has been obtained previously by Gross and by Bernstein, but the method of derivation is different. A very small electron-ion mass ratio m/M is assumed.

For large values of ϵ and α most of the intensity occurs at small frequencies: If $\sin\phi \gg (m/M)^{\frac{1}{2}}\epsilon$, the main spectrum is continuous as in the absence of a magnetic field; if $(m/M)^{\frac{1}{2}} \ll \sin\phi \ll (m/M)^{\frac{1}{2}}\epsilon$, it consists of lines with spacing about the ion gyrofrequency; if $\sin\phi \ll (m/M)^{\frac{1}{2}}$, it consists mainly of a line at zero frequency. Weaker spectral lines are obtained which correspond to plasma oscillations, the existence of "frequency gaps" is confirmed for small angles ϕ, and the intensities of the various components are evaluated. For small ϕ, another spectral line is obtained at a "resonance" frequency intermediate between the electron and ion gyrofrequency.

1. INTRODUCTION

In a previous paper[1] (I) the frequency spectrum was derived for the spatial Fourier transform with fixed wave vector **k** of the electron charge density fluctuations in a plasma. The present paper is a continuation of this work, which now includes the effects of a constant magnetic field **B** in an arbitrary direction. As in most of the previous papers, we assume complete thermodynamic equilibrium and neglect collisions throughout the present work. We also assume that kc is sufficiently large for coupling between longitudinal and transverse oscillations to be negligible and consider only longitudinal ones.

The aim of the present work is twofold. The first is to provide a theoretical basis for analyzing experiments on radar backscatter from the upper ionosphere and exosphere. Preliminary experiments have already been performed and more are planned.[2–4] In some of these experiments, at least, the effect of the earth's magnetic field should be detectable. It is by no means certain that deviations from thermal equilibrium are unimportant in the ionosphere and exosphere, but an equilibrium theory is necessary in any case before such deviations can be deduced from observations. The second aim is a more academic one: Considerable theoretical work has already been done in the past on plasma oscillations in a magnetic field for cases in which collisions can be neglected. These previous papers[5,6] have concentrated mainly on the dispersion relation, i.e., on finding possible solutions for the frequency ω in the complex plane. The dispersion relation has the advantage that it gives some information on plasma oscillations independent of the excitation conditions, i.e., even when the oscillations are produced by deviations from thermal equilibrium as is the case in most laboratory experiments. In the present paper, on the other hand, we derive the actual intensities as a function of the *real* frequency variable ω for a specific case, namely complete thermodynamic equilibrium. Although this is only a special case, knowledge of the actual intensities throws some light on some puzzles encountered in the study of the dispersion relation alone. In

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† Permanent address: Cornell University, Ithaca, New York.

[1] E. E. Salpeter, Phys. Rev. **120**, 1528 (1960); hereafter referred to as I.

[2] K. Bowles, National Bureau of Standards Report No. 6070, 1959 (unpublished).

[3] W. E. Gordon, Proc. I. R. E. **46**, 1824 (1958).

[4] V. C. Pineo, L. G. Kraft, and H. W. Briscoe, J. Geophys. Research **65**, 1620 (1960).

[5] E. P. Gross, Phys. Rev. **82**, 232 (1951).

[6] I. B. Bernstein, Phys. Rev. **109**, 10 (1958).

particular, we shall consider some apparent discontinuities as the magnetic field goes to zero [Sec. 4(e)] and the so-called "frequency gaps" which had met with some criticism[7] [Sec. 4(c)]. For these reasons we shall discuss some limiting cases in more detail than the practical ionospheric applications warrant.

We first define some constants and discuss approximations to be made. We consider a fully ionized plasma consisting of electrons of mass m and a single species of positive ions with mass M and atomic charge Z. Let n be the number particle density of electrons, T the temperature (the same for electrons and ions), $\mathbf{k}$ the fixed wave vector of the sinusoidal spatial density variation, $\mathbf{B}$ the constant magnetic field, and $\frac{1}{2}\pi-\phi$ the angle between $\mathbf{k}$ and $\mathbf{B}$. We define six characteristic angular frequencies by

$$\begin{aligned}\omega_e&=k(2\kappa T/m)^{\frac{1}{2}}, & \omega_p&=(4\pi ne^2/m)^{\frac{1}{2}}, & \omega_c&=eB/mc,\\ \omega_i&=k(2\kappa T/M)^{\frac{1}{2}}, & \omega_{pi}&=(4\pi nZe^2/M)^{\frac{1}{2}}, & \omega_{ci}&=ZeB/Mc.\end{aligned} \tag{1}$$

ω_e is characteristic of electron Doppler broadening frequencies, i.e., ω_e^{-1} is typical of times taken by a thermal electron to travel a wavelength k^{-1}, ω_p is the electron plasma frequency and ω_c the electron cyclotron frequency or gyrofrequency. ω_i, ω_{pi}, and ω_{ci} are the corresponding frequencies for the ions. Another characteristic frequency is kc, the frequency of an electromagnetic wave in vacuum of the same wavelength. We shall assume throughout that kc is very much larger than all the six frequencies in Eq. (1) and shall omit completely all terms in $1/kc$, i.e., omit the magnetic induction and Maxwell's displacement current in Maxwell's equation. Consequently, we shall retain no retardation effects, no coupling between transverse and longitudinal oscillations, in short no genuine magnetohydrodynamic effects.

We define next the Debye length D and electron gyroradius R_e by

$$D=(\kappa T/4\pi ne^2)^{\frac{1}{2}}, \quad R_e=v_e/\omega_e, \tag{2}$$

where v_e is the electron thermal velocity (with a similar relation for the ion gyroradius R_i). We shall also require three dimensionless parameters,

$$\begin{aligned}\alpha&=1/kD=\omega_p/\sqrt{2}\omega_e,\\ \epsilon&=\omega_c/\omega_e\sim 1/kR_e,\\ \epsilon_i&=\omega_{ci}/\omega_i=(Zm/M)^{\frac{1}{2}}\epsilon\sim 1/kR_i.\end{aligned} \tag{3}$$

We shall carry out calculations for general values of α, ϵ, and ϵ_i but shall assume throughout that $\Lambda\equiv nD^3\gg 1$ and that the effective collision mean free path $l\sim D\Lambda/\ln\Lambda$ is very large compared with k^{-1}, R_e, R_i, and D. We consequently shall neglect individual two-particle collisions completely and consider as infinitesimal the electrostatic fields produced by the charge density fluctuations at thermal equilibrium.

In Sec. 2 we derive the general formula for the frequency spectrum for arbitrary values of the angle ϕ and the various parameters α, ϵ, Z, and m/M (but always for $m\ll M$, $Z\sim 1$). In Sec. 3 we consider a few simple limiting cases, such as $\phi\to\pi/2$, $\epsilon\to 0$, and $\alpha\to 0$. In Sec. 4 we discuss the special case of $\phi=0$, i.e., propagation direction perpendicular to the magnetic field. In Sec. 5 we consider some cases for nonzero values of the angle ϕ.

[7] L. Oster, Revs. Modern Phys. **32**, 141 (1960).

2. DERIVATION OF THE GENERAL EQUATION

We consider a volume V containing N electrons and N/Z positive ions. We define the spatial Fourier transforms of the electron and ion charge densities for fixed wave vector $\mathbf{k}$ by

$$\begin{gathered}\rho_e(t)=-e\sum_{j=1}^{N}e^{-ikz_j}, \quad \rho_i(t)=-Ze\sum_{j=1}^{N/Z}e^{-ikZ_j},\\ \rho_t(t)=\rho_e(t)+\rho_i(t),\end{gathered} \tag{4}$$

where we have taken the positive z direction parallel to $\mathbf{k}$ and z_j, Z_j refer to the positions of the jth electron and ion, respectively. Since we are neglecting magnetic induction and retardation, the electrostatic field $\mathbf{E}$ consists of a superposition of terms for the different wave vectors $\mathbf{k}$, given by Eq. (7) of I. The $\mathbf{k}$ term in this sum is

$$E_k(\mathbf{r},t)=-i(4\pi/Vk)\rho_t(t)e^{ikz}\mathbf{i}_z, \tag{5}$$

where $\mathbf{i}_z$ is the unit vector in the $\mathbf{k}$ direction [and real parts of the right-hand sides are implied in Eqs. (4) and (5)].

We are interested in the Laplace transform of $\rho_e(t)$,

$$Q_e(\omega)=\int_0^\infty dt\,\rho_e(t)e^{-(i\omega+\gamma)t}, \tag{6}$$

where γ is an infinitesimal real positive constant and ω is angular frequency. The standard method of deriving an expression for $Q_e(\omega)$ starts from the Boltzmann equation without collision term, the so-called Vlasov equation, and such a method was used in I (in the absence of a magnetic field) and by Bernstein.[6] Since there has been some criticism[7] of some mathematical techniques used in the work of Bernstein[6] and of Gross,[5] we shall use an alternative method of derivation, leading to the same result.

