$$I_0A=-(|a|^2-|m|^2)\sin(\theta-\theta_L)+2\,\mathrm{Re}[ic(a^*-m^*)]\cos(\theta-\theta_L)+4\,\mathrm{Re}gh^*\sin\theta_L,$$

$$I_0R'=(|a|^2-|m|^2)\sin(\theta-\theta_L)-2\,\mathrm{Re}[ic(a^*-m^*)]\cos(\theta-\theta_L)+4\,\mathrm{Re}gh^*\sin\theta_L,$$

$$I_0A'=(|a|^2-|m|^2)\cos(\theta-\theta_L)+2\,\mathrm{Re}[ic(a^*-m^*)]\sin(\theta-\theta_L)+4\,\mathrm{Re}gh^*\cos\theta_L.$$

As in the nonrelativistic case there is a relation among the triple scattering parameters, which becomes

$$(A+R')/(A'-R)=\tan\theta_L.$$

The equality between $C_{PP}^{(\mathrm{rel})}$ at θ and $C_{KK}^{(\mathrm{rel})}$ at $(\pi-\theta)$ also holds for identical particles.

the denominator of the expression for α^{JP} should be $2E(2J+1)$. α^{JP} is independent of L.

ACKNOWLEDGMENTS

I am grateful to Dr. R. J. N. Phillips for pointing out an error in sign in my formula for C_{KP}, to Mr. J. Stuttard for assistance with the numerical work, and to the National Research Council of Canada for a Special Scholarship. I would also like to thank Dr. H. P. Stapp for his comments.

PHYSICAL REVIEW VOLUME 121, NUMBER 3 FEBRUARY 1, 1961

Determination of Pion-Pion Scattering Amplitudes Satisfying Dispersion Relations and Unitarity

John W. Moffat
RIAS, Baltimore, Maryland

(Received September 2, 1960)

A method is developed for determining the partial-wave scattering amplitude in terms of the unitarity condition and the known branch cuts and poles of the inverse amplitude. The method is applied to the problem of pion-pion scattering and an implicit solution to the pion-pion partial-wave amplitude is derived for any angular momentum state and for both elastic and inelastic scattering. With the aid of this solution the low-energy resonance behavior of the pion-pion scattering system is studied by neglecting all inelastic processes and concentrating on S and P waves. It is found that a P-wave resonance with a position and width required by nucleon electromagnetic structure can be determined in terms of two parameters. An iteration procedure is described that is applicable when the P wave dominates the equations and this procedure determines the contribution of the unphysical cut. The first iteration of the unphysical cut is numerically integrated on the IBM 709, and the results show that the shift of the resonance position due to the unphysical branch cut can be neglected.

1. INTRODUCTION

It has been conjectured by Mandelstam[1,2] that two-particle scattering amplitudes can be expressed in terms of a double spectral representation. The scattering amplitudes can be analytically continued into the complex plane as a function of the energy and momentum transfer variables and this leads to dispersion relations for the partial-wave amplitudes which satisfy the unitarity condition in a particularly simple form. It would seem that in principle this representation provides a complete dynamical description of scattering systems.

It has become evident that a more reliable description of pion-pion interaction is required if we are to understand the phenomena of strong interactions and the electromagnetic structure of the nucleon.[3,4] Chew and Mandelstam have used the double representation to formulate an approximation method for low-energy elementary particle scattering. By using the unitarity condition and the "effective-range" approximation, Chew and Mandelstam obtain a system of coupled nonlinear integral equations from the partial-wave dispersion relations for pion-pion scattering.[5–7] In the special case of dominant S-wave scattering and also in the case of dominant P-wave scattering, it has been shown that classes of solutions exist for the nonlinear integral equations. For P-wave dominant solutions a cutoff is required due to the singular nature of the Chew-Mandelstam equations, and the unphysical cuts are replaced by a corresponding series of poles.[8]

In the following a general method is developed which determines the partial-wave amplitude for a scattering problem in terms of the known branch cuts and the unitarity condition.[9] The method is applied to the

[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] S. Mandelstam, Phys. Rev. **115**, 1752 (1959).
[3] G. F. Chew, Phys. Rev. Letters **4**, 142 (1960).
[4] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2**, 365 (1959).
[5] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[6] G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev. **119**, 478 (1960).
[7] G. F. Chew, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, California, 1959), Vol. 9, p. 29.
[8] G. F. Chew and S. Mandelstam, University of California Radiation Laboratory Report UCRL-9126, 1960 (unpublished).

problem of pion-pion scattering as this is the simplest process which lends itself to a partial-wave analysis. We begin in Sec. 2 with a brief survey of the basic features of the Chew-Mandelstam analysis of the pion-pion problem. An important result in their analysis is a simple formulation of the unitarity condition in terms of the partial-wave amplitudes. In Sec. 3, the procedure is developed for determining the partial-wave amplitude. We use the partial-wave dispersion relations with subtractions, since these probably represent the physical situation.

