Stark Broadening of Hydrogenic Ion Lines in a Plasma*

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The frequency distributions of the ionized helium lines at 4686 A and 3203 A broadened by the local fields of both ions and electrons in a plasma are calculated in the classical path approximation, which is shown to be always applicable. General formulas are developed for the Stark profiles of lines from multiply ionized hydrogenic systems, and the validity domains of the impact and quasi-static approximation for electron and ion broadening are delineated. The results are compared with the Holtsmark theory and an approximation for high series members.

INTRODUCTION

IN a series of papers the Stark broadening by local fields from ions and electrons in a plasma has been calculated for the most important hydrogen[1] and neutral helium[2] lines and for high series members of hydrogenlike spectra[3] using a generalized impact theory[4,5] to describe the electron effects. In these calculations the electrons were assumed to follow straight classical paths, but the usual adiabatic approximation was not made. Furthermore, except for $H\alpha$, the electron broadening of the lower states of the various hydrogen lines was neglected.

The latter approximation will not be made here,[6] and therefore the impact broadening of lines with highly excited upper and lower states like that of the strong HeII line at 4686 A can now be obtained with equal accuracy. New calculations for the $H\beta$ line of hydrogen give only slightly different profiles, which further justifies neglecting the impact broadening of the lower state for the higher members of the Balmer series in reference 1.

Instead of the straight classical path, hyperbolic trajectories must be used to calculate the impact broadening of ion lines. This will be shown to yield practically the same results for hydrogenic lines as the straight classical path calculations, thus confirming the approximate formulas for the impact broadening of higher hydrogenic lines given in reference 3.

The classical path approximation turns out to be valid even for lines from high-z hydrogenic systems, i.e., a completely quantum mechanical calculation yields the same answer, if carried through with the corresponding approximations. It will be shown that this is also the case for most nonhydrogenic lines.

At high densities the broadening by ions can always be calculated with the quasi-static approximation, but at lower densities a transition into the impact domain may occur before the Doppler broadening becomes significant. However, the two extreme approximations for the Stark broadening overlap smoothly, and therefore the uncertainties in the region, where none of them is applicable, are not very large.

Correlations of ions contributing to the quasi-static broadening, shielding of their fields by the electrons, and interactions between the emitting and broadening ions will be taken into account in the detailed calculations for ionized helium lines following Baranger and Mozer,[7,8] and the relative importance of these effects for higher z lines will be discussed.

For the electron broadening the impact approximation is valid except at very high densities. But impact and quasi-static theory give almost the same answer in the transition region, and again the over-all accuracy of the profiles should not be seriously impaired by this difficulty.

IMPACT BROADENING

If the impact approximation is valid, i.e., at high perturber velocities and low densities, the profile of a line arising from transitions between any of the sublevels α', α'', $\cdots$ of a group a of closely spaced levels to any of the sublevels β', β'' of another group b can be expressed in terms of the matrix elements of the operator (the * means complex, not Hermitian, conjugate)

$$\phi_{ab} \equiv \sum_j P_j [S_{aj}(0) S_{bj}^*(0) - 1] \qquad (1)$$

[see, e.g., Eq. (7) of reference 1, where $S_{aj}(0)$ was called T_{aj}]. Here P_j is the frequency of collisions of

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[6] This approximation was already removed by B. Mozer, Ph.D. thesis, Carnegie Institute of Technology, 1960 (unpublished).

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[8] B. Mozer and M. Baranger, Phys. Rev. **118**, 626 (1960).

type j; the $S_{aj}(0)$ and $S_{bj}(0)$ are the S matrices for collisions taking place at time $t=0$ with the atom or ion being in the a or b state, respectively. With the classical path approximation and the usual perturbation expansion, (1) becomes to the first nonvanishing order[9]

$$\begin{aligned}\phi_{ab}&=2\pi vN\int d\rho\,\rho\{S_a(0)S_b^*(0)-1\}\\&=2\pi vN\int d\rho\,\rho\Big\{\Big(\frac{i}{\hbar}\Big)^2\Big[\int_{-\infty}^{+\infty}dt\,e^{iH_at/\hbar}V_a(t)e^{-iH_at/\hbar}\\&\quad\times\int_{-\infty}^{t}dt'\,e^{iH_at'/\hbar}V_a(t')e^{-iH_at'/\hbar}\\&\quad-\int_{-\infty}^{+\infty}dt\,e^{iH_at/\hbar}V_a(t)e^{-iH_at/\hbar}\\&\quad\times\int_{-\infty}^{+\infty}dt\,e^{-iH_bt/\hbar}V_b^*(t)e^{iH_bt/\hbar}\\&\quad+\int_{-\infty}^{+\infty}dt\,e^{-iH_bt/\hbar}V_b^*(t)e^{iH_bt/\hbar}\\&\quad\times\int_{-\infty}^{t}dt'\,e^{-iH_bt'/\hbar}V_b^*(t')e^{iH_bt'/\hbar}\Big]\Big\},\end{aligned}\tag{2}$$

where v, N, and ρ are the perturber velocity, density, and impact parameter, $V_{a,b}$ the perturbation, and $H_{a,b}$ the unperturbed Hamiltonian. The perturbation Hamiltonian is, for distant collisions, given by the dipole term

$$V_{a,b}(t)=\pm e^2\mathbf{r}_{a,b}\cdot\mathbf{r}(t)\,|\mathbf{r}(t)|^{-3},\tag{3}$$

