which may be written, with the help of Eq. (6):

$$\omega_1 \int D^{-2}\Delta F_0 d^3k = \tfrac{1}{2}\omega_p{}^2 \frac{m}{\hbar} \int d^3k d^3k' \times \frac{\Delta F_0 \Delta F_0'}{|\mathbf{k}-\mathbf{k}'|^2}\left(\frac{1}{DD'}-\frac{1}{D^2}\right). \quad (10)$$

In the long-wavelength limit, an expansion of the terms in Eq. (10) in powers of K is allowed. Keeping only the lowest order terms, Eq. (10) goes over into

$$\omega_1 = \frac{\omega_p{}^2}{8K^2\omega_0} \int d^3k d^3k' [\mathbf{K}\cdot(\mathbf{k}-\mathbf{k}')]^2 |\mathbf{k}-\mathbf{k}'|^{-2} \times \mathbf{K}\cdot\nabla_k F_0(\mathbf{k}) \mathbf{K}\cdot\nabla_{k'} F_0(\mathbf{k}'). \quad (11)$$

The integral in Eq. (11) is easily evaluated at 0°K temperature and the result is:

$$\omega_1 = -\frac{3}{40}\frac{K^2}{k_F{}^2}\frac{\omega_p{}^2}{\omega_0}, \quad (12)$$

where k_F is the Fermi momentum. But since ω_0 is given by the unperturbed plasma frequency ω_p plus small correction terms of order K^2, etc., we see finally that

$$\begin{aligned}\omega^2 &= (\omega_0+\omega_1)^2 = \omega_0{}^2 + 2\omega_0\omega_1 \\ &= \omega_0{}^2 - (3/20)(K^2/k_F{}^2)\omega_p{}^2,\end{aligned} \quad (13)$$

identical with the result of Kanazawa *et al.*[2]

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Analytic Properties of Single-Particle Propagators for Many-Fermion Systems*

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Certain general properties of single-particle propagators for a system of interacting fermions are derived. In addition, the properties of the proper self-energy part $G_k(\zeta)$ which were used in previous work on the ground-state energy and on the Fermi surface are established. In particular, the fact that to all orders of perturbation theory in the interaction, Im $G_k(x-i0^+)$ behaves like $C_k(x-\mu)^2$ $(C_k>0)$ for x very near μ, is proved.

1. GENERAL DISCUSSION OF THE PROPAGATOR

In some recent work[1] on the theory of a system of interacting fermions, certain analytical properties of the so-called "single-particle propagator" were made use of. No proof of those properties was given at that time. It is the purpose of this brief note to establish these properties. For simplicity we shall restrict ourselves to the case of spinless fermions interacting among themselves, but not moving in an external potential. The resulting simplification is mainly notational, and there is no difficulty in extending our results to the more complicated cases.

The single-particle propagator as used in LW was defined as the sum (with appropriate coefficients) of all connected diagrams having a single line entering and leaving. For the purposes of general discussion it is often convenient to have an explicit closed expression for it. As is well known in field theory, such an expression is given as follows.[2] Consider the quantity

$$S_k'(u,u') \equiv \langle T[a_k^\dagger(u)a_k(u')]\rangle. \quad (1)$$

In (1) the quantity a_k is the destruction operator for a particle of momentum k,

$$a_k(u') = e^{u'H}a_k e^{-u'H}, \quad a_k^\dagger(u) = e^{uH}a_k^\dagger e^{-uH}; \quad (2)$$

H is the total Hamiltonian of the system and the angular bracket represents the average of the enclosed quantity with respect to the grand canonical distribution

$$\langle A\rangle \equiv \mathrm{Tr}\,(e^{\beta(\Omega-H-\mu N)}A), \quad \beta = 1/kT. \quad (3)$$

The operation T is the usual Wick chronological operator meaning

$$\begin{aligned} T[a_k^\dagger(u)a_k(u')] &= a_k^\dagger(u)a_k(u'), \quad u>u' \\ &= -a_k(u')a_k^\dagger(u), \quad u<u'. \end{aligned} \quad (4)$$

Equation (1) provides an expression for the propagators in the "temperature" variables u,u', which are constrained to vary between zero and β. From (1) we see that $S_k'(u,u')$ is a function of $u-u'\equiv v$ only:

$$S_k'(v) = \mathrm{Tr}\, e^{\beta(\Omega-H-\mu N)} \begin{cases} e^{vH}a_k^\dagger e^{-vH}a_k, & \beta>v>0 \\ -a_k e^{vH}a_k^\dagger e^{-vH}, & -\beta<v<0. \end{cases} \quad (5)$$

Using (5), we see at once that the quantity

$$S_k'(v)e^{-(i\pi/\beta+\mu)v}$$

is a periodic function of v of period β in the interval

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[1] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960). (We shall refer to this paper as LW.) J. M. Luttinger, Phys. Rev. **119**, 1153 (1960). We shall follow the notation of these papers as far as is practiced.

[2] The representation we shall use here is essentially the same as that of A. A. Abrikosov, L. P. Gorkov, and I. E. Dsyaloshinskii, Soviet Phys.-JETP **36** (9), 636 (1959), except for minor differences of notation and definition.

