Emission of Photoneutrinos and Pair Annihilation Neutrinos from Stars

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Field-theoretic calculations of the cross sections for the photoneutrino process, $\gamma+e^-\rightarrow e^-+\nu+\bar{\nu}$, and the pair annihilation process, $e^++e^-\rightarrow\nu+\bar{\nu}$, are performed in order to obtain the neutrino luminosity of very hot stars ($T_c\gtrsim 5\times10^8$ °K). The energy loss rate which is obtained for the latter process is sufficient to determine the rate of evolution of the stellar core when $T_c\gtrsim2\times10^9$ °K.

I. INTRODUCTION

If the weak interaction has a term of the form $(\bar{\nu}e)(\bar{e}\nu)$ as predicted by the theory of Feynman and Gell-Mann,[1] then there are a large number of processes that result in the production of neutrino-antineutrino pairs. The emission of a neutrino pair is clearly less probable than the emission of a photon by a factor of about $137(m^2G)^2\approx10^{-21}$, and thus is generally unobservable in the laboratory. However, these processes will yield significant energy losses in stars due to the relation between the neutrino and photon mean-free paths:

$$\lambda(\nu)>d\gg\lambda(\gamma),$$

where d is a typical stellar diameter. This relation also means that the neutrino luminosity may be determined solely from the neutrino production rate.

Among the important neutrino emission processes are: the urca process of Gamow and Schonberg,[2,3]

$$e^-+(Z,A)\rightarrow(Z-1,A)+\nu \searrow(Z,A)+e^-+\bar{\nu};\qquad(1)$$

neutrino bremsstrahlung,[4]

$$e^-+(Z,A)\rightarrow e^-+(Z,A)+\nu+\bar{\nu};\qquad(2)$$

the photoneutrino process,[5]

$$\gamma+e^-\rightarrow e^-+\nu+\bar{\nu};\qquad(3)$$

pair annihilation neutrinos,[6]

$$e^-+e^+\rightarrow\nu+\bar{\nu};\qquad(4)$$

and photon-photon neutrinos,[6,6a]

$$\gamma+\gamma\rightarrow\nu+\bar{\nu},\quad\gamma+\gamma\rightarrow\gamma+\nu+\bar{\nu}.\qquad(5)$$

These processes are important because under the appropriate conditions any one of them may release energy comparable with the total thermal energy of a star in a time comparable with a characteristic time of stellar evolution. This property distinguishes these reactions from the ordinary beta decay reactions occurring in element synthesis. The latter can emit at the most about 10% of the released energy in the form of neutrinos. Processes (4) and (5) have further significance in that they will result in rapid stellar collapse if the temperature reaches 2×10^9 °K, and thus may be considered to precipitate supernova explosions.[6]

The energies of interest in stellar processes are always less than 100 Mev, so that interactions involving mesons are not important. It is not profitable to give a detailed comparison of all the neutrino emission rates until those for the photon-photon reactions have been calculated. The urca process and the neutrino bremsstrahlung process have already been discussed in the literature.[2–4] In this paper we will consider the photoneutrino (3) and the pair annihilation (4) processes in detail, obtaining cross sections and energy loss rates for both degenerate and nondegenerate electrons, and in both the relativistic and nonrelativistic limits.

II. PHOTONEUTRINO PROCESS

A. Matrix Elements and Cross Section

The universal Fermi interaction current is

$$J=(\bar{e}\nu)+(\bar{n}p)+(\bar{\mu}\nu)+(\text{strange particles}),$$

the transition amplitude involving $(\bar{\nu}e)(\bar{e}\nu)$, $(\bar{\nu}\mu)(\bar{e}\nu)$, etc. So far in laboratories only the cross terms $(\bar{p}n)(\bar{e}\nu)$(neutron decay), $(\bar{\nu}\mu)(\bar{e}\nu)(\mu$-decay), and

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[1] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
[2] G. Gamow and M. Schönberg, Phys. Rev. **59**, 539 (1941).
[3] H. Y. Chiu, Ann. Phys. (to be published).
[4] B. M. Pontecorvo, J. Exptl. Theoret. (U.S.S.R.) Phys. **36**, 1615 (1959) [translation: Soviet Physics—JETP **9**, 1148 (1959)]; G. M. Gandel'man and V. S. Pinaev, J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 1072 (1959) [translation: Soviet Physics—JETP **10**, 764 (1960)].
[5] R. C. Stabler, thesis, Cornell University (unpublished).
[6] H. Y. Chiu and P. Morrison, Phys. Rev. Letters **5**, 573 (1960).
[6a] *Note added in proof.* M. Gell-Mann has recently shown [Phys. Rev. Letters **6**, 70 (1961)] that the reaction, $\gamma+\gamma\rightarrow\nu+\bar{\nu}$, is forbidden if the weak interaction coupling has the local form given in (6) below.