We start from the equation of motion for a single electron in the absence of collisions,

$$\ddot{\mathbf{r}}=\boldsymbol{\omega}_c\times\dot{\mathbf{r}}-(e/m)\mathbf{E}(\mathbf{r},t),$$

where $\boldsymbol{\omega}_c$ is a vector of magnitude ω_c [Eq. (1)] and direction parallel to the constant magnetic field $\mathbf{B}$. Let u be the velocity component parallel to $\mathbf{B}$ and v the absolute value of the velocity component in the plane perpendicular to $\mathbf{B}$, at time $t=0$. For the z component of $\mathbf{r}(t)$ we have

$$\begin{aligned}z(t)&=z_u(t)+z'(t),\\ z_u(t)&=ut\sin\phi+(v\cos\phi/\omega_c)\sin(\omega_ct-\delta)+z_0,\end{aligned} \tag{7}$$

where δ and z_0 are constants. $z_u(t)$ is the solution of the equation of motion with $\mathbf{E}$ replaced by zero, but satisfying the correct boundary conditions at $t=0$. $z'(t)$ is the solution of the full equation of motion which satisfies $z'=\dot{z}'=0$ at $t=0$. Let $z'(\omega)$ and $\mathbf{A}(\omega)$ be the Laplace transforms of $z'(t)$ and $(e/m)\mathbf{E}(\mathbf{r}(t),t)$, respectively, defined analogously to Eq. (6). By using the identity

$$\int_0^\infty \dot{a}(t)e^{-(i\omega+\gamma)t}dt=-a(0)+(i\omega+\gamma)\int_0^\infty a(t)e^{-(i\omega+\gamma)t}dt$$

twice, we obtain from the equation of motion the unique solution

$$(i\omega+\gamma)^2\mathbf{r}'(\omega)=(i\omega+\gamma)\boldsymbol{\omega}_c\times\mathbf{r}'(\omega)-\mathbf{A}(\omega).$$

This gives

$$z'(\omega)=(\omega-i\gamma)^{-2}\frac{(\omega-i\gamma)^2-\omega_c^2\sin^2\phi}{(\omega-i\gamma)^2-\omega_c^2}A(\omega), \qquad (8)$$

where $\frac{1}{2}\pi-\phi$ is the angle between $\mathbf{k}$ and $\mathbf{B}$.

We need an expression for $e^{-ikz(t)}$ and make use of the fact that the electric field $\mathbf{E}$, and hence $z'(t)$, is extremely small. We use the linearized approximation, i.e., expand $e^{-ikz'}$ and carry only the first two terms. On using the identity

$$e^{ia\sin\theta}=\sum_{n=-\infty}^{\infty}J_n(a)e^{in\theta}, \qquad (9)$$

and Eq. (7), we find

$$e^{-ikz(t)}=e^{-ik(z_0+ut\sin\phi)}\sum_{n=-\infty}^{\infty}J_n(kv\cos\phi/\omega_c)\times e^{in(\omega_ct-\delta)}[1-ikz'(t)]. \qquad (10)$$

The Laplace transform of this expression is

$$-i\sum_{n=-\infty}^{\infty}e^{-i(kz_0+n\delta)}J_n(kv\cos\phi/\omega_c)[\Omega_n^{-1}+kz'(\Omega_n)],$$
$$\Omega_n=\omega+ku\sin\phi-n\omega_c-i\gamma, \qquad (11)$$

where $z'(\omega)$ is given in terms of $A(\omega)$ by Eq. (8). We shall have to sum Eq. (11) over all N electrons which have a random distribution of z_0 and δ, to lowest order. z' and $A(\omega)$ are already infinitesimal and we need carry in Eq. (11) only those lowest order contributions to $A(\omega)$ which survive after averaging. In taking the Laplace transform of the electric field to obtain $A(\omega)$ we need carry only the contribution in Eq. (5) for the given value of the wave vector k and use for e^{ikz} in Eq. (5) the complex conjugate of Eq. (10) with kz' omitted completely. Substitution into the last term of Eq. (11) leads to a double summation over n and n' but only the terms with $n=n'$ will survive the subsequent averaging over values of δ. In the last term in Eq. (11) we can then substitute

$$A(\Omega_n)\to -i(4\pi e/mVk)Q_t(\omega)e^{i(kz_0+n\delta)}J_n(kv\cos\phi/\omega_c),$$

where $Q_t(\omega)$ is the Laplace transform of $\rho_t(t)$. The term involving z' in Eq. (11) then reduces to

$$(4\pi e/mV)Q_t(\omega)\sum_{n=-\infty}^{\infty}\frac{J_n^2(kv\cos\phi/\omega_c)(\Omega_n^2-\omega_c^2\sin^2\phi)}{\Omega_n^2(\Omega_n^2-\omega_c^2)}. \qquad (12)$$

Let $F_e^{(1)}$ and $F_e^{(2)}$ be the one-dimensional and two-dimensional Maxwell distribution functions for electrons at temperature T,

$$F_e^{(1)}(u)=(m/2\kappa T\pi)^{\frac{1}{2}}\exp(-mu^2/2\kappa T),$$
$$F_e^{(2)}(v)=(mv/2\pi\kappa T)\exp(-mv^2/2\kappa T). \qquad (13)$$

To obtain $Q_e(\omega)$ we have to sum Eq. (11) over all electrons, $j=1$ to N. In the second term, given by Eq. (12), we can replace the summation over j by integrations over the distribution functions. We obtain

$$Q_e(\omega)=-ie\sum_{1=j}^{N}\sum_{n=-\infty}^{\infty}e^{-i(kz_0+n\delta)}J_n(kv\cos\phi/\omega_c)\Omega_n^{-1}+H_e(\omega)Q_t(\omega), \qquad (14)$$

where subscripts j are to be understood for z_0, δ, u, and v and

$$H_e(\omega)=-\sum_n\int_{-\infty}^{\infty}F_e^{(1)}(u)du\int_0^\infty F_e^{(2)}(v)dv\times\frac{\omega_p^2(\Omega_n^2-\omega_c^2\sin^2\phi)}{\Omega_n^2(\Omega_n^2-\omega_c^2)}J_n^2\left(\frac{kv}{\omega_c}\cos\phi\right). \qquad (15)$$

The remainder of the derivation of an explicit expression for $|Q_e(\omega)|^2$ follows along similar lines to Sec. 3 of paper I: One obtains an expression similar to our Eq. (14) for Q_i and hence an explicit expression for $Q_t=Q_e+Q_i$. From this one obtains an explicit expression for Q_e, analogous to Eq. (26) of paper I, which involves the electron summation over j and n shown in Eq. (14) plus a similar summation over the N/Z ions. The square of the modulus of this expression involves a double sum over particle indices j and j', in analogy with Eq. (27) of I, as well as a double sum over n and n'. We assume that $\sin\phi$ is nonzero and fixed and allow γ to tend to zero. We single out the terms with $j=j'$ and $n=n'$ and can replace the sum over j by an integral over the distribution functions. As $\gamma\to 0$ these terms are proportional to γ^{-1} since

$$\int_{-\infty}^{\infty}F_e^{(1)}(u)du(\omega+ku\sin\phi-n\omega_c-i\gamma)^{-2}=F_e^{(1)}[(n\omega_c-\omega)/k\sin\phi]\pi/\gamma k\sin\phi, \qquad (16)$$

if $\gamma\ll\omega_e\sin\phi$. The remaining terms with $j\neq j'$ or $n\neq n'$ tend to a constant limit as $\gamma\to 0$ and can be neglected.