$e\mathbf{r}_{a,b}$ being the dipole moment of the emitting atom or ion and $\mathbf{r}(t)$ the radius vector of the perturber, which is assumed to be singly charged. Since $V_{a,b}(t)$ is an odd function of the perturber coordinates, only even terms in the perturbation expansion contribute to the average over angles in (2), which is indicated by $\{\ \}$.

If all perturbers can be treated by the impact approximation, the exponentials in (2) are equal to 1, because in hydrogenic systems the a and b levels are completely degenerate. But also in cases where the ion broadening is quasi-static and thus removes the degeneracy, the exponentials may be replaced by 1, if for electrons the impact theory is valid. This follows from the usual validity criterion for the impact approximation $|\langle\alpha'|\phi|\alpha''\rangle|\rho/v\ll1$, if one realizes that the electron impact broadening is only important for conditions where the ϕ-matrix elements are of the same order as the matrix elements of $H_{a,b}/\hbar$, which include the quasi-static perturbation and therefore determine the quasi-static shifts from the ion fields, and also occur in the exponents.

To calculate the time integrals in (2), the equation of motion of the perturber (mass m'; charge e) in the field of the emitting ion with net charge $(z-1)e$ must be introduced:

$$m'd^2\mathbf{r}(t)/dt^2=\pm(z-1)e^2\mathbf{r}(t)\,|\mathbf{r}(t)|^{-3}.\tag{4}$$

(The principal ions contributing to the impact broadening are hydrogen ions at extremely high temperatures, where only high-z line radiation exists. Therefore, one can usually neglect the motion of the emitting ion.) With (3) and (4) and omitting the exponentials and higher order terms, (2) can finally be written as

$$\begin{aligned}\phi_{ab}=2\pi vN\int d\rho\,\rho\Big\{-\Big(\frac{m'\hbar}{e^2mz(z-1)}\Big)^2&\\\times\Big(\int_{-\infty}^{+\infty}dt\,\mathbf{R}_a\cdot\frac{d^2\mathbf{r}(t)}{dt^2}\int_{-\infty}^{t}dt'\,\mathbf{R}_a\cdot\frac{d^2\mathbf{r}(t')}{dt'^2}&\\-\int_{-\infty}^{+\infty}dt\,\mathbf{R}_a\cdot\frac{d^2\mathbf{r}(t)}{dt^2}\int_{-\infty}^{\infty}dt'\,\mathbf{R}_b\cdot\frac{d^2\mathbf{r}(t')}{dt'^2}&\\+\int_{-\infty}^{+\infty}dt\,\mathbf{R}_b\cdot\frac{d^2\mathbf{r}(t)}{dt^2}\int_{-\infty}^{t}dt'\,\mathbf{R}_b\cdot\frac{d^2\mathbf{r}(t')}{dt'^2}\Big)\Big\},\end{aligned}\tag{5}$$

where the matrix elements of $\mathbf{R}_{a,b}$ are now those for hydrogen in atomic units. The time integrals yield

$$\begin{aligned}\{\quad\}=-\frac{2}{3}\Big[\frac{m'\hbar v}{e^2mz(z-1)}\sin(\tfrac{1}{2}\alpha)\Big]^2&\\\times(\mathbf{R}_a\cdot\mathbf{R}_a-2\mathbf{R}_a\cdot\mathbf{R}_b+\mathbf{R}_b\cdot\mathbf{R}_b),&\end{aligned}\tag{6}$$

if one uses

$$d\mathbf{r}/dt(-\infty)=\mathbf{v},\ d\mathbf{r}/dt(+\infty)=\mathbf{v}\cos\alpha+(v/\rho)\boldsymbol{\varrho}\sin\alpha,$$
$$\{v_\sigma\rho_\nu\}=0,\ \{v_\sigma v_\nu\}=\tfrac{1}{3}\delta_{\sigma\nu}v^2,\text{ and }\{\rho_\sigma\rho_\nu\}=\tfrac{1}{3}\delta_{\sigma\nu}\rho^2.$$

The ρ integral in (5) can be carried through between a maximum and minimum impact parameter with the relation between impact parameter and scattering angle,

$$\rho=\frac{(z-1)e^2}{m'v^2}\frac{\cos(\frac{1}{2}\alpha)}{\sin(\frac{1}{2}\alpha)},\tag{7}$$

i.e., finally

$$\begin{aligned}\phi_{ab}=-\frac{4\pi}{3v}N\Big(\frac{\hbar}{mz}\Big)^2\ln\Big(\frac{\sin(\frac{1}{2}\alpha_{\max})}{\sin(\frac{1}{2}\alpha_{\min})}\Big)&\\\times(\mathbf{R}_a\cdot\mathbf{R}_a-2\mathbf{R}_a\cdot\mathbf{R}_b+\mathbf{R}_b\cdot\mathbf{R}_b).&\end{aligned}\tag{8}$$