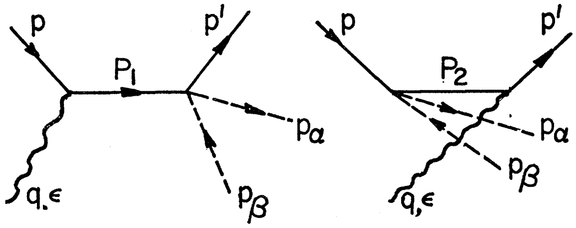

FIG. 1. Feynman diagrams for the photoneutrino process.

$(\bar{\nu}\mu)(\bar{n}p)$ (μ-capture) have been studied and verified by experiments. Owing to the smallness of the cross section ($\approx 10^{-44}$ cm^2) and difficulties in experimental techniques, the existence of square terms such as $(\bar{\nu}e)(\bar{e}\nu)$, etc. has not been verified. However, we assume the existence of such terms in what follows.

In particular the weak interaction has the form[1,7]

$$(8)^{\frac{1}{2}}G(\bar{\psi}_e\gamma_\mu a\psi_\nu)(\bar{\psi}_\nu\gamma_\mu a\psi_e), \tag{6}$$

where $a\equiv\frac{1}{2}(1+i\gamma_5)$, $GM^2=(1.01\pm0.01)\times10^{-5}$, and M is the proton mass. The two Feynman diagrams that must be considered are shown in Fig. 1. On introducing the electromagnetic interaction $e\boldsymbol{\varepsilon}$ and the electron propagator $1/(\boldsymbol{P}-m)$ the matrix elements may be written

$$M_1=-(8)^{\frac{1}{2}}Ge(\bar{u}_\alpha\gamma_\mu a u_\beta)\left(u_{P'}\gamma_\mu a\frac{1}{\boldsymbol{P}_1-m}\boldsymbol{\varepsilon}u_P\right),$$

$$M_2=-(8)^{\frac{1}{2}}Ge(\bar{u}_\alpha\gamma_\mu a u_\beta)\left(u_{P'}\boldsymbol{\varepsilon}\frac{1}{\boldsymbol{P}_2-m}\gamma_\mu a u_P\right).$$

We normalize the spinors by $u^\dagger u=2E$ so that the projection operators have the form $(\boldsymbol{P}+m)$. Then, summing over final states, we have

$$\begin{aligned}|M|^2&\equiv\textstyle\sum_{\rm pol}|M_1+M_2|^2\\&=\tfrac{1}{2}(8G^2e^2)\,\mathrm{Tr}[\gamma_\mu a\boldsymbol{P}_\alpha\gamma_\nu a\boldsymbol{P}_\beta]\\&\qquad\times\mathrm{Tr}[\bar{Q}(\boldsymbol{P}'+m)Q(\boldsymbol{P}+m)],\end{aligned}\tag{8}$$

where

$$Q=\frac{\gamma_\mu a(\boldsymbol{P}_1+m)\boldsymbol{\varepsilon}}{P_1{}^2-m^2}+\frac{\boldsymbol{\varepsilon}(\boldsymbol{P}_2+m)\gamma_\mu a}{P_2{}^2-m^2},$$

and in $\bar{Q}$ the μ's are to be changed into ν's. The computation of the traces is lengthy but may be simplified by the expressions in the Appendix. By choosing Lorentz gauge we have $\epsilon\cdot q=0$. We finally obtain

$$\begin{aligned}(4G^2e^2)^{-1}|M|^2=&\frac{8P'\cdot P_\beta}{P\cdot q}\left[\frac{2(\epsilon\cdot P)^2}{P\cdot q}(P\cdot P_\alpha+q\cdot P_\alpha)-\epsilon^2(q\cdot P_\alpha)-2(\epsilon\cdot P)(\epsilon\cdot P_\alpha)\right]+\frac{8P\cdot P_\alpha}{P'\cdot q}\left[\frac{2(\epsilon\cdot P')^2}{P'\cdot q}(P'\cdot P_\beta-q\cdot P_\beta)\right.\\&\left.-\epsilon^2(q\cdot P_\beta)+2(\epsilon\cdot P')(\epsilon\cdot P_\beta)\right]-\frac{8\epsilon^2}{(P\cdot q)(P'\cdot q)}[(P\cdot P')(q\cdot P_\alpha)(q\cdot P_\beta)+(P\cdot q)(P'\cdot q)(P_\alpha\cdot P_\beta)\\&-(P'\cdot q)(P\cdot P_\beta)(q\cdot P_\alpha)-(P\cdot q)(P'\cdot P_\alpha)(q\cdot P_\beta)]+\frac{16}{(P\cdot q)(P'\cdot q)}[(\epsilon\cdot P')(\epsilon\cdot P)(q\cdot P_\alpha)(q\cdot P_\beta)\\&+(\epsilon\cdot P_\alpha)(\epsilon\cdot P_\beta)(P\cdot q)(P'\cdot q)+(\epsilon\cdot P')(P\cdot q)(\epsilon\cdot P_\alpha)((P'-q)\cdot P_\beta)-(\epsilon\cdot P)(P'\cdot q)(\epsilon\cdot P_\beta)((P+q)\cdot P_\alpha)\\&\quad+(\epsilon\cdot P)(\epsilon\cdot P')(q\cdot P_\beta)(P\cdot P_\alpha)-(\epsilon\cdot P)(\epsilon\cdot P')(q\cdot P_\alpha)(P'\cdot P_\beta)-2(\epsilon\cdot P)(\epsilon\cdot P')(P\cdot P_\alpha)(P'\cdot P_\beta)].\end{aligned}\tag{9}$$