In analogy with Eq. (28) of paper I one obtains an explicit expression for the average, under complete thermodynamic equilibrium, of $|Q_e(\omega)|^2$, the frequency spectrum for the intensity of the electron density variations with fixed wave vector **k**. This expression is

$$|Q_e(\omega)|^2=\pi^{\frac{1}{2}}(Ne^2/\gamma\sin\phi)|1-H_e-H_i|^{-2}$$

$$\times\{|1-H_i|^2\sum_{n=-\infty}^{\infty}X_{ne}\exp(-x_{ne}^2)/\omega_e$$

$$+|H_e|^2Z\sum_{n=-\infty}^{\infty}X_{ni}\exp(-x_{ni}^2)/\omega_i\},\quad(17)$$

where

$$X_{ne}=\int_0^\infty F_e^{(2)}(v)dv\,J_n^2(kv\cos\phi/\omega_c),\quad(18)$$

$$x_{ne}=(n\omega_c-\omega)/\omega_e\sin\phi.\quad(19)$$

and H_e is defined by Eq. (15) with similar definitions for the ion quantities x_{ni}, X_{ni}, and H_i. Equation (17) is our desired expression but Eq. (15) for H_e can be simplified: After rearranging terms in the summation over n and an integration by parts in the u integration, using the explicit expression in Eq. (13) for $F_e^{(1)}$, one obtains an integrand involving Ω_n^{-1}. On using the identity

$$J_{n-1}^2(z)-J_{n+1}^2(z)=(2n/z)dJ_n^2/dz,$$

and integrating by parts over v, one obtains finally

$$H_e(\omega)=-\alpha^2\sum_{n=-\infty}^{\infty}X_{ne}T_{ne},\quad H_i(\omega)=-Z\alpha^2\sum_{n=-\infty}^{\infty}X_{ni}T_{ni},$$

$$T_{ne}(\omega)=1-\int_{-\infty}^{\infty}F_e^{(1)}(u)du\,\omega/\Omega_n\quad(20)$$

$$=1-(\omega/\omega_e\sin\phi)[x_{ne}^{-1}f(x_{ne})-i\pi^{\frac{1}{2}}\exp(-x_{ne}^2)],$$

with an equivalent definition for the ion quantity T_{ni}. The variable x_{ne} is defined by Eq. (19) and $f(x)$ is the tabulated function

$$f(x)=2x\exp(-x^2)\int_0^x\exp(t^2)dt,\quad(21)$$

which was graphed in paper I. X_{ne}, which is independent of the frequency ω, is defined by Eq. (18) and can be expressed in terms of the Bessel function I_n of the first kind of imaginary argument,

$$X_{ne}=\exp(-\cos^2\phi/2\epsilon^2)I_n(\cos^2\phi/2\epsilon^2),\quad(22)$$

which has been tabulated[8] extensively. For the ion quantities x_{ni}, X_{ni}, and T_{ni}, simply replace ω_c, ω_e, and ϵ by ω_{ci}, ω_i, and ϵ_i, respectively.

Equation (17), with Eqs. (19) and (20), is our desired formula for the intensity frequency spectrum, involving infinite series of tabulated functions. The shape of the spectrum depends on five dimensionless parameters $\tan\phi$, α, ϵ, Z, and $\epsilon/\epsilon_i=(M/Zm)^{\frac{1}{2}}$. If these parameters are comparable with unity the series converge rapidly and numerical evaluation of Eq. (17) is simple. For cases of practical interest $Z\sim1$ but $M/m\gg1$. In the remaining sections we discuss approximations obtained for various limiting values of the parameters $\tan\phi$, α, and ϵ, but always for $M\gg m$.

We shall usually express integrals of the intensity distribution $|Q_e(\omega)|^2$ over frequency ω in units of $\pi Ne^2/\gamma$. In Sec. 2 of paper I we had derived expressions for the total integrated intensity from statistical mechanics using purely energy considerations. Since the magnetic field does not contribute to the energy, the total integrated intensity should be the same with or without a magnetic field.

3. SOME SIMPLE LIMITING CASES FOR FINITE ϕ

(a) We consider some limiting values for the parameters α and ϵ for a fixed nonzero value of the angle ϕ, assuming in fact $\sin\phi\gg(m/M)^{\frac{1}{2}}$. The limit of $\alpha\to0$ is particularly simple and corresponds to omitting electrostatic forces entirely and replacing H_e and H_i by zero. The second term in Eq. (17), the "ion component," disappears in this case and we get

$$|Q_e(\omega)|^2=(\pi^{\frac{1}{2}}Ne^2/\omega_e\sin\phi)\sum_{n=-\infty}^{\infty}X_{ne}\exp(-x_{ne}^2),\quad(23)$$

with x_{ne} and X_{ne} defined by Eqs. (19) and (22). Equation (23) had been derived previously and analyzed in detail by Laaspere.[9] The spectrum consists of a series of broadened lines of widths $\sim\omega_e\sin\phi$ and spacing ω_c. If $\epsilon\ll\sin\phi$, the lines overlap strongly and the spectrum reduces to the Gaussian shape it has in the absence of a magnetic field. If $\sin\phi\ll\epsilon\ll1$, the lines are sharp, but the intensity envelope for the line spectrum has approximately the same Gaussian shape. If $\epsilon\gg1$ most of the intensity resides in a central broadened line of width $\sim\omega_e\sin\phi$ and the intensity of the broadened lines at $\pm n\omega_c$ decreases rapidly with increasing n.

(b) Another simple limit is obtained when $\phi\to\frac{1}{2}\pi$ for arbitrary values of ϵ and α. In the limit of $\cos\phi/\epsilon=0$ one finds from Eq. (22) that $X_{0e}=X_{0i}=1$ and $X_{ne}=X_{ni}=0$ for $n\neq0$. In this case the functions $H_e(\omega)$ and $H_i(\omega)$ in Eq. (20) reduce to the functions $G_e(\omega)$ and $G_i(\omega)$, defined in paper I, Eq. (29). Equation (17) for the frequency spectrum then reduces exactly to Eq. (28) of paper I, the equivalent expression in the absence of a magnetic field. This is to be expected physically when the magnetic field is parallel to the wave vector since the motion of charges along a line of force is unaffected by the magnetic field.

(c) We consider next the limit of vanishing magnetic field, $\epsilon\to0$, for fixed values of α and ϕ. More specifically, we require $\epsilon_i\ll\epsilon\ll\sin\phi$. Since $\epsilon\ll1$ the series over X_{ne} in Eqs. (17) and (20) converge very slowly and we require asymptotic expansions in ϵ^{-1}. If a number of terms in these asymptotic expansions are required one

[8] Trans. Am. Inst. Elec. Engrs. **60**, 135 (1941).

[9] T. Laaspere, Ph.D. thesis, Cornell University, 1960 (unpublished).

can use contour integration with the method of steepest descent, etc., as used by Gross[5] and Bernstein.[6] We shall evaluate only the leading term and can use a less standard but simpler method for this purpose: Over the important range of integration in Eq. (18) the argument of the Bessel function J_n is of the order of $\epsilon^{-1}\gg 1$. The Bessel function of large argument and large order can be approximated by[10]

$$J_n(z)\approx\cos[n(\tan\beta-\beta)-\pi/4](\tfrac{1}{2}\pi z\sin\beta)^{-\frac{1}{2}}, \quad (24)$$
$$\cos\beta\equiv n/z,$$

if $|n/z|<1$. If $|n/z|>1$ and $n\gg 1$, $z\gg 1$ the Bessel function is extremely small and can be neglected.[11] Since $n\gg 1$ the factor $\cos[\ ]$ in Eq. (24) oscillates rapidly as z varies and, for substitution into the integrand of Eq. (18), we can replace its square by the average value $\frac{1}{2}$. If $g(z)$ is any smoothly and slowly varying function of z without singularities for $z\gg 1$ we can make the replacement

$$\sum_{n=-\infty}^{\infty} J_n^2(z)g(n)\rightarrow(2\pi)^{-1}\int_0^{2\pi}d\beta\, g(z\cos\beta). \quad (25)$$

We substitute Eq. (18) into Eqs. (17) and (20) and replace the summation over n by the integral in Eq. (25). This gives a three-dimensional integral[12] over u, v, and $\cos\beta$ for $H_e(\omega)$. Using the fact that the product $F^{(1)}(u)F^{(2)}(v)$ is simply the three-dimensional Maxwell distribution function which is spherically symmetrical, a change of coordinate system gives the general relation

$$\int\int dudv\, F^{(1)}(u)F^{(2)}(v)\sum_n J_n^2(kv\cos\phi/\omega_c)$$
$$\times y(ku\sin\phi-n\omega_c\cos\phi)\rightarrow\int dt\, F^{(1)}(t)y(kt), \quad (26)$$

where y is an arbitrary function. With the use of Eq. (26) the expression for $H_e(\omega)$ reduces to Eq. (25), paper I, for the function $G_e(\omega)$. The summation appearing explicitly in Eq. (17) can be reduced in a similar manner leading to

$$\sum_n X_{ni}\exp(-x_{ni}^2)\rightarrow\exp(-\omega^2/\omega_i^2), \quad (27)$$

with a similar expression for the electron quantities. With these substitutions Eq. (17) reduces exactly to Eq. (28) of paper I.

We have thus derived the physically reasonable result that, for fixed nonzero angle ϕ, the limit of the frequency spectrum as $\epsilon\rightarrow 0$ is identical with the spectrum for zero magnetic field, $\epsilon=0$. The case of $\phi\rightarrow 0$, as well as $\epsilon\rightarrow 0$, is more difficult and will be discussed in Sec. 4(e). The difficulty for $\sin\phi\ll 1$ lies in Eq. (25). Although Eq. (24) is a good approximation when $\epsilon\ll 1$, no matter what ϕ is, Eq. (25) holds only when $g(n)\approx g(n+1)$, where $g(n)$ stands for $T_{ne}(\omega)$ or $\exp(-x_{ne}^2)$. This holds only if $\epsilon\ll\sin\phi$, in which case the functions T_{ne} of ω overlap for adjacent values of n.