[9] In an earlier version of this paper the cross term in a, b was erroneously left out, as was pointed out to the authors by M. Baranger. This term had already been considered in reference 6. Its importance is minimized if, as in the present calculations, parabolic wave functions are used. In that case it affects even the line profile of the HeII 4686 A line only by ~5%, i.e., only to a negligible extent.

The smallest scattering angle will correspond to an impact parameter of the order of the Debye radius ρ_D or the mean distance $r_0=[(4\pi/3)N]^{-\frac{1}{3}}$, whichever is larger. (For $r_0>\rho_D$ the Debye theory is invalid, and by cutting off at r_0 one then only counts the closest of several simultaneous impacts.) Equation (7) yields accordingly

$$\sin(\tfrac{1}{2}\alpha_{\min})\approx\frac{(z-1)e^2}{m'v^2\rho_{\max}}\left[1+\left(\frac{(z-1)e^2}{m'v^2\rho_{\max}}\right)^2\right]^{-\frac{1}{2}}. \quad (9)$$

If the perturbation theory breaks down for large angles, $\alpha_{\max}$ must be chosen in such a way that the absolute value of the first contributing term in the expansion for the matrix elements of $S_{aj}(0)S_{bj}^*(0)$ does not exceed 1, which gives with (6) and using $(a)^4$ for a typical matrix element of $\mathbf{R}_a\cdot\mathbf{R}_a-2\mathbf{R}_a\cdot\mathbf{R}_b+\mathbf{R}_b\cdot\mathbf{R}_b$

$$\sin(\tfrac{1}{2}\alpha_{\max})\approx\left(\frac{3}{2}\right)^{\frac{1}{2}}\frac{e^2mz(z-1)}{m'\hbar va^2}. \quad (10a)$$

In situations where the classical path approximation breaks down first, impact parameters smaller than the de Broglie wavelength must be excluded (see the Appendix for further discussion), and one should use instead of (10a)

$$\sin(\tfrac{1}{2}\alpha_{\max})\approx\left[1+\left(\frac{\hbar v}{(z-1)e^2}\right)^2\right]^{-\frac{1}{2}}\approx 1. \quad (10b)$$

This follows from (7) with the requirement $m'v\rho_{\min}\approx\hbar$, if one recognizes that typical electron velocities are $v=3\times10^7 z$ [cm sec^{-1}], because electron temperatures have to be of the order of z^2 [ev] for sufficient ionization and excitation. But $\sin(\frac{1}{2}\alpha_{\max})=1$ is obviously identical with the limit of the classical path theory, and for $z\gtrsim 2$ one can accordingly use the latter to calculate the weak collision contribution to the impact broadening of hydrogenic lines. The same conclusion had already been reached for hydrogen lines[1] (a quantum mechanical calculation for this case is carried through in the Appendix). It also holds for most nonhydrogenic lines, because for small impact parameters (corresponding to short durations of the perturbations), where one might expect the classical path approximation to break down, the hydrogenic formulas are usually applicable, since the exponentials in (2) can then almost always be neglected. In all these cases only the strong collision contribution ($\rho<\rho_{\min}$) may have to be calculated by a completely quantum mechanical theory. This contribution is of the order $\pi vN\rho_{\min}^2$ with $\rho_{\min}$ corresponding to $\alpha_{\max}$. It is negligible for hydrogenic lines if $\ln[\sin(\frac{1}{2}\alpha_{\max})/\sin(\frac{1}{2}\alpha_{\min})]$ is large compared to 1 in cases where $\alpha_{\max}<\pi$ and zero for $\alpha_{\max}=\pi$.

That in the impact domain the logarithm will usually be large compared to 1 in the first case can be seen from (9) and (10a) which yield for small $\alpha_{\min}$ (low densities)

$$\frac{\sin(\frac{1}{2}\alpha_{\max})}{\sin(\frac{1}{2}\alpha_{\min})}\approx\left(\frac{3}{2}\right)^{\frac{1}{2}}\frac{mv\rho_{\max}z}{\hbar a^2}\approx\frac{1}{\Delta\omega}\frac{v}{\rho_{\max}}\approx\frac{1}{\Delta\omega\Delta t}\gg 1.$$

($\Delta\omega\approx\hbar a^2/(m\rho_{\max}{}^2z)$ is a typical value of the quasistatic splitting and $\Delta t\approx\rho_{\max}/v$ is characteristic for the duration of a collision, and finally $\Delta\omega\Delta t$ is much smaller than 1 in the impact domain).