The total cross section is given in terms of $|M|^2$ by

$$\sigma=\frac{2\pi}{(2E)(2q)}\int|M|^2\frac{1}{(2\pi)^6}\delta^4(P+q-P'-P_\alpha-P_\beta)\frac{d^3P_\alpha}{2E_\alpha}\frac{d^3P_\beta}{2E_\beta}\frac{d^3P'}{2E'}.\tag{10}$$

The integrations over the neutrino momenta may be carried out with the aid of a relation due to Lenard[8]:

$$\int\frac{d^3P_\alpha}{2E_\alpha}\frac{d^3P_\beta}{2E_\beta}\delta^4(X-P_\alpha-P_\beta)P_\alpha{}^\mu P_\beta{}^\nu=\frac{\pi}{24}[2X^\mu X^\nu+g^{\mu\nu}X\cdot X].\tag{11}$$

If we evaluate the resulting expression in the center-of-mass system we have

$$\epsilon\cdot P=0,\quad \epsilon^2=-1,\quad |\mathbf{q}|=|\mathbf{P}|,\quad \langle(\epsilon\cdot P')^2\rangle_{\rm av}=\tfrac{1}{2}[\mathbf{P}'^2-(\mathbf{P}'\cdot\mathbf{q}/|\mathbf{q}|)^2].\tag{12}$$

After integration over d^3P', the cross section becomes

$$\begin{aligned}\sigma(\mathrm{ND})=\frac{G^2e^2}{96\pi^3}\frac{E^5}{P^3}\Bigg\{&\frac{P}{E+P}\left[\frac{5}{2}+\frac{65}{6}\left(\frac{P}{E}\right)+\frac{1}{3}\left(\frac{P}{E}\right)^2-\frac{233}{9}\left(\frac{P}{E}\right)^3-\frac{101}{6}\left(\frac{P}{E}\right)^4+\frac{17}{6}\left(\frac{P}{E}\right)^5+\frac{16}{9}\left(\frac{P}{E}\right)^6\right]\\&+\ln\left(\frac{P+E}{m}\right)\left[-\frac{5}{2}-\frac{25}{3}\left(\frac{P}{E}\right)+\frac{53}{6}\left(\frac{P}{E}\right)^2+\frac{62}{3}\left(\frac{P}{E}\right)^3-\frac{7}{2}\left(\frac{P}{E}\right)^4-7\left(\frac{P}{E}\right)^5+\frac{5}{2}\left(\frac{P}{E}\right)^6\right]\Bigg\}.\end{aligned}\tag{13}$$

[7] We set $\hbar=c=k$ (Boltzmann's constant)$=1$. We also define $A_\mu B_\mu\equiv A\cdot B\equiv A_0B_0-\mathbf{A}\cdot\mathbf{B}$, $\boldsymbol{A}\equiv\gamma\cdot A$, and $A^2=A\cdot A$ where appropriate.
[8] A. Lenard, Phys. Rev. **90**, 968 (1953).

In Eq. (13), "ND" refers to the evaluation of the cross section for the case of nondegenerate electrons.

No approximations have been made in obtaining Eq. (13). However, it is useful to have simpler expressions for the cross section in the low-energy (NR), and high-energy (ER) limits. If P/E and $\ln[(P+E)/m]$ are expanded in powers of P/m, the first nonvanishing term in $\sigma(\mathrm{ND})$ is proportional to $(P/m)^7$:

$$\sigma(\mathrm{ND,NR})=\frac{e^2G^2}{96\pi^3}\frac{E^5}{P^3}\left[\frac{96}{35}\left(\frac{P}{m}\right)^7+\frac{64}{105}\left(\frac{P}{m}\right)^8+\cdots\right]$$
$$\rightarrow\frac{(e^2/\hbar c)(Gm^2)^2}{35\pi^3}\left(\frac{P}{mc}\right)^4\left(\frac{\hbar}{mc}\right)^2, \quad (14)$$

where the latter expression is given in conventional units. Feynman has obtained the same formula for the cross section in this limit.[9] In the extreme relativistic limit,

$$\sigma(\mathrm{ND,ER})\rightarrow\frac{(e^2/\hbar c)(Gm^2)^2}{9\pi^3}\left(\frac{E}{mc^2}\right)^2\left(\frac{\hbar}{mc}\right)^2$$
$$\times\left[\ln\left(\frac{2E}{mc^2}\right)-\frac{55}{48}\right]. \quad (15)$$