(d) We consider next the case of $\epsilon_i\ll\sin\phi$ but with ϵ not necessarily small. In this case we can again make the replacement $H_i(\omega)\rightarrow G_i(\omega)$ and use Eq. (27) for the ions, but cannot in general make such replacements for the electrons. For the first term in Eq. (17) we must carry out the summations but the second term can be simplified: According to Eq. (27) this term is most important in the range $\omega\sim\omega_i\sim\omega_e(m/M)^{\frac{1}{2}}$. Since $\epsilon\gtrsim 1$, we have $\omega\ll\omega_c$ and, from Eq. (20), $T_{ne}(\omega)$ can be replaced by unity for $n\neq 0$. Using the identity

$$\sum_{n=-\infty}^{\infty} J_n^2(z)=\sum_{n=-\infty}^{\infty} X_{ne}=1, \quad (28)$$

we derive from Eq. (20)

$$H_e(\omega)\approx-\alpha^2[1-X_{0e}f(x_{0e})+iX_{0e}(\pi)^{\frac{1}{2}}x_{0e}\exp(-x_{0e}^2)], \quad (29)$$

where $x_{0e}=\omega/\omega_e\sin\phi$. Now if, in addition to $\epsilon_i\ll\sin\phi$, we also have $(m/M)^{\frac{1}{2}}\ll\sin\phi$, then $x_{0e}\ll 1$ for the important range of $\omega\sim\omega_i$. In this case we have $H_e(\omega)\approx-\alpha^2$, the same value as the function $G_e(\omega)$ has for $\omega\sim\omega_i\ll\omega_e$. In this case then the second term in Eq. (17) reduces to the second term in Eq. (34) of paper I, the so-called "ion component" (which is the dominant term if $\alpha>1$). This "ion component" is thus unaffected by the magnetic field as long as the double inequality

$$\epsilon_i^2\ll\sin^2\phi\gg m/M \quad (30)$$

holds, even if ϵ is larger than unity.

4. THE LIMIT OF $\phi\rightarrow 0$

In this section we consider the magnetic field, and hence ϵ and ϵ_i [defined in Eq. (3)], as fixed and nonzero and let the angle ϕ approach zero. The formulas derived below require for their approximate validity not merely $\sin\phi\ll 1$ but the more stringent conditions

$$(m/M)^{\frac{1}{2}}\ll\sin\phi\ll(m/M)\epsilon\sim(m/M)^{\frac{1}{2}}\epsilon_i. \quad (31)$$

Our derivation of the general equation, Eq. (17), required that $\gamma\ll\omega_i\sin\phi$, so we let the infinitesimal parameter γ tend to zero first for fixed ϕ and secondly let ϕ tend to zero.[13] In this limit we shall obtain a series of sharp lines. The "central line" at $\omega=0$ has

[10] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1952), Chap. 8.

[11] These statements are incorrect over a narrow range of values of n, $|n-z|\lesssim z^{\frac{1}{3}}$, but this narrow range is found to contribute little to the summations over n and can be neglected.

[12] The three-dimensional integral is the expression for $G_e(\omega)$ in terms of a cylindrical polar coordinate system with arbitrary direction of the axis.

[13] One could also put $\phi=0$ first for fixed γ and second let γ tend to zero. The form of the expression derived in this manner appears to differ from the expression to be derived below. Nevertheless, for all the simple special cases described previously for $\phi=0$ [E. E. Salpeter, J. Geophys. Research (to be published)], the two expressions give identical results and the two expressions are probably equivalent.

somewhat different properties than the others and we discuss it first.

(a) The Central Line

We consider the angle ϕ as nonzero and fixed but extremely small so that Eq. (31) is satisfied. We call the "central line" the intensity obtained from Eq. (17) by carrying only the terms with $n=0$ in the summation which is shown explicitly. These terms are important only for frequencies $\omega \sim \omega_e \sin\phi$ or $\omega_i \sin\phi$, so that [from Eq. (31)] $\omega \ll \omega_{ci}$, ω_{ce}. In Eq. (20) each T_{ni} and T_{ne} with $n\neq 0$ can then be replaced by unity. In this case H_e reduces to the expression in Eq. (29) and H_i reduces to a similar expression. For the first term in Eq. (17) which involves X_{0e} we require the frequency range $\omega \sim \omega_e \sin\phi \gg \omega_i \sin\phi$ and H_i reduces further to $-\alpha^2(1-X_{0i})$. One then finds that this term reduces to a multiple of $\Gamma_{\beta_e}(\omega/\omega_e \sin\phi)$, where $\Gamma_\beta(x)$ is the function defined by Eq. (35) of paper I and

$$\beta_e^2 = X_{0e}\alpha^2[1+Z\alpha^2(1-X_{0i})+\alpha^2(1-X_{0e})]^{-1}.$$

In the limit of $\phi \to 0$ we are interested only in the integral over frequency ω of the intensity $|Q_e(\omega)|^2$ since the frequency line becomes infinitely sharp. From the work of paper I one can show that

$$\int dx\, \Gamma_\beta(x) = (\pi)^{\frac{1}{2}}(1+\beta^2)^{-1} \qquad (32)$$

for any β. By denoting the integrated intensity I_{0e} in units of $\pi N e^2/\gamma$, we find

$$I_{0e} = X_{0e}[1+Z\alpha^2(1-X_{0i})]^2[1+Z\alpha^2(1-X_{0i}) +\alpha^2(1-X_{0e})]^{-2}(1+\beta e^2)^{-1}. \qquad (33)$$

For the second term in Eq. (17) which involves X_{0i} we have $\omega \sim \omega_i \sin\phi \ll \omega_e \sin\phi$ and H_e reduces further to $-\alpha^2$. This term also reduces to a multiple of an expression of the form $\Gamma_{\alpha e}(\omega/\omega_i \sin\phi)$. The integrated intensity I_{0i} of this term (in units of $\pi N e^2/\gamma$) is found to be

$$I_{0i} = X_{0i} Z\alpha^4[1+Z\alpha^2(1-X_{0i})+\alpha^2]^{-1} \times[1+(Z+1)\alpha^2]^{-1}. \qquad (34)$$

If the magnetic field is weak enough so that $\epsilon_i \ll \epsilon \ll 1$ we have $X_{0i} \ll X_{0e} \ll 1$ and the intensity $I_{0i}+I_{0e}$ of the "central line" is only of the order of X_{0e} times the total intensity of the whole spectrum. On the other hand, if the magnetic field is strong enough so that $\epsilon \gg 1$ (even though ϵ_i may be smaller than unity) the "central line" contains the bulk of the total intensity: If $\epsilon \gg 1$ we have $X_{0e} \approx 1$, and making this replacement we obtain from Eqs. (33) and (34)

$$I_{0i}+I_{0e} \approx (1+Z\alpha^2)[1+(Z+1)\alpha^2]^{-1}. \qquad (35)$$

According to Eq. (13) of paper I the right-hand side of Eq. (35) represents the total intensity of the whole spectrum and the error in the equality in Eq. (35) is only of relative order of magnitude $(1-X_{0e}) \gg 1$.

The result that for $\epsilon \gg 1$ most of the intensity resides in the central line may seem somewhat surprising for the case $\alpha \gg 1$ and $\epsilon_i < 1$: In this case the ion's gyroradius is larger than the wavelength and the electron charge density is coupled to that of the ions, and one might have expected a pattern of width ω_i consisting of lines with spacing $\sim \omega_{ci}$. This is not the case, however, under the simplifying approximations we have made throughout this paper. With the propagation direction $\mathbf{k}$ exactly perpendicular to the magnetic field the electrons can move no further than an electron gyroradius R_e in this direction, in our approximation. The net ion charge density is coupled to the electron density to within distances of the order of the Debye length D and the ion charge density is thus confined to within distances of the order of R_e or D, which are both smaller than the wavelength. This confinement of the electrons and ions depends very sensitively on the approximations made, and deviations to be expected under practical circumstances will be discussed in Sec. 6.

(b) The General Limit of $\phi \to 0$

We consider now Eq. (17) for $\omega \neq 0$ as the angle ϕ decreases towards zero. The factor $\exp(-x_{ne}^2)$ is very small unless $|\omega - n\omega_c| \lesssim \omega_e \sin\phi$. However, in this narrow frequency range T_{ne} in Eq. (20), and hence H_e, is extremely large as $\phi \to 0$. Since H_e occurs only in the denominator in the factor multiplying $\exp(-x_{ne}^2)$, this frequency range gives a vanishingly small contribution to the intensity and we assume $|\omega - n\omega_c| \ll \omega_e \times \sin\phi$. The factors $\exp(-x_{ne}^2)$ are then extremely small, but so is the imaginary part of the denominator $(1-H_e-H_i)$ and we shall obtain nonzero intensity only near those frequencies at which the real part of this denominator vanishes. The same arguments apply for the terms involving x_{ni} and we also have $|\omega - n\omega_{ci}| \gg \omega_i \sin\phi$.