The only approximation whose validity remains to be checked is the use of the dipole interaction in (3). It will be adequate if only values of $|\mathbf{r}(t)|$ larger than $a^2\hbar^2/(zme^2)$, the Bohr radius of the upper state, contribute significantly. This was shown to be the case for hydrogen[1] ($z=1$). For $z>1$ the distance of closest approach for electrons is shorter because of the Coulomb attraction. This effect is, however, compensated by the fact that the electrons will for the same reason only spend a shorter time near the perturbed ion for any given impact parameter. Accordingly this approximation is also expected to be valid for the evaluation of the ϕ operator for high-z hydrogenic systems [obviously (3) is a very good approximation for impact broadening ions]. That the choice of the dipole interaction does not introduce serious errors was already suggested by the fact that no cutoff was necessary in situations where (10b) applies.

From (8), (9), and (10a) or (10b) follows finally for the ϕ operator which describes the impact broadening of hydrogenic lines for $(\frac{3}{2})^{\frac{1}{2}}e^2mz(z-1)/(m'\hbar va^2)<1$:

$$\phi_{ab}=-\frac{4\pi}{3v}N\left(\frac{\hbar}{mz}\right)^2(\mathbf{R}_a\cdot\mathbf{R}_a-2\mathbf{R}_a\cdot\mathbf{R}_b+\mathbf{R}_b\cdot\mathbf{R}_b)\times\left\{\frac{1}{2}\left[1-\left((\tfrac{3}{2})^{\frac{1}{2}}\frac{e^2mz(z-1)}{m'\hbar va^2}\right)^2\right]+\ln\left((\tfrac{3}{2})^{\frac{1}{2}}\frac{mv\rho_{\max}z}{\hbar a^2}\left[1+\left(\frac{(z-1)e^2}{m'v^2\rho_{\max}}\right)^2\right]^{\frac{1}{2}}\right)\right\}, \quad (11a)$$

and for $(\frac{3}{2})^{\frac{1}{2}}e^2mz(z-1)/(m'\hbar va^2)>1$:

$$\phi_{ab}=-\frac{4\pi}{3v}N\left(\frac{\hbar}{mz}\right)^2(\mathbf{R}_a\cdot\mathbf{R}_a-2\mathbf{R}_a\cdot\mathbf{R}_b+\mathbf{R}_b\cdot\mathbf{R}_b)\times\ln\left(\frac{m'v^2\rho_{\max}}{(z-1)e^2}\left[1+\left(\frac{(z-1)e^2}{m'v^2\rho_{\max}}\right)^2\right]^{\frac{1}{2}}\right), \quad (11b)$$

if one adds an estimate for the strong collision term ($\pi N v\rho_{\min}{}^2$).

These ϕ_{ab} must still be averaged over the velocity distribution. It turns out that the velocities are usually in the range where (11a) is valid, and comparison with Eq. (21) in reference 1 (in this equation the strong collision term was neglected) shows that the same equation also applies to hydrogen ($z=1$). The second factor under the logarithm in (11a) is normally close

to 1 (also for $z>1$), i.e., the hydrogen formula can be used in most cases to calculate the weak collision contribution to the impact broadening of hydrogenic ion lines, and the Coulomb interactions between emitting ions and perturbers have no significant effect on the impact broadening, especially if one realizes the uncertainties in the choice of $\rho_{\max}$.

QUASI-STATIC BROADENING

The perturber motion can be neglected at high densities and low temperatures, if the frequency characterizing the perturbation is much smaller than a typical value of the quasi-static splitting $5\hbar N^{\frac{2}{3}}a^2/(mz)$, i.e., if

$$N\gg(zmv/5a^2\hbar)^3. \tag{12}$$

Then the quasi-static (half) half-width of a line will be

$$\Delta\omega_s\approx 5(a^2\hbar/zm)N^{\frac{2}{3}}, \tag{13}$$

using the estimates in reference 3 (where a and b are interchanged) for the matrix elements and neglecting the broadening of the lower states and interactions between the perturbers.

From this relation follows an approximate value for the maximum density $N_{\max}$ at which a discrete line can be observed (Inglis-Teller relation[10]), by equating $\Delta\omega_s$ with half the distance of two neighboring hydrogenic lines in the same series,

$$\Delta\omega_a=\frac{me^4z^2}{4\hbar^3}\left(\frac{1}{a^2}-\frac{1}{(a+1)^2}\right)\approx\frac{me^4z^2}{2a^3\hbar^3}, \tag{14}$$

namely

$$N_{\max}\approx 3\times10^{-2}\left(\frac{me^2}{\hbar^2}\right)^3\frac{z^{9/2}}{a^{15/2}}\approx 2\times10^{23}\frac{z^{9/2}}{a^{15/2}}. \tag{15}$$

At this density the validity criterion (12) is always fulfilled for broadening by ions, whereas the electron broadening may remain in the impact domain.