In Eq. (15) terms of the order $(m/E)^2$ in the brackets have been neglected.

B. Energy Loss for Nondegenerate Electrons

To obtain the mean energy loss per event, we weight the differential cross section by $E_\alpha+E_\beta=E+q-E'$ and integrate as before over the outgoing states. We find

$$\mathcal{E}(\mathrm{ND})=\frac{e^2G^2}{96\pi^3}\left(\frac{E^5}{P^2}\right)\left\{\left[\frac{203}{60}+\frac{731}{60}\left(\frac{P}{E}\right)-\frac{179}{90}\left(\frac{P}{E}\right)^2\right.\right.$$
$$-\frac{2627}{90}\left(\frac{P}{E}\right)^3-\frac{13\,451}{900}\left(\frac{P}{E}\right)^4+\frac{1831}{300}\left(\frac{P}{E}\right)^5$$
$$\left.+\frac{56}{225}\left(\frac{P}{E}\right)^6\right]+\ln\left(\frac{P+E}{m}\right)\left[-\frac{203}{60}\left(\frac{E}{P}\right)\right.$$
$$-\frac{731}{60}+\frac{187}{60}\left(\frac{P}{E}\right)+\frac{133}{4}\left(\frac{P}{E}\right)^2+\frac{175}{12}\left(\frac{P}{E}\right)^3$$
$$\left.\left.-\frac{59}{4}\left(\frac{P}{E}\right)^4-\frac{21}{4}\left(\frac{P}{E}\right)^5+\frac{11}{4}\left(\frac{P}{E}\right)^6\right]\right\}. \quad (16)$$

In the NR limit $\mathcal{E}(\mathrm{ND})\rightarrow P\sigma(\mathrm{ND,NR})$, or,

$$\mathcal{E}(\mathrm{ND,NR})=\frac{e^2G^2}{35\pi^3}\left(\frac{P^5}{m^2}\right), \quad (17)$$

[9] R. P. Feynman (private communication).

and in the ER limit we have

$$\mathcal{E}(\mathrm{ND,ER})=\frac{17e^2G^2}{90\pi^3}E^3\left[\ln\left(\frac{2E}{m}\right)-\frac{2723}{2040}\right]. \quad (18)$$

On transforming to the laboratory system, the total energy loss is given by

$$\mathcal{E}_t=\frac{1}{\rho}\int|\mathbf{v}|dn(\mathbf{q})dn(\mathbf{P})\mathcal{E}(\mathbf{P},\mathbf{q}), \quad (19)$$

where dn represents a momentum distribution, $\mathbf{v}$ the relative velocity ($=c$ here), and ρ the density. We finally have

$$\mathcal{E}_t(\mathrm{ND,NR})=8\pi^3e^2G^2N_eT^8/525m^2\rho$$
$$=(8\times10^{-2}/\mu_e)T_8{}^8\ \mathrm{erg/g\text{-}sec}, \quad (20)$$

where $\mu_e\equiv\langle A/Z\rangle_{\mathrm{av}}$ is the mean molecular weight of the electrons and $T_n\equiv T/10^n$ °K. Equation (20) is correct to the lowest order in T/m.

For $T\gg m$, we obtain

$$\mathcal{E}_t(\mathrm{ND,ER})$$
$$\approx\frac{833e^2G^2\zeta(9/2)N_eT^6}{512\pi^4\rho}\left[\ln\left(\frac{2T}{m}\right)+2\psi\left(\frac{9}{2}\right)-\frac{3131}{2040}\right]$$
$$\approx(2\times10^{13}/\mu_e)T_{10}{}^6(\log_{10}T_{10}+1.6)\ \mathrm{ergs/g\text{-}sec}. \quad (21)$$

Here $\psi(k)=(d/dk)\ln\Gamma(k)$ is the psi function.

C. Energy Loss for Degenerate Electrons

In stellar matter at very high densities, $\rho\gtrsim10^6$ g/cc, the energy loss from the photoneutrino process will be greatly reduced due to the degeneracy of the electrons. For simplicity we shall only consider the extreme degenerate case (ED). Also, as the loss for this process becomes negligible for $kT\ll E_F\ll m$, where E_F is the Fermi energy, we shall restrict ourselves to the extreme relativistic limit ($E_F\gg m$).