$(-\beta,\beta)$, and therefore may be taken as periodic everywhere, since only this interval comes into our results. [To establish this we need only compare $S_k'(v)$ at v and $v-\beta$ $(0<v<\beta)$, making use of the properties of destruction operators and the cyclic invariance of the trace.] Therefore we may write

$$S_k'(v)=\frac{1}{\beta}\sum_{l=-\infty}^{\infty} S_k'(\zeta_l)e^{\zeta_l v}, \tag{6}$$

where

$$\zeta_l=\frac{2\pi i}{\beta}l+\frac{i\pi}{\beta}+\mu. \tag{7}$$

The "Fourier coefficient" $S_k'(\zeta_l)$ is exactly what we called the true single-particle propagator in LW. We want to investigate its analytic properties as a function of the complex variable ζ, in the limit of zero temperature. From (6) we have

$$S_k'(\zeta_l)=\int_0^\beta e^{-\xi_l v}S_k'(v)dv. \tag{8}$$

To obtain a more useful expression for $S_k'(\zeta_l)$ we introduce the exact eigenstates of H:

$$H\psi_{N\alpha}=E_{N\alpha}\psi_{N\alpha}. \tag{9}$$

$\psi_{N\alpha}$ is the exact eigenfunction of an N-particle system; α being all the other quantum numbers necessary to specify the state completely. Then, for $v>0$, we may write

$$\begin{aligned}S_k'(v)&=\sum_{N\alpha} e^{\beta(\Omega-E_{N\alpha}+\mu N)}(N\alpha|e^{vH}a_k^\dagger e^{-vH}a_k|N\alpha)\\ &\sum_{N\alpha\alpha'} e^{\beta(\Omega-E_{N\alpha}+\mu N)}e^{v(E_{N\alpha}-E_{N-1,\alpha'})}B_{N\alpha\alpha'}{}^k,\end{aligned} \tag{10}$$

where

$$\begin{aligned}B_{N\alpha\alpha'}{}^k&\equiv|(N\alpha|a_k^\dagger|N-1\alpha')|^2\\ &=|(N-1\alpha'|a_k|N\alpha)|^2\geqslant 0.\end{aligned} \tag{11}$$

Inserting (10) in (8) we obtain, after a little rearrangement,

$$S_k'(\zeta_l)=\sum_{N\alpha\alpha'} e^{\beta(\Omega-E_{N\alpha}-\mu N)} \left\{\frac{B_{N\alpha\alpha'}{}^k}{\zeta_l-(E_{N\alpha}-E_{N-1\alpha'})}+\frac{B_{N+1\alpha'\alpha}{}^k}{\zeta_l-(E_{N+1\alpha'}-E_{N\alpha})}\right\}, \tag{12}$$

where

$$B_{N+1\alpha'\alpha}{}^k=|(N+1\alpha'|a_k^\dagger|N\alpha)|^2\geq 0. \tag{13}$$

In the limit of zero temperature, ζ_l becomes a continuous variable (which we shall denote by ζ from now on). Further, for an N-particle system only the ground state (denoted by $\alpha=0$) will contribute to the sum in (12), the others being exponentially smaller. Therefore, in the zero-temperature limit we may write (12) as

$$S_k'(\zeta)=\sum_{\alpha'}\left\{\frac{|(N-1\alpha'|a_k|N0)|^2}{\zeta-(E_{N0}-E_{N-1\alpha'})}+\frac{|(N+1\alpha'|a_k^\dagger|N0)|^2}{\zeta-(E_{N+1\alpha'}-E_{N0})}\right\}, \tag{14}$$

where E_{N0} is the exact ground-state energy of the N-particle system. The expression (14) for $S_k'(\zeta)$ may be written in another form, which is sometimes useful:

$$S_k'(\zeta)=\left(N0\left|a_k^\dagger\frac{1}{\zeta-E_{N0}+H}a_k+a_k\frac{1}{\zeta-H+E_{N0}}a_k^\dagger\right|N0\right). \tag{15}$$

Certain analytic properties of the propagator may be seen at once from (14). Since all the energy differences in the denominators are real numbers, the expression for $S_k'(\zeta)$ can only become singular if ζ is on the real axis. Therefore *$S_k(\zeta)$ is analytic everywhere in the complex ζ plane with the possible exception of the real axis.* On the real axis, in the limit of an infinitely large system, each point is usually a limit point of poles, and therefore the real axis is in general a branch line.

It is sometimes convenient to decompose $S_k'(\zeta)$ into two parts,

$$S_k'\equiv S_k^+ + S_k^-, \tag{16}$$

$$S_k^+(\zeta)\equiv\sum_{\alpha'}\frac{|(N+1\alpha'|a_k^\dagger|N0)|^2}{\zeta-(E_{N+1\alpha'}-E_{N0})}, \tag{17}$$

$$S_k^-(\zeta)\equiv\sum_{\alpha'}\frac{|(N-1\alpha'|a_k|N0)|^2}{\zeta-(E_{N0}-E_{N-1\alpha'})}. \tag{18}$$

Since the ground-state energy of a system with $N+1$ particles exceeds that of a system of N particles by the chemical potential μ, $E_{N+1\alpha'}-E_{N0}\geq\mu$. Similarly $E_{N0}-E_{N-1\alpha'}\leq\mu$. Therefore the function S_k^+ is analytic everywhere except on the portion of the real axis between μ and ∞, while S_k^- is analytic everywhere except on the portion of the real axis from $-\infty$ to μ.