Each x_{ne} and x_{ni} approaches infinity as $\phi \to 0$ and the function f in Eq. (20) can be replaced by its asymptotic value of unity. We are interested in those frequencies at which the real part of $(1-H_e-H_i)$, which we denote by Y, vanishes. By using Eq. (28), we find

$$Y(\omega) \equiv 1 - 2\alpha^2 \sum_{n=1}^{\infty} \left[X_{ne} \frac{n^2\omega_c^2}{\omega^2 - n^2\omega_c^2} + ZX_{ni} \frac{n^2\omega_{ci}^2}{\omega^2 - n^2\omega_{ci}^2} \right] = 0. \qquad (36)$$

This dispersion relation has been derived previously by Gross[5] and by Bernstein.[6] Let $\omega = \omega_\nu$ be one of the (real) roots of this dispersion relation. In the numerator of Eq. (17) we can put $H_e = 1 - H_i$ and neglect the infinitesimal part ξ of $(1-H_e-H_i)$. The numerator is then found to be proportional to ξ itself and the intensity spectrum in the vicinity of $\omega = \omega_\nu$ is a Lorentzian-shaped line with maximum at $\omega = \omega_\nu$ and half-width proportional to ξ. In the limit of $\phi \to 0$ the expression ξ approaches zero, the spectral line becomes infinitely sharp and its integrated intensity approaches a limit independent of the value of ξ. By denoting this integrated intensity I_ν in units of $\pi Ne^2/\gamma$, one finds

$$I_\nu = (1-H_i)^2/\alpha^2 Y'\omega_\nu, \tag{37}$$

where $Y' \equiv dY/d\omega$ evaluated at $\omega=\omega_\nu$.

(c) The "Frequency Gaps"

As ω is increased by ω_{ci} from any value, the function $Y(\omega)$ encounters a singularity once and covers the region from $-\infty$ to $+\infty$. Thus a frequency interval of $\approx\omega_{ci}$ contains one root[14] of the dispersion relation. Similarly, there are other roots with a larger spacing of $\approx\omega_c$ and we consider these roots first. We assume $\epsilon \gg (m/M)^{\frac{1}{2}}$ so that these roots occur at frequencies large compared with ω_{ci} and ω_i. In Eq. (36) we omit the term involving X_{ni} entirely and in Eq. (37) we replace H_i by zero.

As discussed previously by Gross and by Bernstein, the dispersion relation exhibits "frequency gaps," i.e., some frequencies near multiples of ω_c are never encountered as roots of the dispersion relation. For instance, consider ϵ as fixed, allow α to vary from zero to infinity, and let $a_\nu(\alpha) \equiv \omega_\nu/\omega_c$. For any value of α there will be one root a_n between n and $n+1$, where n is any positive integer. For $\alpha=0$ we have $a_n=n$ and as α increases so does $a_n(\alpha)$ but a_n approaches a limit, less than $n+1$ by a finite amount, as $\alpha\to\infty$. If $\epsilon\gg1$ we can approximate X_{ne} and the dispersion relation by

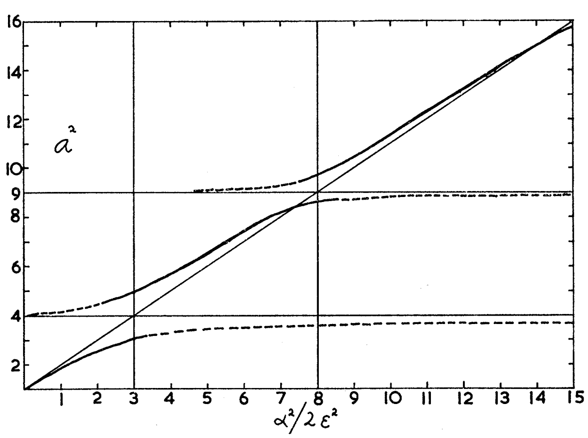

FIG. 1. The roots a^2 of the dispersion relation as a function of $(\alpha^2/2\epsilon^2)$ for $\epsilon=2$. The diagonal line represents $[1+(\alpha^2/2\epsilon^2)]$.

[14] If we do not proceed to the limit $\phi\to0$, but consider $\sin\phi$ as nonzero and small compared with unity, Eq. (36) is replaced by a more complicated expression for $Y(\omega)$ which is unique and has no singularities. Nevertheless $Y(\omega)$ has large negative minima and large positive maxima and the roots of the dispersion relation are very similar to those for $\phi=0$ as long as $\sin\phi\ll1$. The lines are no longer infinitely sharp but the integrated intensities are still approximated closely by Eq. (36).

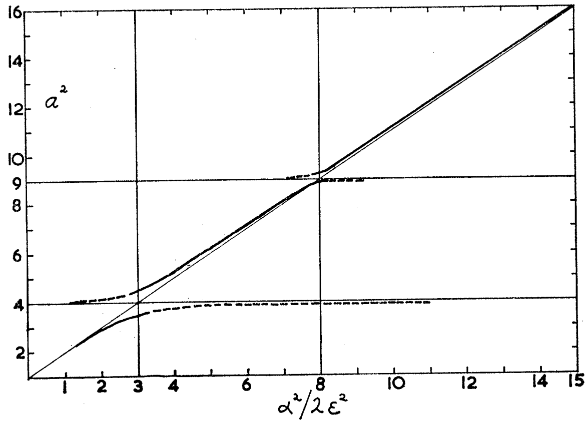

FIG. 2. The roots a^2 of the dispersion relation as a function of $(\alpha^2/2\epsilon^2)$ for $\epsilon=4$.

$$X_{ne}\approx[n!(2\epsilon)^2!]^{-1}[1-(2\epsilon^2)^{-1}], \quad a\equiv\omega/\omega_c,$$

$$Y\equiv 1-\frac{\alpha^2}{2\epsilon^2}\left(1-\frac{1}{2\epsilon^2}\right)\left[\frac{1}{a^2-1}+\frac{1}{2\epsilon^2(a^2-4)}+\frac{3}{32\epsilon^4(a^2-9)}+\cdots\right]=0. \tag{38}$$

For the first few roots, a^2 is plotted against α^2 for $\epsilon=2$ and 4 in Figs. 1 and 2. For $\epsilon\gg1$ the first frequency gap lies between $a^2=4-(3/2\epsilon^2)$ and $a^2=4$, the second gap between $a^2=9-(3/4\epsilon^4)$ and $a^2=9$, etc. If all but one inverse power of ϵ^2 is omitted in Eq. (38), one obtains only a single approximate root

$$a_a^2\approx1+(\alpha^2/2\epsilon^2), \quad \omega_a^2\approx\omega_c^2+\omega_p^2. \tag{39}$$

Of the infinite number of actual roots, there is always one very close to the value given by Eq. (39) if ϵ is very large.

For $\epsilon\gg1$ and a given value of α, the intensity of each of the infinite number of lines can be obtained from Eq. (37) by putting $H_i=0$ and using Eq. (38) for $Y(\omega)$. We assume at first that the approximate expression, Eq. (39), does not accidentally give a value for a_a very close to an integer. Consider first that root of the actual dispersion relation, Eq. (38), with a close to a_a. Since $\epsilon\gg1$, all terms in the series in square brackets for Y in Eq. (38) are small except the first one, and the same is true of the derivative Y'. Carrying only the first term in this series for Y' and using Eqs. (37) and (39), we obtain for the integrated intensity of the two symmetric lines at $\omega=\pm\omega_a$ (in units of $\pi Ne^2/\gamma$),

$$I_a=(\alpha^2+2\epsilon^2)^{-1}. \tag{40}$$

If $\alpha\gg\epsilon$ (in addition to $\epsilon\gg1$), the frequency and intensity of this pair of lines reduces to that of the plasma frequency lines in the absence of a magnetic field. Consider next the line corresponding to any of the other roots of Eq. (38). In this case a is extremely close to some integer n and the term involving $(a^2-n^2)^{-1}$ in the series in Eq. (38) is comparable with the first term. In the equivalent series for Y' the corresponding term involves $(a^2-n^2)^{-2}$ and is very much larger than the first term. From Eq. (37) the intensity of all these other lines is then negligibly small compared with I_a in Eq. (40).