The integration of Eq. (9) over the neutrino and antineutrino momenta may be carried out as in the non-degenerate case by using Lenard's result, Eq. (11). Averaging over the polarization of the photon should not be carried out in the manner indicated by Eq. (12), however. Use of the center-of-mass frame is inconvenient because the integrals over the initial and final electron momenta would involve nonspherical Fermi surfaces. After some algebra it may be shown that an invariant expression for the average over photon polarization is given by

$$\langle\epsilon\cdot P\rangle_{\mathrm{av}}=0,\quad \epsilon^2=-1,\quad \langle(\epsilon\cdot P)(\epsilon\cdot P')\rangle_{\mathrm{av}}=0,$$
$$\langle(\epsilon\cdot P')^2\rangle_{\mathrm{av}}=\frac{1}{2(P\cdot q)^2}\{2(P\cdot q)(P'\cdot q)(P\cdot P') \quad (22)$$
$$-m^2[(P\cdot q)^2+(P'\cdot q)^2]\}.$$

The resulting differential cross section will vanish for $\mathbf{P}'=\mathbf{P}$; however, as the low-energy electron states are filled, the phase space vanishes unless $|\mathbf{P}'|\approx|\mathbf{P}'|$. Therefore we must find an approximation for $|\mathbf{P}-\mathbf{P}'|\approx T$. On letting

$$\Delta_1=(P-P')\cdot q,\quad \Delta_2=P\cdot P'-m^2, \tag{23}$$

and expanding $|M|^2$ in powers of Δ_1 and Δ_2, we find

$$d\sigma(\mathrm{ED})=\frac{8\pi G^2e^2}{3(2\pi)^5(2E)(2q)}\frac{4(\Delta_1-\Delta_2)d^3P'}{2E'\{\exp[(E_F-E')/T]+1\}} \tag{24}$$

to lowest order in Δ_1 and Δ_2 [and hence to lowest order in (T/E_F)]. For the phase space factor in Eq. (24) we have used the density of unfilled states.

To obtain the mean energy loss we multiply Eq. (24) by $(E+q-E')$ and integrate over d^3P'. This integration and the integration over the initial electron density are obtained only to lowest order in T/E_F and m/E_F. Finally

$$\mathcal{E}_t(\mathrm{ED,ER})=\frac{5\times10^5}{\mu_e}\left(\frac{m}{E_F}\right)^3T_9{}^7(1+5T_9{}^2)\frac{\mathrm{ergs}}{\mathrm{g\text{-}sec}}, \tag{25}$$

where we have put $\rho=ME_F{}^3(\mu_e/3\pi^2)$.

We summarize the results for the photoneutrino process in Figs. 2 and 3. In these graphs we have arbitrarily set $\mu_e=3$, corresponding to a mixture of heavy elements and neutrons. In the semirelativistic region we have joined the extreme relativistic and the nonrelativistic forms by a smooth curve. It is estimated that the result is accurate to within a factor of 1.5.

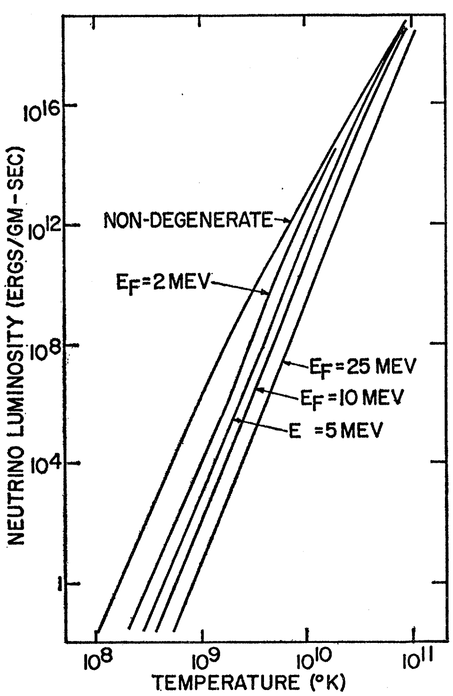

FIG. 2. The rate of energy loss in the photoneutrino process as a function of temperature.

III. PAIR ANNIHILATION NEUTRINOS

A. Density of Electron-Positron Pairs

When kT becomes comparable to mc^2, collisions between photons and electrons or nuclei can be accompanied by pair production. Even *in vacuo* at 10^9 °K, where photon-photon collisions will be a source of pairs, the relaxation time for equilibrium to be established between radiation and electron-positron pairs is less than 10^{-11} sec. Thus the number density of pairs may be obtained from statistical considerations alone.