If we go over to an infinite system so that the energy levels become continuous, we may write

$$S_k^+(\zeta)=\int_\mu^\infty \frac{\rho_k^+(\xi)}{\zeta-\xi}d\xi, \tag{19}$$

$$S^-(\zeta)=\int_{-\infty}^\mu \frac{\rho_k^-(\xi)}{\zeta-\xi}d\xi, \tag{20}$$

where

$$\rho_k^\pm(\xi)\geq 0. \tag{21}$$

Introducing a function

$$\begin{aligned}\rho_k(\xi)&=\rho_k^+(\xi),\quad \xi>\mu\\ &=\rho_k^-(\xi),\quad \xi<\mu,\end{aligned} \tag{22}$$

we may write

$$S_k'(\zeta)=\int_{-\infty}^{\infty}\frac{\rho_k(\xi)}{\zeta-\xi}d\xi, \tag{23}$$

$$\rho_k(\xi)\geq 0.$$

We call the expression (23) the *spectral representation* of the propagator and the quantity $\rho_k(\xi)$ the *spectral density*. For noninteracting particles $\rho_k=\delta(\xi-\epsilon_k)$.

Another simple property that follows at once from (14) is that for $|\zeta|$ very large

$$\begin{aligned}S_k'(\zeta)&=\frac{1}{\zeta}\sum_{\alpha'}(|(N-1\alpha'|a_k|N0)|^2\\&\qquad+|(N+1\alpha'|a_k^\dagger|N0)|^2)\\&=\frac{1}{\zeta}(N0|a_k^\dagger a_k+a_ka_k^\dagger|N0)\\&=1/\zeta.\end{aligned} \tag{24}$$

Equivalent to this is the result

$$\int_{-\infty}^{\infty}\rho_k(\xi)d\xi=1, \tag{25}$$

which we obtain from (23).

The spectral density is related to the discontinuity of $S_k'(\zeta)$ as we cross the real axis. Consider $\zeta=x+i\eta$ as η approaches zero.

$$\begin{aligned}\lim_{\eta\to 0}S_k'(x+i\eta)&=\lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{\rho_k(\xi)}{x-\xi+i\eta}d\xi\\&=\int_{-\infty}^{\infty}P\frac{\rho_k(\xi)}{x-\xi}\mp i\pi\frac{\eta}{|\eta|}\int_{-\infty}^{\infty}\rho_k(\xi)\delta(x-\xi)d\xi.\end{aligned}$$

Therefore

$$S_k'(x-i0^+)-S_k'(x+i0^+)=2\pi i\rho_k(x). \tag{26}$$

That is, the real part of S_k' is continuous as we cross the real axis, while the imaginary part undergoes a jump which is proportional to the spectral density at that point.

From the reality of everything but ζ in (14), it follows that

$$[S_k'(\zeta)]^*=S_k'(\zeta^*), \tag{27}$$

so that the values of the function in (say) the lower half plane are just the complex conjugates of those at the mirror image point in the upper half plane.

Lastly we have

$$\begin{aligned}S_k'(x+iy)&=\int_{-\infty}^{\infty}\frac{\rho_k(\xi)}{x-\xi+iy}d\xi\\&=\int_{-\infty}^{\infty}\frac{\rho_k(\xi)(x-\xi)}{(x-\xi)^2+y^2}-iy\int_{-\infty}^{\infty}\frac{\rho_k(\xi)}{(x-\xi)^2+y^2}d\xi.\end{aligned} \tag{28}$$

Therefore

$$\operatorname{Im}S_k'(x+iy)=-y\int_{-\infty}^{\infty}\frac{\rho_k(\xi)}{(x-\xi)^2+y^2}d\xi. \tag{29}$$

Since the integrand in (29) is positive we have

$$\begin{aligned}\operatorname{Im}S_k'(x+iy)&>0\quad\text{if } y<0\\&<\quad\text{if } y>0.\end{aligned} \tag{30}$$

2. ANALYTIC PROPERTIES OF THE PROPER SELF-ENERGY PART

In LW we wrote the propagator in the form

$$S_k'(\zeta)=\frac{1}{\zeta-\epsilon_k-G_k(\zeta)}, \tag{31}$$

where ϵ_k is the unperturbed single-particle energy and $G_k(\zeta)$ was the proper self-energy part. $G_k(\zeta)$ had a very simple expression in terms of diagrams, and some of its properties played an essential role in the above-mentioned work.

Since

$$G_k(\zeta)=\zeta-\epsilon_k-[S_k'(\zeta)]^{-1}, \tag{32}$$

it follows that $G_k(\zeta)$ is regular at infinity, since from (24)

$$[S_k'(\zeta)]^{-1}=\zeta+a_k+b_k/\zeta+\cdots, \tag{33}$$

and therefore

$$G_k(\zeta)=-(\epsilon_k+a_k+b_k/\zeta+\cdots). \tag{34}$$

Further, since the imaginary part of $S_k'(\zeta)$ never vanishes unless we are on the real axis [see (29)], $S_k'(\zeta)$ can have no complex zeros. Therefore $[S_k'(\zeta)]^{-1}$ can have no complex poles. From the analyticity of $S_k'(\zeta)$ it then follows that $G_k(\zeta)$ *is analytic everywhere in the complex plane with the possible exception of the real axis.*