The resonance behavior of the pion-pion scattering system is studied in Sec. 4. A derivation of the P-wave phase shift shows that a pion-pion resonance can develop in the P state in terms of two parameters. By neglecting the discontinuity of the inverse amplitude across the left-hand cut, we obtain the one-pole approximation to the P-wave amplitude used by Frazer and Fulco to study the pion form factor and nucleon structure.[4,10]

In Sec. 5, an iteration method is developed in order to study the effect of the unphysical branch-cut on the resonance behavior of the P-wave amplitude. Finally, in Sec. 6, the results of a calculation carried out on the IBM-709 computer are described. These results determine to a first iteration the resonance shift produced by the left-hand branch cut. It is shown that for a resonance position and width required by the electromagnetic structure of the nucleon this shift of the resonance is small.

2. PARTIAL-WAVE ANALYSIS OF PION-PION SCATTERING

In the problem of pion-pion scattering there are no spins and all three channels of Fig. 1 correspond to pion scattering. A charge degree of freedom is associated with each pion and this charge degree of freedom is described by an appropriate index with values 1, 2, 3. The ingoing four-momenta and isotopic spin indices are (p_1,α) and (p_2,β) and the outgoing are $(-p_3,\gamma)$ and $(-p_4,\delta)$. The variables used in Mandelstam's double dispersion relations are defined by[1]

$$\begin{aligned} s&=(p_1+p_2)^2=(p_3+p_4)^2=4(q^2+\mu^2),\\ \bar{t}&=(p_1+p_4)^2=(p_2+p_3)^2=-2q^2(1+\cos\theta),\\ t&=(p_1+p_3)^2=(p_2+p_4)^2=-2q^2(1-\cos\theta),\end{aligned} \tag{1}$$

where q is the magnitude of the pion momentum in the barycentric system and θ is the barycentric scattering angle. There is the auxiliary condition on the three variables s, t, and $\bar{t}$:

$$s+t+\bar{t}=4\mu^2. \tag{2}$$

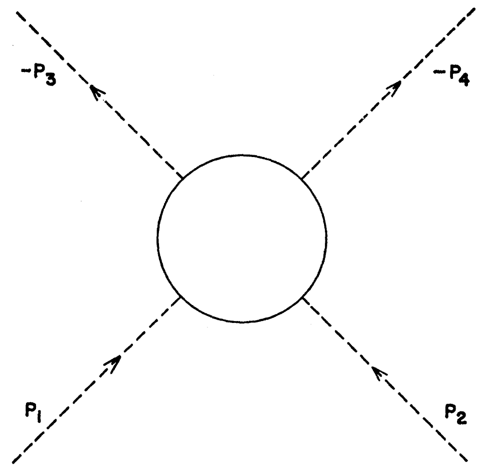

FIG. 1. Pion-pion elastic scattering, $\pi+\pi\leftrightarrow\pi+\pi$.

Thus only two of the three variables are counted as independent.

In view of charge independence there are only three invariants in the pion-pion problem corresponding to the three possible values of isotopic spin $I=0$, 1, 2. The three invariants A, B, and C can be expressed in terms of the scattering amplitude A^I for states of well-defined isotopic spin:

$$\begin{aligned} A^0&=3A+B+C,\\ A^1&=B-C,\\ A^2&=B+C.\end{aligned} \tag{3}$$

The partial-wave expansion of the pion amplitude A^I as a function of q^2 and $\cos\theta$ is given by

$$A^I(q^2,\cos\theta)=\sum_{\substack{l\text{ even, } I=0,2\\ l\text{ odd, } I=1}}(2l+1)A_l{}^I(q^2)P_l(\cos\theta). \tag{4}$$

In virtue of crossing symmetry only even powers of $\cos\theta$ occur for $I=0$, 2 and only odd powers of $\cos\theta$ for $I=1$. In order to take into account the unitarity condition and establish a connection between the amplitudes $A_l{}^I$ and the phase shifts $\delta_l{}^I$, Chew and Mandelstam write

$$A_l{}^I(q^2)=\frac{(q^2+\mu^2)^{\frac{1}{2}}}{q}\exp(i\delta_l{}^I)\sin\delta_l{}^I, \tag{5}$$

where the phase shifts are real for $s<16\mu^2$ $(q^2<3\mu^2)$ and complex at higher energies above the onset of two-pion production. A general expression for the unitarity condition at all energies is given by

$$\operatorname{Im}A_l{}^I(q^2)=\frac{q}{(q^2+\mu^2)^{\frac{1}{2}}}R_l{}^I|A_l{}^I(q^2)|^2, \tag{6}$$

where $R_l{}^I$ is the ratio of the total to the elastic partial-wave cross section.