Consider now those special values of α/ϵ for which the solution a_a of Eq. (39) lies very close to an integer n, i.e., a_a lies in (or very near) one of the "frequency gaps." The exact dispersion relation then has two roots for a close to a_a and n, one just below and the other just above the "frequency gap" near n. In these cases $|a-n|$ is sufficiently small so that the term involving $(a^2-n^2)^{-2}$ in the series for Y' is comparable with the first term, whereas the term involving $(a^2-n^2)^{-1}$ in the series for Y is much smaller than the first term. The two lines on either side of the frequency gap then have comparable intensities, each somewhat smaller than I_a in Eq. (40).

Oster[7] has questioned the existence of the "frequency gaps" found by previous workers[5,6] on the grounds of some alleged ambiguities in the derivation when $\phi=0$ and a equals an integer, but the different method of the present paper also leads to these gaps. Furthermore, essentially the same gaps are obtained (although the lines are not infinitely sharp) as long as $\omega_c \sin\phi$ is small compared with the width of a frequency gap—and the intensity function is unique and divergence-free for any nonzero ϕ. Although the "frequency gaps" exist, they do not lead to any discontinuities in the frequency spectrum as α is varied for constant ϵ: Consider the average of frequency over all the lines with the line intensity as weighting function. In general only one line is prominent and the average is close to $\omega_c a_a(\alpha)$. When a_a as a function of α crosses a "frequency gap" the intensity of the line below the gap decreases continuously whereas the intensity of the line above increases. As a consequence the "frequency average" is a continuous function of α and always close to a_a (all for $\epsilon\gg1$).

(d) The "Ion Resonance" for Large ε

In the preceding subsection we have analyzed the dispersion relation, Eq. (36), for frequencies $\omega\gtrsim\omega_c\gg\omega_{ci}$, ω_i. For this frequency range we had omitted the second series involving X_{ni} in Eq. (36) entirely, but now we consider lower frequencies $\omega<\omega_c$ and carry the terms in X_{ni}. We assume at the moment that $\epsilon\gg1$, replace X_{1e} by $(2\epsilon)^{-2}$ according to Eq. (38), and replace X_{ne} by zero for $n>1$. The dispersion relation has an infinite number of roots but, in this frequency range also, all roots but one (or a few) occur at frequencies extremely close to multiples of ω_{ci} (if $\epsilon\gg1$) and have very low intensities. We assume $\alpha\gg1$ and shall find that the roots with appreciable intensity occur for frequencies ω much larger than ω_i and ω_{ci}, no matter what the value of ϵ_i. In the denominator of the term multiplying X_{ni} in Eq. (36) we then replace $(\omega^2-n^2\omega_{ci}^2)$ by ω^2.

For X_{ni} defined in analogy with Eq. (22), one can easily derive the general Bessel function identity

$$\sum_{n=1}^{\infty} n^2X_{ni}=\cos^2\phi/4\epsilon_i^2, \tag{41}$$

which holds for any value of ϵ_i. By using these approximations and Eqs. (3) and (41), the dispersion relation reduces to

$$1-(\alpha^2/2\epsilon^2)[(a^2-1)^{-1}+(m/Ma^2)]=0, \quad a\equiv\omega/\omega_c. \tag{42}$$

With $m\ll M$ the only root for $|a|<1$ occurs for $|a|\ll1$, and replacing (a^2-1) by -1 this root occurs at a frequency given by

$$\omega_r^2=\omega_i^2\frac{\epsilon^2\alpha^2}{\alpha^2+2\epsilon^2}=\frac{M}{m}\omega_{ci}^2\frac{\alpha^2}{\alpha^2+2\epsilon^2}=\frac{m}{M}\frac{\omega_c^2\times\omega_p^2}{\omega_c^2+\omega_p^2}. \tag{43}$$

With the same approximations one easily finds for the intensity I_r of the pair of lines at $\omega=\pm\omega_r$, from Eq. (37),

$$I_r=\alpha^2(\alpha^2+2\epsilon^2)^{-1}(2\epsilon^2)^{-1}. \tag{44}$$

As in the previous subsection our approximation breaks down if ω_r happens to be very close to a multiple of ω_{ci} and two lines would be obtained instead of one. For $\epsilon\gg1$ this happens only over very narrow frequency ranges. Some results for $\epsilon\sim1$ will be discussed in Sec. 4(f).

The physical significance of the resonance lines at $\pm\omega_r$ is seen most easily if $\epsilon\gg\alpha\gg1$, in which case ω_r reduces to $\omega_{pi}=(m/M)^{\frac{1}{2}}\omega_p$, the plasma frequency for ions imbedded in a fixed background of negative charge. This comes about in our approximation because the electrons are confined to distances of about an electron gyroradius in the direction perpendicular to the magnetic field. As for the "central line," the "ion resonance" line at $=\pm\omega_r$ is very sensitive to deviations from our simple approximations, caused by electron collisions, etc. (see Sec. 6).

(e) The Limit of $\varepsilon\rightarrow0$

For any nonzero value of the angle ϕ we have shown in Sec. 2 that the frequency spectrum reduces to the spectrum without a magnetic field in the limit $\epsilon\rightarrow0$, as is to be expected on physical grounds even if we afterwards allow ϕ to tend to zero. A seemingly different limit is obtained if we first put $\phi=0$ and afterwards take the limit $\epsilon\rightarrow0$. As has been pointed out previously,[6] this limiting procedure corresponds to a "set of measure zero" of angles since we require $\sin\phi\ll\epsilon$ for its validity. This case is nevertheless of academic interest since a study of the general dispersion relation alone seems to imply contradictory results for the two limiting procedures: For $\epsilon=0$ the roots are complex and the imaginary (damping) term remains finite even as $\phi \to 0$. For $\phi=0$ the roots are real (infinitely sharp lines) and remain so as $\epsilon \to 0$. For the explicit case of thermal equilibrium discussed in the present paper, we can now show that the same physical frequency spectrum is in fact obtained by the two limiting procedures.

For $\phi=0$ we are dealing with a line spectrum but the spacing of the lines is of order ω_c or ω_{ci} which tends to zero as ϵ does. In this limit the line spectrum is physically indistinguishable from a continuous spectrum and we need the envelope to the line spectrum in this limit, as obtained from Eq. (37). We merely outline the derivation: We need simplified expressions for $Y=1-H_e-H_i$ in Eq. (36), in analogy with Eq. (25) but with $g(n)$ replaced by the singular function $(a-n)^{-1}$. For such a singular function the terms in the summation with n fairly close to a have to be treated separately. Otherwise making similar approximations we find, instead of Eq. (25), in the limit of very large a and z,

$$\sum_{n=-\infty}^{\infty} J_n{}^2(z)(n-a)^{-1} \to (2\pi)^{-1}[R+I \cot\pi a] \times \int_0^{2\pi} d\beta (z \cos\beta - a + i\gamma)^{-1}, \quad (45)$$

where R and I denote the real and imaginary parts of the integral. With H_e given by the first series in Eq. (36) and using Eqs. (18) and (45), one finds for $\epsilon \to 0$

$$H_e \to G_e{}^{(R)} + G_e{}^{(I)} \cot\pi a, \quad a=\omega/\omega_c, \quad (46)$$

where G_e is the function of ω defined in paper I Eq. (25), and (R), (I) denote real and imaginary part. One obtains an equivalent expression for H_i. As $\epsilon \to 0$ we also have $\omega_c \to 0$ and in evaluating the derivative Y' we need only carry the derivative of $\cot\pi a$. On substituting into Eq. (37) one finds, after some algebra, that the intensity I_ν of the line spectrum (after smoothing over the infinitesimally small frequency range between lines) reduces to the intensity of the continuous spectrum in the absence of a magnetic field, which was described in paper I.

(f) The "Ion Resonance" for Small ϵ_i

In Sec. 4(d) we found a single pair of "ion resonance" lines at frequency $\pm\omega_r$, given by Eq. (43), if $\epsilon \gg 1$. In Sec. 4(e) we found a closely spaced line spectrum whose envelope is the "ion component" of the field-free continuous spectrum, if $\epsilon < 1$. The intermediate region of $(m/M)^{\frac{1}{2}} \ll \epsilon \ll (M/m)^{\frac{1}{2}}$ can be investigated as follows: We consider only $\omega \ll \omega_c$ (but disregard the central line $\omega=0$) and can replace H_e by $-\alpha^2(1-X_{0e})$. Since $\epsilon_i \ll 1$ we replace H_i by the expression equivalent to Eq. (46). Substituting into Eq. (37) then gives as envelope to the line spectrum a multiple of the function $\Gamma_\beta(\omega/\omega_i)$, defined in Eq. (35) of paper I, with

$$\beta^2 = Z\alpha^2[1+\alpha^2(1-X_{0e})]^{-1}. \quad (47)$$

By using Eq. (32), one finds for the total integrated intensity I_i of this part of the spectrum

$$I_i = \beta^4(1-X_{0e})^2(\beta^2+1)^{-1}. \quad (48)$$

If $\epsilon \ll 1$ then $X_{0e} \ll 1$ and, if we omit X_{0e} altogether in Eqs. (47) and (48), we have again the field-free "ion component" which is almost a Gaussian curve if $\alpha \gg 1$ and an almost flat-topped broad curve if $\alpha \gg 1$. If $\epsilon \sim 1$ then $1-X_{0e}$ is somewhat, but not very much, smaller than unity and β is slightly larger than for the field-free case. If ϵ is appreciably larger than unity, then $1-X_{0e}$ can be approximated by $(2\epsilon^2)^{-1}$ and β is then appreciably larger than unity if $\alpha \gg 1$. In this case Γ_β is a fairly sharply peaked function with small width and the maxima occur at frequencies near $\pm\omega_r$. If ϵ (as well as α) is very much larger than unity then also $\beta \gg 1$. The width of Γ_β is then much less still than the spacing ω_{ci} of the line spectrum and we have a single pair of lines, as found in Sec. 4(d).