Following Landau and Lifshitz,[10] we have

$$\mu_-+\mu_+=\mu_\gamma,$$

where μ is the chemical potential for the particle concerned. The chemical potential for the photons is zero. Thus $\mu_-=-\mu_+\equiv\mu$. In the absence of interacting matter, $\mu_-=\mu_+=0$ (the same is true for $T\to\infty$). For the general case $\mu\neq0$, we have

$$n_-=n_0+n_+=\frac{8\pi}{(2\pi\hbar)^3}\int_0^\infty\frac{p^2dp}{\exp[(E-\mu)/T]+1},$$
$$n_+=\frac{8\pi}{(2\pi\hbar)^3}\int_0^\infty\frac{p^2dp}{\exp[(E+\mu)/T]+1}. \tag{26}$$

In the nondegenerate case $\exp[(E\pm\mu)/(kT)]\gg1$ in the region where the maximum of the integrand occurs. Thus

$$n_0+n_+=\frac{1}{\pi^2\hbar^3}\exp(\mu/T)\int_0^\infty\exp(-E/T)p^2dp,$$
$$n_+=\frac{1}{\pi^2\hbar^3}\exp(-\mu/T)\int_0^\infty\exp(-E/T)p^2dp. \tag{27}$$

The solution is:

$$n_+=-(n_0/2)+[(n_0/2)^2+n_v{}^2]^{\frac{1}{2}}, \tag{28}$$

where

$$n_v\equiv\frac{m^3c^3}{\pi^2\hbar^3}f(\beta),$$
$$f(\beta)\equiv\int_0^\infty\exp[-\beta(1+x^2)^{\frac{1}{2}}]x^2dx,\quad \beta\equiv mc^2/kT, \tag{29}$$

and n_v is the value of n_+ in the limit $n_0\to0$ (vacuum). On writing $x=\sinh\theta$ and partially integrating $f(\beta)$, we

[10] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (translation by R. F. Peierls and E. Peierls, Pergamon Press, New York, 1958), p. 325 ff.

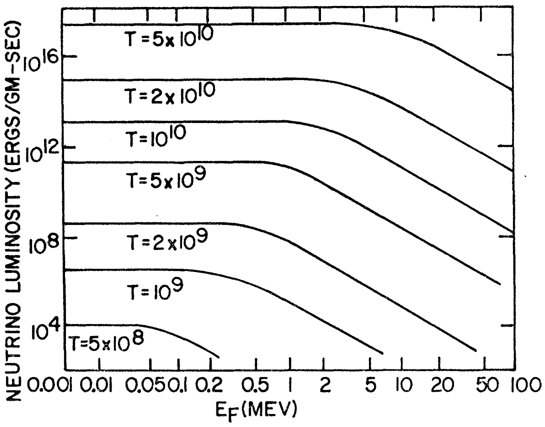

FIG. 3. The rate of energy loss in the photoneutrino process as a function of the electron Fermi energy.

have

$$f(\beta)=\frac{1}{\beta}\int_0^\infty e^{-\beta\cosh\theta}\cosh 2\theta d\theta$$
$$=\frac{1}{\beta}K_2(\beta), \tag{30}$$

where $K_2(\beta)$ is the modified Hankel function of second order. Tabulated forms of $K_2(\beta)$ can be found in existing literature.[11] For $\beta\rightarrow\infty$ we write, in Eq. (29), $(1+x^2)^{\frac{1}{2}}\approx 1+\frac{1}{2}x^2$ and obtain

$$f(\beta)\rightarrow\frac{\pi^{\frac{1}{2}}}{4}\left(\frac{2}{\beta}\right)^{\frac{3}{2}}e^{-\beta}\equiv f_a(\beta). \tag{31}$$

For $\beta\rightarrow 0$ we write $(1+x^2)^{\frac{1}{2}}\approx x$, and obtain

$$f(\beta)\rightarrow\frac{2}{\beta^3}\equiv f_b(\beta). \tag{32}$$

Since $1+\frac{1}{2}x^2>(1+x^2)^{\frac{1}{2}}>x$, we have

$$f_a(\beta)<f(\beta)<f_b(\beta). \tag{33}$$

The degenerate case $\mu/kT\gg 1$ can be solved in the limit $n_0\gg n_+$. Then $\mu\rightarrow E_F$, the Fermi energy for degenerate electrons. From Eq. (26) we have

$$n_+\approx\frac{m^3c^3}{\pi^2\hbar^3}\exp\left(-\frac{E_F}{T}\right)f(\beta)=\exp\left(-\frac{E_F}{T}\right)n_v. \tag{34}$$

It is interesting to note that for the nondegenerate case the product $n_+n_-=n_v^2$ is independent of n_0, although n_+ and n_- each depend on n_0. This result cannot be extended to the case $E_F\gtrsim kT$.