Equation (15) seems to overestimate the maximum density because electron broadening was neglected. This will however be offset by correlations between the ions, whose importance is characterized[7,8] by the magnitude of r_0/ρ_D, where r_0 is the mean distance of the perturbers defined by $(4\pi/3)r_0^3N=1$ and ρ_D the Debye radius $[kT/(4\pi e^2N)]^{\frac{1}{2}}$. This parameter will be a maximum for $N_{\max}$, i.e., with (15) and $T_z\approx T_1z^2$

$$\frac{r_0}{\rho_D}<1.25\left(\frac{m}{kT_1}\right)^{\frac{1}{2}}\frac{e^2}{\hbar a^{5/4}z^{\frac{1}{4}}}. \tag{16}$$

Accordingly the perturbing ion-ion correlation and the electron shielding effects decrease slowly with increasing z and a.

For $z>1$ also the interactions between broadening and emitting ions must be considered.[8] They will be important if the Coulomb energy of a perturber in the field of the emitting ion at the mean distance is of the same order as its kinetic energy, i.e., if

$$\frac{(z-1)e^2}{r_0kT}\approx\left(\frac{r_0}{\rho_D}\right)^2(z-1)\approx 1. \tag{17}$$

This correlation therefore becomes stronger at higher z. It should be noted that (17) is actually an overestimate for the ratio of the electrostatic interaction and kinetic energies, because the ratio will be smaller by a factor $\exp(-r_0/\rho_D)$, which accounts for the Debye shielding by the electrons. However, one must expect that the correlations between emitting and broadening ions reduce the quasi-static broadening significantly at high densities, thus making (15) an underestimate of the limiting density for low a but high z lines, unless electron broadening over-compensates this effect. (That correlations between emitting ions and perturbing electrons are not important follows from the similarity of the straight and the hyperbolical classical path results.)

The question whether the quasi-static approximation for broadening by ions is valid in all cases can be answered by comparing $\Delta\omega_s$ from (13) with the Doppler width of a hydrogenic line

$$\Delta\omega_d=0.833\left(\frac{2kT}{m_zc^2}\right)^{\frac{1}{2}}\frac{me^4z^2}{2\hbar^3}\left(\frac{1}{b^2}-\frac{1}{a^2}\right). \tag{18}$$

This gives for the density N_{ds} where quasi-static and Doppler broadening are equal with $m_z\approx 2m_1z$, $T\approx T_1z^2$ $\approx z^2$[ev], and neglecting $1/a^2$ versus $1/b^2$,

$$N_{ds}\approx\frac{1}{40}\left(\frac{kT_1}{m_1c^2}\right)^{\frac{3}{4}}\left(\frac{me^2}{\hbar^2}\right)^3\frac{z^{21/4}}{a^3b^3}\approx 3\times10^{16}\frac{z^{21/4}}{a^3b^3}, \tag{19}$$

assuming that the quasi-static approximation were still valid. Comparison of (19) with (12) shows that this is strictly justified if $z^{-\frac{1}{4}}a/b$ is much larger than ~ 20. But as will be shown at the end of this section, the quasi-static approximation may still be used at densities much lower than those indicated by (12), and (19) can therefore usually serve to estimate the density at which Doppler and Stark widths are comparable.

In the transition region between impact and quasi-static domain, where (12) is a near equality, i.e., for

$$v\approx 5\frac{a^2\hbar}{zm}N^{\frac{2}{3}}, \tag{20}$$

Eq. (8) yields for the ϕ_{ab} operator

$$\begin{aligned}\phi_{ab}&\approx-\frac{4\pi\hbar}{15a^2mz}N^{\frac{2}{3}}\mathbf{R}_a\cdot\mathbf{R}_a\ln\left(\frac{\sin(\frac{1}{2}\alpha_{\max})}{\sin(\frac{1}{2}\alpha_{\min})}\right)\\&\approx-5\frac{\hbar}{a^2mz}N^{\frac{2}{3}}\mathbf{R}_a\cdot\mathbf{R}_a,\end{aligned} \tag{21}$$

[10] D. R. Inglis and E. Teller, Astrophys. J. **90**, 439 (1939).

because the logarithm is about 6 if (20) is fulfilled. Again using $(a)^4$ as an estimate for a typical matrix element of $\mathbf{R}\cdot\mathbf{R}$, one sees that then the ϕ-matrix elements are very close to the quasi-static half-width given by (13).

In case of isolated lines the impact half-width is equal to the diagonal ϕ-matrix element, but here the width would be larger by a factor of the order of the principal quantum number, because of the degeneracy. Actually in the transition region some quasi-static broadening will also be present, which removes the degeneracy, thus reducing the importance of the off-diagonal ϕ-matrix elements, which correspond to the impact-induced transitions between the various Stark components.

If one schematically splits the ion field into a high- and low-frequency component, one can therefore say that in the transition region each component will approximately contribute $\frac{1}{2}\Delta\omega_s$ as given by (13) to the half-width, i.e., no large errors are expected if one uses the simpler quasi-static approximation for the total field. As long as an intermediate theory is lacking, one should do this even for lower densities (because in this range impact and quasi-static broadening depend similarly on the density) unless the impact approximation for ions yields significantly smaller widths.