5. SOME CASES FOR NONZERO ϕ

We shall consider only some cases for nonzero values of the angle ϕ. We first of all restrict ourselves to such values of α and of the magnetic field so that

$$\alpha \gg 1; \quad (m/M)^{\frac{1}{2}} \ll \epsilon_i \gtrsim 1, \quad (49)$$

from which $\epsilon \gg 1$ follows. In Sec. 4 we have seen that the spectrum is essentially that for $\phi=0$ if the inequality in Eq. (31) holds. One can also show that the less stringent single inequality $\sin\phi \ll (m/M)^{\frac{1}{2}}$ is in fact sufficient. Some aspects of the spectrum change radically when this inequality is violated. This change is most marked for the "ion component" and we discuss this first.

(a) The "Normal Ion Component"

We now return to the general expression, Eq. (17), and denote by "normal ion component" the spectrum obtained from the second series of terms (involving X_{ni}) in Eq. (17) in the restricted frequency range $|\omega| \lesssim \omega_i \gtrsim \omega_{ci}$. Consider first the term with $n=0$ which contributes to the "central line" discussed in Sec. 4(a). It can be shown easily that Eq. (34) is a good approximation to the intensity I_{0i} as long as $\sin\phi \ll \epsilon_i$ even if $\sin\phi$ is not smaller than $(m/M)^{\frac{1}{2}}$. For $Z=1$, $\alpha \gg 1$ this expression reduces to $I_{0i}=[X_{0i}/2(2-X_{0i})]$ compared with a total intensity of $\frac{1}{2}$ for the whole spectrum.

Consider next the terms involving X_{ni} with $n \neq 0$ in this frequency range and assume again $\sin\phi \ll \epsilon_i$. In this region of ω and ϕ the function H_i does not depend very sensitively on the relation between ϕ and $(m/M)^{\frac{1}{2}}$, which is much less than ϵ_i. However, the function H_e is given by Eq. (29) which depends quite strongly on the value of x_{0e}. If $\sin\phi \ll (m/M)^{\frac{1}{2}}$ we have $x_{0e} \gg 1$ over the frequency range in question, $-H_e$ is much smaller than α^2 and there is very little intensity in the frequency

TABLE I. Frequency ω_ν for the first few lines in the "normal ion spectrum," in units of ω_{ci}.

ϵ_i \ ν	1	2	3	4	5
0.5	1.15	2.18	3.10	4.03	5.01
1	1.14	2.04	3.01		

range $|\omega|\lesssim\omega_i$ except for the central line. However, if $\sin\phi\gg(m/M)^{\frac{1}{2}}$ we have $x_{0e}\ll1$ and H_e can be replaced by $-\alpha^2$.

Since $\sin\phi\ll\epsilon_i$ the imaginary part of H_i is small and we get a series of fairly sharp lines and methods similar to those of Sec. 4(b) can be used. However, with $H_e=-\alpha^2$ the dispersion relation is given by

$$Y(\omega)=1-\alpha^2[1-2Z\sum_{n=1}^{\infty}X_{ni}n^2\omega_{ci}^2(\omega^2-n^2\omega_{ci}^2)^{-1}]=0, \quad (50)$$

instead of Eq. (36). If the double inequality

$$(m/M)^{\frac{1}{2}}\ll\sin\phi\ll\epsilon_i\sim(m/M)^{\frac{1}{2}}\epsilon \quad (51)$$

holds, the frequency of the centers of the fairly sharp lines in the frequency region $|\omega|\lesssim\omega_i$ is then given by the roots of this dispersion relation. The integrated intensity of each line is again given by Eq. (37).

In Table I we give the values of ω_ν for the first few lines for $\epsilon_i=0.5$ and 1 with $\alpha\gg1$ and $Z=1$. As ν increases the value of ω_ν approaches $\nu\omega_{ci}$ more and more closely. In Table II we give the intensity I_ν of the line at $+\omega_\nu$ (the line at $-\omega_\nu$ has the same intensity) together with the intensity $I_{0i}=\frac{1}{2}X_{0i}(2-X_{0i})^{-1}$ for the central line. For all values of ϵ_i does the intensity fall off very rapidly as ω_ν becomes much larger than both ω_i and ω_{ci}. In our approximation of $\alpha\gg1$ the sum of the intensities of these lines (including those at $-\omega_\nu$) equals $\frac{1}{2}$, the total intensity of the whole spectrum. If $\epsilon_i\gg1$ most of the intensity is in the central line. For $\epsilon_i=0.5$ the intensity envelope to the line spectrum is almost flat-topped, and for ϵ_i appreciably less than 0.5 we have many lines with spacing very close to ω_{ci} and with intensity envelope very close to the field-free ion component of the spectrum, as was shown in Sec. 3(d).

The line spectrum discussed above is almost independent of ϕ as long as it satisfies the double inequality in Eq. (51) (which is possible only when $\epsilon\gg1$). As $\sin\phi$ approaches ϵ_i the lines broaden and begin to overlap, and for $\sin\phi\gg\epsilon_i$ (if $\epsilon_i\ll1$) the spectrum is continuous and identical with the field-free spectrum. When $\sin\phi\sim(m/M)^{\frac{1}{2}}$ the function $-H_e$ is somewhat less than α^2

TABLE II. Intensities in the "normal ion spectrum" (multiplied by 2).

ϵ_i	I_{0i}	I_{1i}	I_{2i}	I_{3i}	I_{4i}	I_{5i}
0.5	0.183	0.169	0.145	0.075	0.015	0.003
1	0.475	0.213	0.045	0.004		

and the intensity of each line, other than the central one, is smaller than in the region of Eq. (51). For $\sin\phi\ll(m/M)^{\frac{1}{2}}$ only the central line survives and the value of I_{0i} is unchanged. It should be remembered that there is an additional contribution I_{0e} to the central line intensity, discussed in Sec. 4(a). For $\sin\phi\ll(m/M)^{\frac{1}{2}}$ this intensity is given by Eq. (33), but this intensity decreases as $\sin\phi$ increases and is negligible if $\sin\phi\gg(m/M)^{\frac{1}{2}}$.

The physical reason for this dependence on the ratio $\sin\phi/(m/M)^{\frac{1}{2}}$ is as follows. The effective component of the thermal drift velocity of the electrons in the direction of the propagation vector is of order $\omega_e\sin\phi$ since the electron's gyroradius is much smaller than the wavelength. On the other hand, the ion's gyroradius is comparable with or larger than the wavelength, so that the effective thermal drift velocity of the ions is of order $\omega_i\sim\omega_e(m/M)^{\frac{1}{2}}$. For the spectrum at low frequencies (and for the "resonance lines" discussed in the next subsection) it is important which of these two effective drift velocities is the larger.

(b) The Plasma Oscillation and "Resonance" Line

In the preceding subsection we have seen that the main part of the spectrum is concentrated into a frequency range $|\omega|\lesssim\omega_i$ if both α and ϵ are large compared with unity. However, there are also some components of the spectrum at frequencies much larger than ω_i whose total intensity is small but which may still be of interest since they occur in the form of a few sharp lines. We now consider this part of the spectrum for arbitrary values of the angle ϕ.

The expression for H_i in Eq. (20) can be simplified if $|\omega|\gg\omega_i$. The values of X_{ni} are negligibly small unless $n\omega_{ci}<\omega_i$ and we can expand in powers of the small quantity $n\omega_{ci}/\omega$. We then also have $x_{ni}\gg1$ for the important range of values of n, the function $f(x_{ni})$ can be replaced by $[1-(2x_{ni}^2)^{-1}]$ and the imaginary part of H_i is very small. By keeping only the lowest order nonvanishing terms in the expansion in powers of $n\omega_{ci}/\omega$ and using the Bessel function identities, Eqs. (28) and (41), we find for the real part of H_i

$$H_i^{(R)}=Z\alpha^2\omega_i^2/2\omega^2 \quad (52)$$

independent of the value of ϕ.