From (32) and (27) we also have at once that

$$[G_k(\zeta)]^*=G_k(\zeta^*). \tag{35}$$

We now obtain a spectral representation for $G_k(\zeta)$ analogous to that of $S_k'(\zeta)$. By means of (34) we see that for large $|\zeta|$, $G_k(\zeta)$ becomes a constant g_k, which is real from (35). Then the function

$$\bar{G}_k(\zeta)\equiv G_k(\zeta)-g_k \tag{36}$$

is analytic everywhere in the upper half plane and vanishes for large (ζ). If we apply Cauchy's theorem to a contour consisting of a line just above the real axis and closed by an infinite semicircle in the upper half plane, we obtain at once

$$\bar{G}_k(\zeta)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\bar{K}_k(\xi)-i\bar{J}_k(\xi)}{\xi-\zeta}d\xi, \tag{37}$$

where we have put

$$\lim_{\eta\to 0^+}\bar{G}_k(\xi+i\eta)=\bar{K}_k(\xi)-i\bar{J}_k(\xi). \tag{38}$$

If we apply (37) to a point $\zeta=\xi'+i\eta'$ immediately above the real axis we get

$$\bar{K}_k(\xi')-i\bar{J}_k(\xi')=\frac{1}{2\pi i}\int_{-\infty}^{\infty}[\bar{K}_k(\xi)-i\bar{J}_k(\xi)]\times\left(P\frac{1}{\xi-\xi'}+i\pi\delta(\xi-\xi')\right)d\xi',$$

or

$$\bar{K}_k(\xi')-i\bar{J}_k(\xi')=\frac{1}{\pi i}\int_{-\infty}^{\infty}[\bar{K}_k(\xi)-i\bar{J}_k(\xi)]P\frac{1}{\xi-\xi'}d\xi'. \tag{39}$$

Equating real and imaginary parts on both sides of (39) we get the "dispersion" relations

$$\bar{K}_k(\xi')=-\frac{1}{\pi}\int_{-\infty}^{\infty}P\frac{\bar{J}_k(\xi)}{\xi-\xi'}d\xi, \tag{40}$$

$$\bar{J}_k(\xi')=\frac{1}{\pi}\int_{-\infty}^{\infty}P\frac{\bar{K}_k(\xi)}{\xi-\xi'}d\xi. \tag{41}$$

From (40) we have

$$\int_{-\infty}^{\infty}\frac{\bar{K}_k(\xi')}{\xi'-\zeta}d\xi'=-\frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\xi'\times\bar{J}_k(\xi)\frac{1}{(\xi'-\zeta)}P\frac{1}{\xi-\xi'}. \tag{42}$$

We may write

$$P\frac{1}{\xi-\xi'}=\frac{1}{2}\lim_{\eta\to0^+}\left(\frac{1}{\xi-\xi'-i\eta}+\frac{1}{\xi-\xi'-i\eta}\right). \tag{43}$$

Therefore

$$\int_{-\infty}^{\infty}\frac{\bar{K}_k(\xi')}{\xi'-\zeta}d\xi'=-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\,\bar{J}_k(\xi)\times\lim_{\eta\to0}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\zeta}\left(\frac{1}{\xi-\xi'+i\eta}+\frac{1}{\xi-\xi'-i\eta}\right)=i\int_{-\infty}^{\infty}d\xi\frac{\bar{J}_k(\xi)}{\zeta-\xi} \tag{44}$$

on closing below.[3] Therefore (37) becomes

$$\bar{G}_k(\xi)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\bar{J}_k(\xi)}{\zeta-\xi}d\xi. \tag{45}$$

[3] The relationship (44) may also be obtained by integrating $\bar{G}_k(\zeta')|(\zeta-\zeta')$ on ζ', along a contour consisting of a line just below the real axis and closed by an infinite semicircle in the lower half plane. For ζ in the upper half plane this gives zero, and leads at once to (44).

Writing (45) in terms of the original proper self-energy part $G_k(\xi)$, we get

$$G_k(\xi)=g_k+\int_{-\infty}^{\infty}\frac{\sigma_k(\xi)}{\zeta-\xi}d\xi, \tag{46}$$

where

$$\sigma_k(\xi)=\frac{1}{\pi}J_k(\xi), \tag{47}$$

$$G_k(x+i0^+)=K_k(x)-iJ_k(x). \tag{48}$$

Equation (46) is the spectral representation of $G_k(\zeta)$ and $\sigma_k(\xi)$ is its spectral density.

The spectral density $\sigma_k(\xi)$ is non-negative. To see this we specialize (32) for $\zeta=x\pm i\eta$, as $\eta\to0^+$,

$$\lim_{\eta\to0^+}\frac{1}{i}[G_k(x-i\eta)-G(x+i\eta)]=\lim_{\eta\to0^+}\left\{\frac{1}{S_k'(x+i\eta)}-\frac{1}{S_k'(x-i\eta)}\right\}\frac{1}{i}\geq0$$

by (30). Therefore

$$J_k(x)\geq0. \tag{49}$$

From (46) we then have $\sigma_k(\xi)\geq0$.