Similar considerations show that, in the cases where the impact approximation breaks down for electrons, a theory treating part of the field (the ion field) as quasi-static and using the impact approximation for the rest of the field (the electron field), would yield almost the same width as a quasi-static calculation for the total field. Therefore one can use the quasi-static approximation for electrons where it results in a narrower line than the impact approximation.

COMPLETE LINE PROFILES

At low densities, where the impact approximation is valid for all perturbers, the line profile is given by the generalized impact formula[1,2,4,5]

$$I_{ab}(\omega)=-\frac{1}{\pi}\mathrm{Re}\sum_{\alpha'\alpha''\beta'\beta''\sigma}\langle\alpha'|\mu_\sigma|\beta'\rangle\langle\beta''|\mu_\sigma|\alpha''\rangle \times\langle\alpha'|\langle\beta'|[i\omega-i(H_a-H_b)/\hbar+\phi_{ab}]^{-1}|\alpha''\rangle|\beta''\rangle. \quad (22)$$

The matrix elements of μ_σ are the dipole moments for the various polarizations and those of $(H_a-H_b)/\hbar$ are the frequencies of the unperturbed lines. Because ϕ_{ab} is essentially inversely proportional to the perturber velocity, the electron contribution is here entirely negligible, i.e., (11a) or (11b) can be used with m' being the mass of the perturbing ions.

Equation (22) can be approximated by a dispersion profile[3] with a half-width given by the ϕ matrix element following from (8) with $\frac{1}{3}a^5$ as an estimate for the sum of the contributing matrix elements of $\mathbf{R}_a\cdot\mathbf{R}_a-2\mathbf{R}_a\cdot\mathbf{R}_b+\mathbf{R}_b\cdot\mathbf{R}_b$. This width is equal to the Doppler width in Eq. (18) for densities greater than

$$N_{di}\approx 4\times10^{-2}\left(\frac{kT_1}{m_1}\right)\frac{m^3e^4z^{11/2}}{c\hbar^5a^5}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\approx 4\times10^{16}\frac{z^{11/2}}{a^5b^2}, \quad (23)$$

again using 6 for the logarithm and assuming hydrogen ions as perturbers and $T\approx z^2[\mathrm{ev}]$, $m_z=2m_1z$.

As pointed out in the previous section, the impact approximation, i.e., Eq. (22), can be used up to densities where the impact half-width reaches the quasi-static width given by (13). Using similar estimates for ϕ, this yields for the limiting density

$$N_{si}\approx\frac{1}{4}\left(\frac{mv_1}{\hbar}\right)^3\frac{z^6}{a^9}\approx\tfrac{1}{2}\times10^{17}\frac{z^6}{a^9}, \quad (24)$$

with $v_z=v_1z=7\times10^5z$ [cm sec^{-1}] as the velocity of perturbing hydrogen ions. Comparison with (23) shows that this is larger than N_{di} if $b^2z^{\frac{1}{2}}/a^4$ is larger than 0.8. Usually N_{si} will be smaller than N_{di}; and one should use (19) instead of (23) to estimate the density at which Doppler and Stark broadening are equal.

For densities higher than N_{si} the line profile is given by

$$I(\omega)=\int dF\, W(F)I_e(\omega,F), \quad (25)$$

that is by the average of the electron impact profile $I_e(\omega,F)$ described by an equation analogous to (22) over the ion field-strength F. At low densities the Holtsmark distribution function can be used for $W(F)$, and at high densities the distribution functions modified for correlation and shielding effects are applicable, which were discussed in the previous section. The electron impact profile depends on the ion field-strength because the matrix elements of $(H_a-H_b)/\hbar$ in Eq. (22) now represent the frequencies of the various Stark components, which are split by the quasi-static ion fields.

Equation (25) can be used up to densities where the electron impact width approaches the quasi-static width, i.e., according to estimates corresponding to those leading to (24) up to densities smaller than

$$N_{se}\approx\tfrac{1}{2}\times10^{22}z^6/a^9, \quad (26)$$

which is a factor 10^5 larger than the corresponding quantity N_{si} for the ion broadening. N_{se} is only larger than the Inglis-Teller limit [Eq. (15)] for cases where $12a<z$; i.e., for most of the important lines from low-z hydrogenic ions one must use the quasi-static approximation for all perturbers at densities approaching the Inglis-Teller limit. For this reason, and because the electron broadening tends to compensate the reduction