Since $\epsilon\gg1$ we can replace X_{0e} by $[1-(\cos^2\phi/2\epsilon^2)]$ and X_{1e} by $\cos^2\phi/4\epsilon^2$, according to Eq. (38), and omit X_{ne} with $n>1$ altogether in Eq. (20). We shall find that we are interested only in frequencies ω such that both ω and $|\omega|-\omega_c$ are much larger than $\omega_e\sin\phi$. This is the case for any value of ϕ as long as $\alpha\gg1$ and $\epsilon\gg1$. We can then replace $f(x_{0e})$ by $[1-(2x_{0e}^2)^{-1}]$ and $f(x_{1e})$ by unity. The imaginary part of H_e is then very small, for any angle ϕ, and its real part is given by

$$H_e^{(R)}=(\alpha^2/2\epsilon^2)(a^2-\sin^2\phi)/a^2(a^2-1), \quad (53)$$

where $a=\omega/\omega_c$.

Since the imaginary parts of H_e and H_i are both small we obtain very narrow lines at frequencies given by the roots of a real dispersion relation. In our approximation the dispersion relation has only two roots for a^2, which are given (after dropping a small term in a^2m/M) by

$$Y(\omega)\equiv 1-\frac{\alpha^2}{2\epsilon^2}\left[\frac{\cos^2\phi}{a^2-1}+\frac{(m/M)+\sin^2\phi}{a^2}\right]=0, \quad a=\omega/\omega_c;$$

$$2a^2=\left(1+\frac{\alpha^2}{2\epsilon^2}\right) \pm\left[\left(1+\frac{\alpha^2}{2\epsilon^2}\right)-2\frac{\alpha^2}{\epsilon^2}\left(\frac{m}{M}+\sin^2\phi\right)\right]^{\frac{1}{2}}. \quad (54)$$

The intensity of each of the two pairs of lines is given by Eq. (37) with the expression in Eq. (54) for $Y(\omega)$.

Of the two solutions for a^2, one is always larger than unity and one smaller than unity. If ϕ is close to $\pi/2$, one root is close to $a^2=\alpha^2/2\epsilon^2$, i.e., $\omega=\pm\omega_p$, the plasma frequency in the absence of a magnetic field. In this case the other root is close to $a^2=1$, Y' in Eq. (37) is very large, and the corresponding intensity is very small. If either $\sin\phi\ll 1$ or $\epsilon\ll\alpha$ or both, the square root in Eq. (54) can be expanded and the two roots simplify. One root is then given by

$$\omega^2=\omega_p^2+\omega_c^2\cos^2\phi+\sin^2\phi\,\omega_c^4/(\omega_p^2+\omega_c^2), \quad (55)$$

which corresponds to a plasma oscillation in the presence of a magnetic field. For $\sin\phi\ll 1$ the frequency and intensity of this pair of lines reduce to the expressions given in Eqs. (39) and (40).

The other root is given by

$$\omega_r^2=[(m/M)+\sin^2\phi]\omega_c^2\omega_p^2/(\omega_c^2+\omega_p^2), \quad (56)$$

and its intensity by

$$I_r=\frac{\alpha^2}{2\epsilon^2(\alpha^2+2\epsilon^2)}\left[1-\frac{\sin^2\phi}{\sin^2\phi+(m/M)}\left(1+\frac{2\epsilon^2}{\alpha^2}\right)\right]^2. \quad (57)$$

Unlike the plasma oscillation line, this "resonance line" changes its character depending on whether $\sin^2\phi$ is larger or smaller than m/M, for the physical reasons discussed in the preceding subsection. If $\sin^2\phi\ll m/M$ the results reduce to those of Sec. 4(d). If $(m/M)\ll\sin^2\phi\ll 1$ the resonance frequency in Eq. (56) becomes independent of the ion mass M and the intensity I_r is given approximately by

$$I_r=2\epsilon^2/\alpha^2(\alpha^2+2\epsilon^2). \quad (58)$$

6. DISCUSSION

The cases likely to be encountered in the near future in ionospheric radar backscatter fall into one of two categories. For the Lincoln Laboratories and the proposed Puerto Rico experiments with rather high radar frequencies the angle ϕ is not extremely small, the parameter ϵ is slightly larger than unity ($\epsilon_i\ll 1$), and the parameter α somewhat larger than ϵ. In such cases the frequency spectrum to be expected is relatively straightforward: The main part of the spectrum is essentially the same as the "ion component" in the absence of a magnetic field. The fairly weak "plasma oscillation" lines occur not at $\pm\omega_p$ but at the slightly modified frequencies given by Eq. (55). In addition we have a fairly weak pair of "resonance lines" at frequencies $\pm\omega_r$ given by Eq. (57) with intensity given by Eq. (58). This pair of lines is absent in the absence of a magnetic field.

Another class of proposed experiments deals with very small values of the angle ϕ, i.e., radar beam almost perpendicular to the magnetic field, and uses low enough radar frequencies so that α and ϵ are both very much larger than unity and ϵ_i only slightly less than unity. In this case the "plasma oscillation" lines and the "resonance lines" discussed in Secs. 4(d) and 5(b) are very weak. If $\sin^2\phi$, although much smaller than unity, is larger than the mass ratio m/M, the bulk of the spectrum is as discussed in Sec. 5(a). For $\phi=0$ the bulk of the spectrum would reside in the "central line" discussed in Sec. 4(a), if all the approximations made in this paper were strictly valid. In practice, however, the following complications may arise.

In ionospheric radar backscatter experiments we are dealing with a wave-pulse of finite length and width and thus a finite volume V for which the radar signal measures the Fourier components of the electron charge density fluctuations. There is then a certain spread, not only in the absolute value but also the direction of the propagation vector **k**. This results in a certain spread in the value of $\sin\phi$ which is very small compared to unity (since very many wavelengths are contained in the volume V), but not necessarily smaller than $(m/M)^{\frac{1}{2}}$.

Even if the spread in the angle ϕ is negligibly small, we have to consider the effect of electron collisions which we have neglected throughout this paper. We are considering cases where the collision mean free path l is large compared with all the linear dimensions such as wavelength, Debye length, and ion gyroradius R_i. For most components of the spectrum [including the discussion of the "frequency gaps" in Sec. 4(c)] this is sufficient to justify the complete neglect of collisions, but our treatment of the "central line" [and the "resonance lines" in Secs. 4(d) and 5(b)] requires more stringent inequalities: Our treatment for $\phi=0$ requires that the effective mobility of the electrons in the propagation direction **k** be less than the thermal velocity of the ions. After each collision an electron can move its position along the **k** direction by a distance of the order of an electron gyroradius R_e. By a random walk process, the collisions then enable an electron to drift along the **k** direction even though this direction is perpendicular to the magnetic field. In the absence of neutral atoms the collision mean free path l is of order $D\Lambda/\ln\Lambda$, where $\Lambda = nD^3$. We are assuming throughout that $\Lambda \gg 1$, but for the electron mobility to be less than the ion mobility we require the more stringent inequality

$$(\epsilon^2/\alpha)\Lambda/\ln\Lambda \gg (M/m)^{\frac{1}{2}}. \tag{59}$$

If this inequality is not satisfied, the results are more complicated than the results for $\phi=0$ derived in this paper but should be qualitatively similar to our results for $(m/M) \ll \sin^2\phi \ll 1$.

Because $\Lambda \gg 1$ we have used the random phase approximation throughout this paper and neglected any coupling with Fourier components of the density fluctuations with propagation vectors different from the constant $\mathbf{k}$. This coupling is indeed weak but does produce weak electric fields in directions other than $\mathbf{k}$ which can, in the presence of the magnetic field, cause a slow drift of the electrons in the $\mathbf{k}$ direction. This slow electron drift will also affect the "central line" unless Λ is sufficiently large for the inequality in Eq. (59) to hold.

If the spread of $\sin\phi$ around zero is sufficiently small and if Λ is sufficiently large, most of the frequency spectrum is contained in the very sharp "central line." This implies that both the electron and ion density fluctuations in a direction perpendicular to the magnetic field cannot move through distances much larger than the Debye length D or the (small) electron gyroradius R_e. It should be remembered that we are dealing with density fluctuations with a rather special geometry, which would not necessarily apply to macroscopic density variations in the magnetic containment problem, say: The setup of a radar backscatter experiment selects out of all possible density fluctuations only sinusoidal ones, with a definite propagation vector $\mathbf{k}$, which extend over very many wavelengths (the width of the radar beam) in directions perpendicular to $\mathbf{k}$. "End effects," which may be of importance in the magnetic containment of macroscopic density variations with more complicated geometry, can be neglected in our case.

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