Using the same argument that led to (30) from (23) we can extend (49), and obtain

$$\begin{aligned}&\operatorname{Im}G_k(\zeta)>0,\quad \operatorname{Im}\zeta<0,\\&\operatorname{Im}G_k(\zeta)<0,\quad \operatorname{Im}\zeta>0.\end{aligned} \tag{50}$$

3. BEHAVIOR OF $J_k(x)$ NEAR $x=\mu$

The analytic properties of $G_k(\zeta)$ which we have discussed till now are in essence rather trivial, being direct consequences of the expression for the propagator at zero temperature as an average over the exact ground state of certain operator [see (15)]. No use was made of perturbation theory, and all the results given in the previous section are independent of the nature of the interaction between the particles, it being assumed only that there is a lowest state.

We now consider another property, of which extensive use was made in LW. This is that as x approaches μ,

$$J_k(x)=C_k(x-\mu)^2,\quad C_k\geq0. \tag{51}$$

We have not succeeded in finding necessary and sufficient conditions on the interaction between the particles for which (51) is valid. It certainly cannot be valid in general because one of its consequences[1] is the existence of a sharp Fermi surface, which is *certainly* not present for some systems of fermions with attractive forces between the particles.[4]

[4] Examples are a crystal of molecular deuterium (the deuterium atoms being fermions) and the Bardeen, Schrieffer, Cooper ground state in the theory of superconductivity.

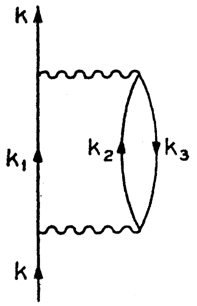

FIG. 1. A simple second order diagram contributing to the proper self-energy part.

We shall now demonstrate (51) under the assumption that one can use perturbation theory on the strength of the interaction. According to LW(40), $G_k(\zeta)$ is given by

$G_k(\zeta)=[$all possible skeleton diagrams with the unperturbed propagator S_k replaced by the true propagator S_k']. (52)

The skeleton diagrams are all proper self-energy diagrams without self-energy parts inserted into any of the particle lines. Examples are found in Fig. 2(b) and Fig. 2(c) of LW, the last two diagrams of Fig. 2(c) *not* being allowed skeleton diagrams for $G_k(\zeta)$.

To see what is involved in the proof let us first consider the simplest diagrams. The first order diagrams of Fig. 2(b) in LW give rise simply to real numbers independent of ζ and therefore do not contribute to $J_k(x)$. The lowest order diagram which contributes is second order, the simplest one being illustrated in Fig. 1. Its contribution $G_k^a(\zeta)$ is proportional to

$$\left(\frac{1}{2\pi i}\right)^2 \sum_{k_1k_2k_3} |(kk_3|v|k_1k_2)|^2 \times\int d\zeta_1\int_{\mu-i\infty}^{\mu+i\infty} d\zeta_2\int d\zeta_3 \frac{\delta(\zeta_1+\zeta_2-\zeta_3-\zeta)}{(\zeta_1-\epsilon_{k_1})(\zeta_2-\epsilon_{k_2})(\zeta_3-\epsilon_{k_3})}, \quad (53)$$

according to the general rules of LW. Doing the integral in (53) we obtain

$$G_k^a(\zeta)\propto \sum_{k_1k_2k_3} |(kk_3|v|k_1k_2)|^2 \times\frac{\theta^-(\epsilon_{k_1})\theta^-(\epsilon_{k_2})\theta^+(\epsilon_{k_3})+\theta^+(\epsilon_{k_1})\theta^+(\epsilon_{k_2})\theta^-(\epsilon_{k_3})}{\zeta-(\epsilon_{k_1}+\epsilon_{k_2}-\epsilon_{k_3})}, \quad (54)$$

where

$$\begin{aligned}\theta^+(x)&=1, \quad x>\mu\\ &=0, \quad x<\mu,\\ \theta^-(x)&=1, \quad x<\mu\\ &=0, \quad x>\mu.\end{aligned} \quad (55)$$

Therefore writing

$$\lim_{\eta\to 0^+} G_k^a(x-i\eta)=K_k^a(x)+iJ_k^a(x), \quad (56)$$

we get for $J_k^a(x)$

$$J_k^{(a)}(x)\propto \sum_{k_1k_2k_3} |(kk_3|v|k_1k_2)|^2\delta(x-(\epsilon_{k_1}+\epsilon_{k_2}-\epsilon_{k_3})) \times\{\theta^-(\epsilon_{k_1})\theta^-(\epsilon_{k_2})\theta^+(\epsilon_{k_3})+\theta^+(\epsilon_{k_1})\theta^+(\epsilon_{k_2})\theta^-(\epsilon_{k_3})\}. \quad (57)$$

Now consider the part of J_k^a which arises from the first term in the curly bracket of (57). Because of the θ's we have the inequality

$$\epsilon_{k_1}+\epsilon_{k_2}-\epsilon_{k_3}\leq\mu, \quad (58)$$

the equality sign only holding at the limits of the allowed values of ϵ_{k_1}, ϵ_{k_2}, ϵ_{k_3}. Therefore, because of the δ function this term gives nothing if x is greater than μ. For $x<\mu$ but very close to it, this contribution to $J_k^a(x)$ must be very small since ϵ_{k_1}, ϵ_{k_2}, ϵ_{k_3} can only vary very slightly without making the argument of the δ-function negative. If we introduce as integration variables

$$\epsilon_{k_1}=\mu-t_1, \quad \epsilon_{k_2}=\mu-t_2, \quad \epsilon_{k_3}=\mu+t_3. \quad (59)$$