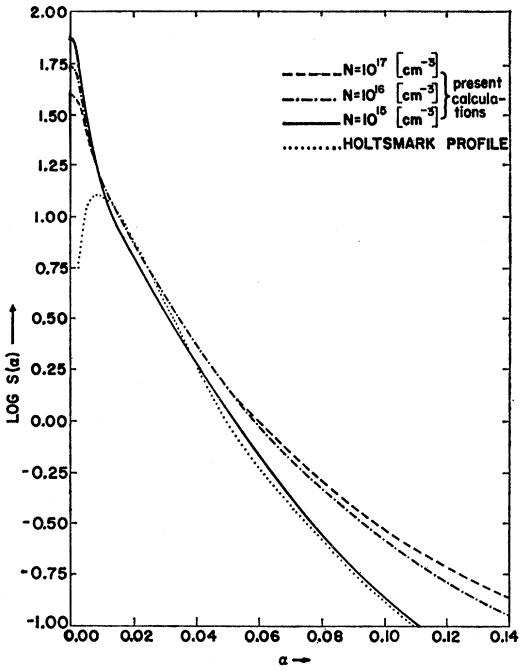

FIG. 1. Comparison of the present calculations with the Holtsmark profile of HeII 4686 A at $T=40\,000$ [°K]. [$\alpha=\Delta\lambda(\mathrm{A})/F_0(\mathrm{cgs})$.]

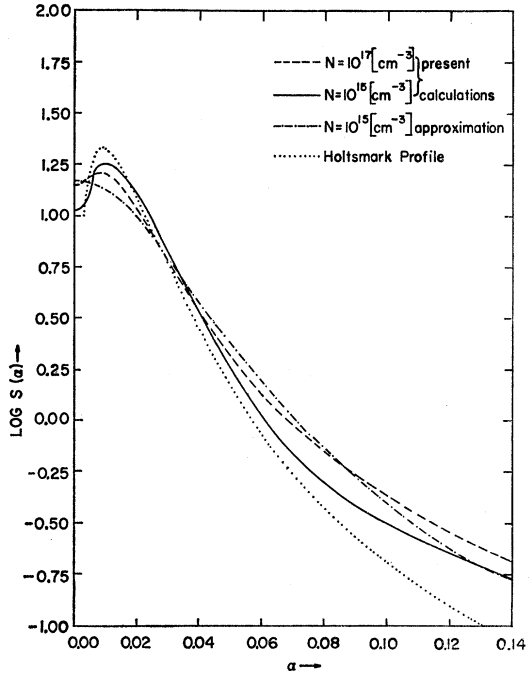

FIG. 2. Comparison of the present calculations with the Holtsmark profile and an approximation for high series members for HeII 3203 A at $T=40\,000$ [°K] [$\alpha=\Delta\lambda(\mathrm{A})/F_0(\mathrm{cgs})$].

of the quasi-static broadening by correlation and shielding effects, the Inglis-Teller formula is therefore essentially correct in most applications.

RESULTS AND DISCUSSION

Detailed calculations in the usual reduced wavelength-scale $\alpha=\Delta\lambda/(2.61eN^{\frac{2}{3}})$ for densities from 10^{15} to 10^{18} [cm^{-3}] and temperatures from 5000 to 80 000 [°K] were carried through for the ionized helium lines at 4686 A and 3203 A,[11,12] with the quasi-static approximation for ions and the impact approximation for electrons. For both lines, these approximations are reasonably valid for the whole density and temperature range considered. At lower densities where one must correct for Doppler broadening the Holtsmark profile is a fair approximation. At higher densities one should use the quasi-static theory for both ions and electrons, taking the Coulomb interactions into account.

In Figs. 1 and 2 the present calculations are compared with the Holtsmark theory (which neglects the electron broadening and interactions between the charged particles) for an intermediate temperature (the profiles are very insensitive to the temperature). As in the case of $H\beta$,[1] the agreement with the Holtsmark profile is sometimes better than expected, e.g., in the case of Wulff's experiment[13] the Holtsmark theory for HeII 4686 A gives an electron density $N=3.9\times10^{16}$ [cm^{-3}], whereas comparison with the present theory results in $N\approx2.5$ [cm^{-3}]. Obviously the additional broadening by electron impacts is also here partially compensated by the narrowing of the quasi-static profile resulting from the Coulomb interactions.

For higher series members and low densities the approximate simplified formulas given in reference 3 should be applicable. That this begins to become true already for the $5\rightarrow3$ transition (HeII 3203 A) is demonstrated by the fair agreement on the line wings between the curve corresponding to this approximation in Fig. 2 with the result of the present calculation for $N=10^{15}$ [cm^{-3}]. (The differences in the line core are partially due to the correlation and shielding effects and will be masked by Doppler broadening, especially at still lower densities.)

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APPENDIX

It will be demonstrated that a completely quantum mechanical calculation leads to the same result as the classical path calculation, if the corresponding approxi-

[11] K. Y. Shen, Ph.D. thesis, University of Maryland, College Park, 1960 (unpublished).
[12] A complete set of profiles will be published in a U. S. Naval Research Laboratory Report.
[13] H. Wulff, Z. Physik **150**, 614 (1958).

mations are made. As an example a so-called one-state completely degenerate case[14,15] with $z=1$ is considered. Then the matrix elements of the ϕ operator can be written as

$$\langle\alpha'|\phi_a|\alpha''\rangle=\sum_j P_j\langle\alpha'|S_a(0)-1|\alpha''\rangle = -\tfrac{1}{2}Nv\{\sigma_{\alpha'\alpha''}\}, \quad (1A)$$

where $S_a(0)$ is now the quantum mechanical scattering matrix and $\{\sigma_{\alpha'\alpha''}\}$ the average of the cross section for electron scattering which causes a transition of the atom from α' to α''. The sum is over all states of the perturber.