B. Energy Loss Rate

The transition amplitude for the pair annihilation process (4) clearly involves only the weak coupling $(\bar{\nu}e)(\bar{e}\nu)$. The matrix element thus consists of a single term and may be written:

$$M=(8)^{\frac{1}{2}}G(\bar{u}_\alpha\gamma_\mu a u_\beta)(\bar{u}_{P'}\gamma_\mu a u_P) \tag{35}$$

in the same notation as Eq. (7) above. The cross section is easily obtained and is

$$\sigma v=\frac{G^2}{6\pi EE'}[m^4+3m^2(P\cdot P')+2(P\cdot P')^2], \tag{36}$$

where v is the relative velocity of the pair. Equation (36) reduces to the previously given result[6] in the c.m. system

$$\sigma v=1.5\times 10^{-45}[(2E/m)^2-1]\text{ cm}^2. \tag{37}$$

The mean energy loss per annihilation is just $\mathcal{E}=(E+E')\sigma$. Substituting $\mathcal{E}$ and the positron and electron number densities into Eq. (19), we have

$$\mathcal{E}_t(\text{ND})=\frac{2G^2m^9}{\pi^5\rho}\left[\frac{K_2(\beta)}{\beta}\right]^2,$$
$$\mathcal{E}_t(\text{ED})=\frac{2G^2m^6}{\pi^3M\mu_e}\frac{K_2(\beta)}{\beta}\exp\left[\frac{-(2mE_F+m^2)^{\frac{1}{2}}}{T}\right], \tag{38}$$

where E_F is the nonrelativistic Fermi energy. In obtaining Eqs. (38) we have approximated the total neutrino energy by $2mc^2$ so that Eqs. (38) are valid only in the nonrelativistic and semirelativistic regions.

The extreme relativistic results are obtained by the use of Eq. (32) in the integral over the initial momenta distributions. We find

$$\mathcal{E}_t(\text{ND,ER})=\frac{7G^2m^9}{6\pi\rho}\left(\frac{T}{m}\right)^9. \tag{39}$$

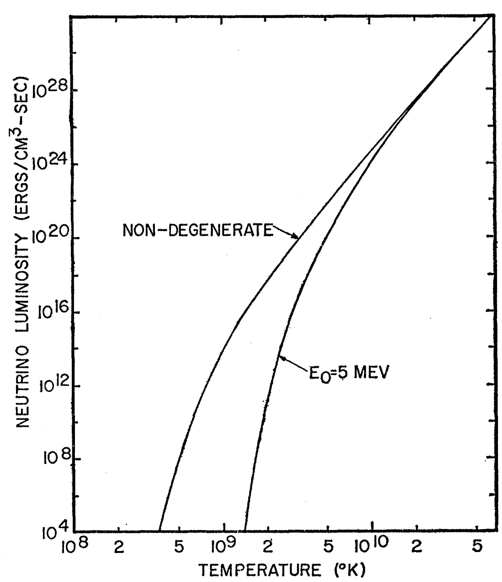

FIG. 4. The rate of energy loss in the pair annihilation process as a function of temperature.

[11] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, 1945), 2nd ed., p. 182 ff.

TABLE I. Summary of photoneutrino and pair annihilation energy loss rates. (ρ in g/cm^3; $T_n \equiv T/10^n$.)

Temperature and density region ($c=k=1$)	Photoneutrino energy loss (ergs/g-sec)	Pair annihilation energy loss (ergs/g-sec)
Nonrelativistic, nondegenerate ($E_F \lesssim T \ll m$)	$(8\times10^6/\mu_e)T_9{}^8$	$(4.8\times10^{18}/\rho)T_9{}^3e^{-2m/T}$
Extreme relativistic, nondegenerate ($m \ll T$, $E_F \lesssim T$)	$(2\times10^{13}/\mu_e)T_{10}{}^6[\log_{10}T_{10}+1.6]$	$(4.3\times10^{24}/\rho)T_{10}{}^9$
Nonrelativistic, extreme degenerate ($T \ll E_F \ll m$)	$(<10^2)$	$(4.5\times10^6 T_8{}^{\frac{3}{2}}/\mu_e)\exp[-(2m+E_F)/T]$
Extreme relativistic, extreme degenerate (T, $m \ll E_F$)	$(5\times10^5/\mu_e)(1+5.0T_9{}^2)T_9{}^7(m/E_F)^3$	$(1.4\times10^{11}/\mu_e)T_9{}^4(E_F/m)^2\exp(-E_F/T)$

In Fig. 4 we show the pair annihilation energy loss rate in ergs/cm^3 sec for the nondegenerate case and for $E_F=5$ Mev. The rates must be divided by ρ to obtain the loss rate per gram of matter.