We may write the corresponding contribution to $J_k^a(x)$ for $\mu-x$ very small and positive as proportional to

$$u^2\int_0^u\int_0^u\int_0^u dt_1dt_2dt_3\,\delta(t_1+t_2+t_3-u), \quad (60)$$

where

$$u\equiv(\mu-x). \quad (61)$$

Changing the integration variables by a factor u

$$t_i=u\tau_i \quad (62)$$

(60) becomes

$$u^2\int_0^1\int_0^1\int_0^1 d\tau_1d\tau_2d\tau_3\,\delta(\tau_1+\tau_2+\tau_3-1)\propto u^2, \quad (63)$$

which is what we wanted to prove. Similarly the second term in the curly brackets of (57) vanishes for $x<\mu$ and goes as $(\mu-x)^2$ for x greater than μ but very close to it. Therefore

$$J_k^{(a)}\propto(x-\mu)^2 \quad (64)$$

for x very near μ.[5]

To complete the proof of (51) one has to do two things: generalize the treatment to include the true propagator rather than the unperturbed propagator used in (53) and generalize the treatment of (53) to arbitrary skeleton diagrams. The former is very simple.

[5] This result is not valid for one-dimensional systems. The reason is that in this case the momentum conservation implied by the matrix element $(kk_3|v|k_1k_2)$ is sufficient to determine the energy ϵ_{k_3} in terms of ϵ_{k_1} and ϵ_{k_2}, so that they no longer are independent variables. Having one less independent variable gives rise to one less factor of $(\mu-x)$, and we find $J_k\propto|\mu-x|$ instead of (64).

Instead of (53) we get

$$\left(\frac{1}{2\pi i}\right)^2 \sum_{k,k_2,k_3} |(kk_3|v|k_1k_2)|^2$$

$$\times \int\int\int_{\mu-i\infty}^{\mu+i\infty} d\zeta_1 d\zeta_2 d\zeta_3\, \delta(\zeta_1+\zeta_2-\zeta_3-\zeta)$$

$$\times S_{k_1}'(\zeta_1) S_{k_2}'(\zeta_2) S_{k_3}'(\zeta_3). \quad (65)$$

Inserting the spectral representation (23) into (65) we obtain

$$\left(\frac{1}{2\pi i}\right)^2 \sum_{k_1k_2k_3} |(kk_3|v|k_1k_2)|^2$$

$$\times \int\int\int_{\infty}^{\infty} d\xi_1 d\xi_2 d\xi_3 \rho_{k_1}(\xi_1)\rho_{k_2}(\xi_2)\rho_{k_3}(\xi_3)$$

$$\times \int\int\int_{\mu-i\infty}^{\mu+i\infty} d\zeta_1 d\zeta_2 d\zeta_3 \frac{\delta(\zeta_1+\zeta_2-\zeta_3-\zeta)}{(\zeta_1-\xi_1)(\zeta_2-\xi_2)(\zeta_2-\xi_3)}. \quad (66)$$

The ζ integrations in (66) are exactly of the same form as those in (53), the ϵ's being replaced by the ξ's. Therefore the contribution of (66) to $J_k(x)$ is proportional to

$$\sum_{k_1k_2k} |(k_1k_3|v|k_1k_2)|^2$$

$$\int\int\int_{-\infty}^{\infty} d\xi_1 d\xi_2 d\xi_3 \rho_{k_1}(\xi_1)\rho_{k_2}(\xi_2)\rho_{k_3}(\xi_3)$$

$$\times\{\theta^-(\xi_1)\theta^-(\xi_2)\theta^+(\xi_3)+\theta^+(\xi_1)\theta^+(\xi_2)\theta^-(\xi_3)\}$$

$$\times\delta(x-(\xi_1+\xi_2-\xi_3)). \quad (67)$$

Now the identical reasoning that led to (64) when applied to the ξ variables instead of the ϵ's tells us once again that the only contribution to (67) comes from ξ_i near μ. From (26) and (31) we have

$$\rho_k(\xi)=\frac{1}{\pi}\frac{J_k(\xi)}{[\xi-\epsilon_k-K_k(\xi)]^2+J_k^2(\xi)}. \quad (68)$$

For ξ very near μ, assuming for the moment that $J_k(\xi)$ does obey (51),

$$\rho_k(\xi)=\delta(\xi-\epsilon_k-K_k(\xi)). \quad (69)$$

For ξ very near μ this has solutions in k, for k very near the Fermi surface.[1] After doing the k integrations we are therefore left with an integration ξ_i, which is of the same form as our previous integrations on the ϵ's. Therefore the same reasoning that led to (64) shows that once again we obtain a contribution to $J_k(x)$ proportional to $(x-\mu)^2$. Equation (52) is actually an implicit equation for $G_k(\zeta)$; what we have shown (at least for the simplest skeleton diagram) is that if we assume the property (51) we again obtain it, so that we have found a consistent solution. It is clear that this technique is general. That is, if we can show that for all skeleton diagrams with unperturbed propagators (51) is valid, then it is valid when the true propagators are used.