Using the interaction Hamiltonian (3) and plane waves as perturber wave functions (corresponding to the straight classical path), one obtains for the scattering amplitudes in the Born approximation (corresponding to the weak-collision approximation), integrating over the volume between two concentric spheres with radii $r_{\min}$ and $r_{\max}$,

$$\begin{aligned} f_{\alpha'\alpha'''}&=\frac{me^2}{2\pi\hbar^2}\langle\alpha'|\mathbf{r}_a|\alpha'''\rangle\cdot\int\mathbf{r}|\mathbf{r}|^{-3}e^{i\mathbf{K}\cdot\mathbf{r}}d\tau \\ &=\frac{me^2}{2\pi\hbar^2}\langle\alpha'|r_a|\alpha'''\rangle_K\int\frac{\cos\vartheta}{r^2}e^{iKr\cos\vartheta}r^2\sin\vartheta d\vartheta d\varphi dr \\ &=\frac{me^2}{\hbar^2}\langle\alpha'|r_a|\alpha'''\rangle_K\int\left[\frac{e^{iKr\cos\vartheta}}{iK}\right]_{r_{\min}}^{r_{\max}}\sin\vartheta d\vartheta \\ &\approx\begin{cases}0; & K<2/r_{\max},\\ \dfrac{2me^2}{i\hbar^2K}\langle\alpha'|r_a|\alpha'''\rangle_K; & 2/r_{\max}<K<2/r_{\min},\\ 0; & 2/r_{\min}<K.\end{cases} \quad (2A)\end{aligned}$$

[14] M. Baranger, Phys. Rev. **111**, 481 (1958).
[15] M. Baranger, Phys. Rev. **112**, 855 (1958).

Here $\mathbf{K}$ ($|\mathbf{K}|=K$) is the difference of the initial ($\mathbf{k}_i$) and final ($\mathbf{k}_f$) propagation vectors and $\langle\alpha'|r_a|\alpha'''\rangle_K$ the matrix element of the component of the atomic electron coordinate vector in the $\mathbf{K}$ direction. The total cross section is then (with $|\mathbf{k}_i|\approx|\mathbf{k}_f|\approx k_0$) given by

$$\begin{aligned}\sigma_{\alpha'\alpha''}&=\int f_{\alpha'\alpha'''}f_{\alpha'''\alpha'}d\Omega=\int f_{\alpha'\alpha'''}f_{\alpha'''\alpha''}\frac{2\pi K}{k_0^2}dK \\ &=\frac{8\pi m^2e^4}{\hbar^4k_0^2}\langle\alpha'|r_a|\alpha'''\rangle_K\langle\alpha'''|r_a|\alpha''\rangle_K\ln\frac{K_{\max}}{K_{\min}}. \quad (3A)\end{aligned}$$

The largest possible value for $K_{\max}$ would obviously be $K_{\max}=2k_0=2mv/\hbar$. But according to (2A) this value should only be used if $K_{\max}$ is not larger than $2/r_{\min}$, i.e., if $r_{\min}$ were smaller than the de Broglie wavelength $\lambda\!\!\!^{-}$. In all other cases $2/r_{\min}$ should be adopted for $K_{\max}$. It also follows from (2A) that $K_{\min}$ will be of the order $2/r_{\max}$, and accordingly one has, for $r_{\min}>\lambda\!\!\!^{-}$,

$$K_{\max}/K_{\min}\approx r_{\max}/r_{\min}. \quad (4A)$$

Averaging over the directions and substitution of (3A) and (4A) into (1A) yields finally for the ϕ operator, with $k_0=mv/\hbar$ and $\mathbf{r}_a=(\hbar^2/me^2)\mathbf{R}_a$,

$$\phi_a=-\frac{4\pi}{3v}\left(\frac{\hbar}{m}\right)^2\ln\left(\frac{r_{\max}}{r_{\min}}\right)(\mathbf{R}_a\cdot\mathbf{R}_a), \quad (5A)$$

which agrees with the classical path result [see Eq. (21) in reference 1], if one identifies $r_{\max}$ with the Debye radius and $r_{\min}$ with the impact parameter, where the perturbation theory breaks down. This is legitimate, because the minimum impact parameter is larger than the de Broglie wavelength by the square of the principal quantum number [see Eq. (19) in reference 1]. If this were not the case the only effect of quantum mechanics would be to introduce $\lambda\!\!\!^{-}$ as minimum impact parameter under the logarithm.