IV. CONCLUSIONS

All the limiting forms for the pair annihilation and photo-neutrino luminosities are summarized in Table I. It should be noted that the ER-ED results apply for the case $T<m\ll E_F$ as well as $m<T\ll E_F$. Except in anomalous cases the pair annihilation energy loss greatly exceeds that of the photoneutrino process.

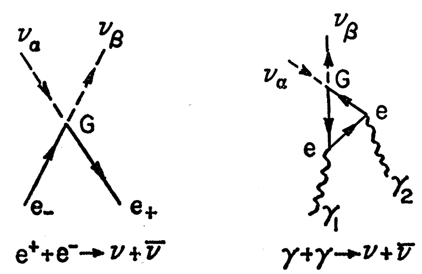

FIG. 5. Feynman diagrams for the pair annihilation process, $e^- + e^+ \to \nu + \bar{\nu}$, and the photon-photon process, $\gamma+\gamma \to \nu+\bar{\nu}$.[6a]

We will not discuss in detail here the effect of this neutrino emission on stellar evolution.[12] A calculation of the rates for the photon-photon processes, $\gamma+\gamma\to\nu+\bar{\nu}$[6a] (see Fig. 5) and $\gamma+\gamma\to\gamma+\nu+\bar{\nu}$, is necessary before the total neutrino energy loss can be determined. Present estimates[6] give contributions from these processes of the same order as those from the pair annihilation process. The pair annihilation neutrino emission alone, however, will greatly accelerate the evolution of a stellar core. If we take the ER-ND results for $\mathcal{E}_t$, and aT^4 for the specific heat per unit volume, the cooling time is

$$\tau \approx \frac{C_v}{\mathcal{E}_t} \approx 2\times10^6 T_9{}^{-5}\rho \text{ sec.} \tag{40}$$

From Fig. 4 we see that Eq. (40) is accurate for $T>2.5\times10^9$ °K providing the electrons are nondegenerate.

τ may be compared with the time for free-fall collapse,

$$\tau_{\text{ff}} \approx (G_{\text{gr}}\rho)^{-\frac{1}{2}}, \tag{41}$$

where G_{gr} is the gravitation constant. On setting $\tau\approx\tau_{\text{ff}}$, we see that collapse occurs when

$$T_9{}^5 \approx 5\times10^5 \rho_6{}^{\frac{1}{2}}, \tag{42}$$

where $\rho_6 \equiv \rho/10^6$ g-cm^{-3}. This temperature is generally higher than the temperature at which the "phase change" from iron to helium conversion occurs, or 5–7$\times10^9$ °K.

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APPENDIX

The following identities have been used in the evaluation of the traces for the photoneutrino process:

$$\text{Tr}[\gamma_\mu \boldsymbol{A}\gamma_\nu \boldsymbol{B}]\,\text{Tr}[\gamma_\mu \boldsymbol{C}\gamma_\nu \boldsymbol{D}] = 32[(A\cdot C)(B\cdot D)+(A\cdot D)(B\cdot C)],$$

$$\text{Tr}[\gamma_\mu \boldsymbol{A}\gamma_\nu \boldsymbol{B}]\,\text{Tr}[\gamma_\mu \boldsymbol{C}\gamma_\nu \boldsymbol{D}\gamma_5] = 0,$$

$$\text{Tr}[\gamma_\mu \boldsymbol{A}\gamma_\nu \boldsymbol{B}\gamma_5]\,\text{Tr}[\gamma_\mu \boldsymbol{C}\gamma_\nu \boldsymbol{D}\gamma_5] = 32[(A\cdot C)(B\cdot D)-(A\cdot D)(B\cdot C)],$$

$$\begin{aligned}\text{Tr}[\boldsymbol{ABCD}\gamma_5]\,\text{Tr}[\boldsymbol{MNPQ}\gamma_5] &= 16(A_\alpha B_\beta C_\gamma D_\delta \epsilon_{\alpha\beta\gamma\delta})(M_\mu N_\nu P_\rho Q_\sigma \epsilon_{\mu\nu\rho\sigma})\\ &= 16[(A\cdot M)(B\cdot N)(C\cdot P)(D\cdot Q)-(A\cdot M)(B\cdot N)(C\cdot Q)(D\cdot P)\\ &\quad +(A\cdot M)(B\cdot P)(C\cdot Q)(D\cdot N)-(A\cdot M)(B\cdot P)(C\cdot N)(D\cdot Q)+\cdots\\ &\quad -(A\cdot Q)(B\cdot P)(C\cdot N)(D\cdot M)].\end{aligned}$$

In the last equation there are 24 terms on the right-hand side corresponding to all indicated permutations. $\epsilon_{\alpha\beta\gamma\delta}$ is the completely antisymmetric unit tensor of rank 4.

[12] See reference 6.