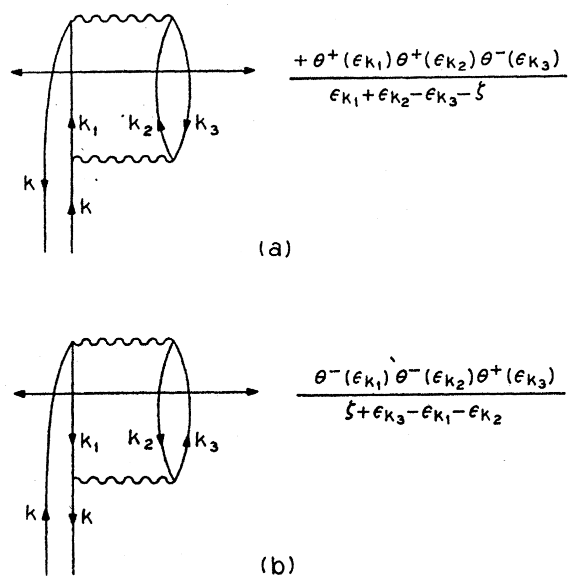

FIG. 2. Ordered diagrams corresponding to the diagram of Fig. 1. The expressions to the right of a diagram are proportional to the contribution of this ordering to the proper self-energy part.

We next have to investigate the ζ dependence of the all skeleton diagram when the unperturbed propagators are used. This is actually well known and just corresponds to using the Goldstone type of time-dependent perturbation theory[6] for $G_k(\zeta)$, dropping all diagrams with self-energy parts. It is also very easy to obtain from our propagator formalism: essentially all one has to do is to write the δ functions which represent "ζ conservation" in terms of the usual Fourier integral representation. In (53), for example, putting $\zeta_i=\mu+iy_i$, the δ function is equivalent to $\delta(y_1+y_2-y_3-y)$, which may be written

$$\delta(y_1+y_2-y_3-y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i(y_1+y_2-y_3-y)t}. \quad (70)$$

The t's which one introduces in this fashion are just the time variables of Goldstone perturbation theory. The result may be stated very simply. Draw a proper self-energy diagram with all possible vertical orderings of the interactions (n! such orderings for a diagram of nth order), the incoming and outgoing k lines being drawn from below. Each ordering contributes a term

[6] J. Goldstone, Proc. Roy. Soc. (London) A293, 267 (1957).

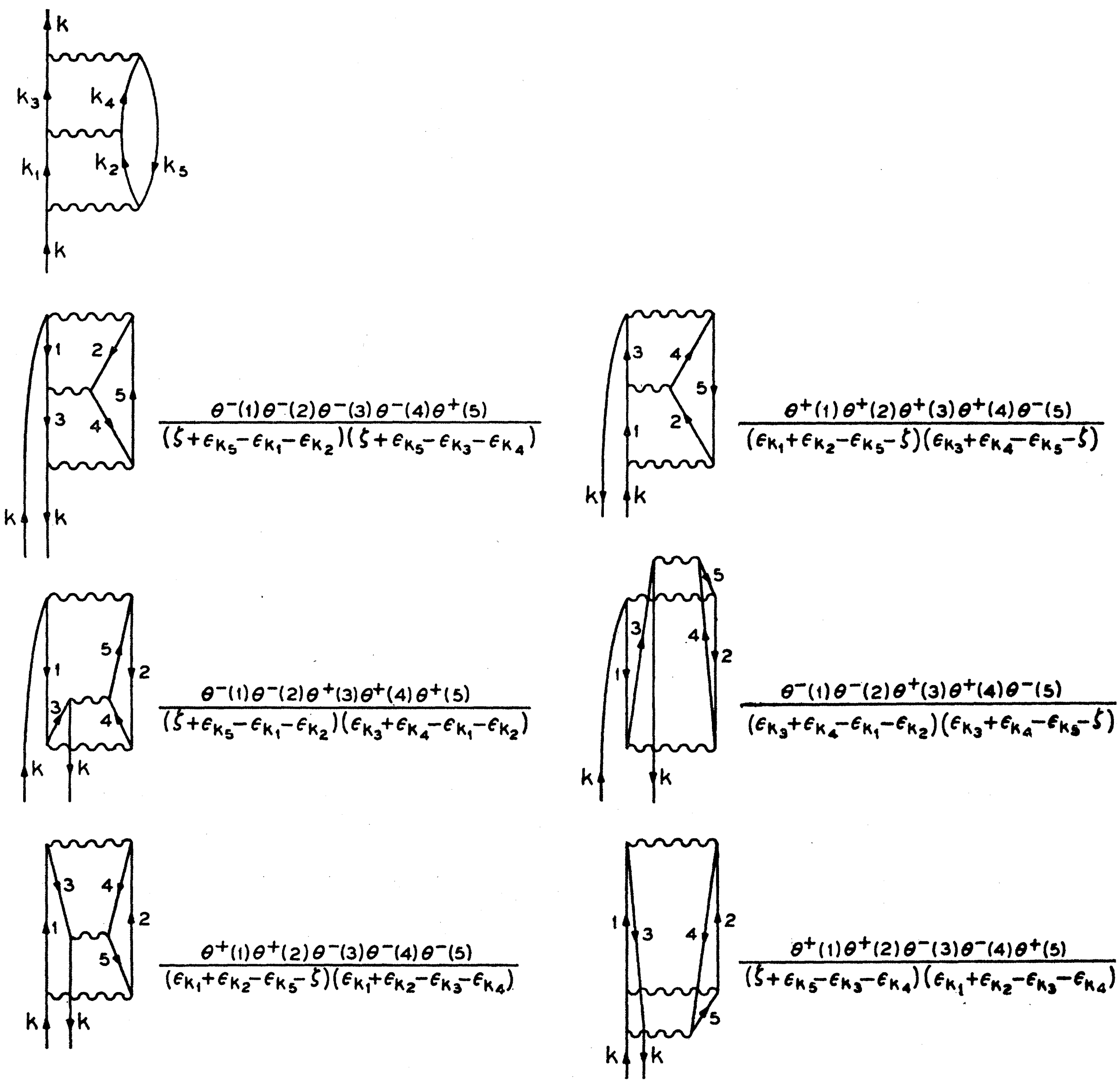

FIG. 3. A more complicated proper self-energy diagram, and all its possible orderings. The expressions to the right of an ordered diagram is proportional to the contribution of this ordering to the proper self-energy part.

to G_k which is proportional to (*a*) a factor of $\theta^+(\epsilon_{k'})$ for each ascending line k', and a factor $\theta^-(\epsilon_{k''})$ for each descending line k'' (*b*) a denominator which consists of a product of factors which represent the "energy" of the situation between all successive interactions. This "energy" is computed as follows: for each ascending line k', we have an energy $+\epsilon_{k'}$, for each descending line k'' and energy $-\epsilon_{k''}$; for an ascending k (the external momentum) an "energy" $+\zeta$, for a descending k and "energy" $-\zeta$. These results are illustrated in Fig. 2 for the simple diagram of Fig. 1. In Fig. 3 they are given for a slightly more complicated diagram.

The fact that we are not including self-energy diagrams attached to any internal line means that the same states are not repeated after some interactions, and therefore that we have no repeated denominators. The general form of a typical contribution is therefore (apart from numerical factors, matrix elements and sums on momenta)

$$\prod_i \frac{\Theta_i}{\zeta-\epsilon_i}. \tag{71}$$

In (71), ϵ_i is the difference in energy between a collection of descending lines and a collection of ascending lines, and Θ_i insures that the ascending lines have energies greater than μ, while the descending lines have energies less than μ. That is, if

$$\epsilon_i=(\epsilon_{k_1}+\epsilon_{k_2}+\cdots+\epsilon_{k_n})-(\epsilon_{k_1'}+\epsilon_{k_2'}+\cdots+\epsilon_{k_m'}), \tag{72}$$

then

$$\Theta_i=\theta^-(\epsilon k_1)\cdots\theta^-(\epsilon k_n)\theta^+(\epsilon k_1')\cdots\theta^+(\epsilon k_m'). \quad (73)$$

By means of the "reality" condition (35) we see at once that the coefficient of (71) must be real. Now we want the imaginary part of this for $\zeta=x-i0^+$. Since

$$\frac{1}{x-\epsilon_i-i0^+}=P\frac{1}{x-\epsilon_i}+i\pi\delta(x-\epsilon_i), \quad (74)$$

the imaginary part of (71) must contain *at least* one factor of $\delta(x-\epsilon_i)$. The number of lines in an ordered skeleton diagram (including the entering and leaving external line k) which are ascending is equal at any point to the number of lines which are descending. This is because an interaction either doesn't change the direction of two lines entering it or if one line enters at one end then a line leaves at that end and ascending and descending lines are created at the other end. A diagram begins, of course, with equal number going up and down (the external k lines). Since each situation which corresponds to a denominator of the form $\zeta-\epsilon_i$ has only one external k line present, the number of ascending and descending internal lines must differ by ± 1. [In (72), $n-m=\pm 1$.] Take the case $n-m=+1$. Then by (73)

$$\epsilon_i\leqslant n\mu-m\mu=\mu. \quad (75)$$

We get no contribution from such a term unless $x<\mu$. For x very close to, but less than, μ, we again get a contribution only when each of the ϵ's is very near μ. Introducing

$$\epsilon k_i=\mu-t_i, \quad \epsilon k_i'=\mu+t_i' \quad (80)$$

the contribution to $J_\kappa(x)$ has a factor

$$\int_0^u\cdots\int\delta(t_1+\cdots+t_{m+1}+t_1'+\cdots+t_m'-u)$$
$$\times dt_1\cdots dt_{m+1}dt_1'\cdots dt_m', \quad (u=\mu-x). \quad (81)$$

Changing variables to $t_i=u\tau_i$, $t_i'=u\tau_i'$, we see at once that (81) is proportional to u^{2m}. Exactly the same argument also gives u^{2m} when the number of ascending lines is $m+1$ and of descending lines m. Now m is at least unity, since m equal zero would correspond to having a diagram which just has a single internal line present at one point. Such a diagram is just what we exclude by considering the *proper* self-energy part $G_k(\zeta)$ rather than the total self-energy diagram. Therefore every single skeleton diagram contributes to $J_k(x)$ an amount which, for x very near μ, behaves like

$$(\mu-x)^{2m} \quad m\geq 1.$$

Therefore we have shown (51) [the non-negativity of C_k follows from (49)] to arbitrary order in perturbation theory.

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