---

[9] A different approach to this problem has been developed by considering dispersion relations for the inverse amplitude [G. Feldman, P. T. Matthews, and A. Salam, Nuovo cimento **16**, 549 (1960); J. Moffat, Nuclear Phys. **18**, 75 (1960)].

[10] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).

It follows from (4) that

$$A_l^I(q^2)=\tfrac{1}{2}\int_{-1}^{+1} d(\cos\theta)A^I(q^2,\cos\theta)P_l(\cos\theta). \qquad (7)$$

By using (7) the dispersion relations for a given partial-wave amplitude (i.e., a particular angular-momentum state) can be derived by a projection of the Mandelstam double dispersion relations. The projection of a given partial-wave corresponds to an integration for fixed s over either dt or $d\bar{t}$. In the case of pion-pion scattering, there occur no poles and for equal masses all the branch points lie on the real axis. There are three cuts corresponding to the three channels of diagram 1, but in this problem the left-hand or "unphysical" cut corresponds to a superposition of two cuts. The right-hand or "physical" cut extends in q^2 from 0 to ∞, while the left-hand cut extends from $-\mu^2$ to $-\infty$. The partial-wave amplitude is discontinuous across the cut, and the discontinuity is twice the imaginary part of the limit as the cut is approached. This imaginary part is positive definite on the physical cut and is given by the nonlinear relation (6). The physical amplitude is of course real in the interval $-\mu^2<q^2<0$.

In the case of pure elastic scattering when $q^2<3\mu^2$, we have $R_l^I=1$ and the unitarity condition takes on the simple form

$$\mathrm{Im}A_l^I(q^2)=\frac{q}{(q^2+\mu^2)^{\frac{1}{2}}}|A_l^I(q^2)|^2. \qquad (8)$$

The unitarity condition cannot be used to calculate the imaginary part of the amplitude on the left-hand cut for negative q^2. But an application of the "crossing symmetry" permits a calculation of the imaginary part of the amplitude on the unphysical cut in terms of its known value on the physical cut.

3. COMBINATION OF UNITARITY AND DISPERSION RELATIONS

The result of the Chew-Mandelstam analysis of the location of singularities in the pion-pion scattering problem can be expressed by the partial-wave dispersion relations[5]:

$$A_l^I(\nu)=\frac{1}{\pi}\int_{-\infty}^{-1}\frac{d\nu'\,\mathrm{Im}A_l^I(\nu')}{\nu'-\nu}+\frac{1}{\pi}\int_0^{\infty}\frac{d\nu'\,\mathrm{Im}A_l^I(\nu')}{\nu'-\nu}, \qquad (9)$$

where we have introduced the variable $\nu=q^2/\mu^2$. The unitarity condition (5) ensures that the partial-wave amplitude behaves asymptotically like a constant for large ν. It is probably necessary to carry out at least one subtraction in the Mandelstam double representation.[1,11]

At the point of maximum symmetry $s=t=\bar{t}=\frac{4}{3}\mu^2$ the invariants A, B, and C are all equal and real

$$A(\tau,\tau,\tau)=B(\tau,\tau,\tau)=C(\tau,\tau,\tau), \qquad (10)$$

[11] In the case of partial waves, one subtraction will be made in each angular-momentum state l.

where $\tau=\frac{4}{3}\mu^2$. As there are no poles in the pion problem, the coupling constant is not defined through the residue of a pole but through a subtraction at a point $\nu_0=-\sigma$, where σ is a positive number. We shall keep σ as an arbitrary constant for the time-being and determine it later. If the subtraction is carried out at the symmetry point

$$\nu_0=-\tfrac{2}{3}, \qquad (11)$$

we can deduce from (3) that

$$A^0(\nu_0,0)=-5\lambda,\quad A^1(\nu_0,0)=0,\quad A^2(\nu_0,0)=-2\lambda. \qquad (12)$$

We obtain from (9) the subtracted dispersion relations

$$A_l^I(\nu)=a_l^I+\frac{\nu-\nu_0}{\pi}\int_{-\infty}^{-1}\frac{d\nu'\,\mathrm{Im}A_l^I(\nu')}{(\nu'-\nu)(\nu'-\nu_0)}+\frac{\nu-\nu_0}{\pi}\int_0^{\infty}\frac{d\nu'\,\mathrm{Im}A_l^I(\nu')}{(\nu'-\nu)(\nu'-\nu_0)}, \qquad (13)$$

where a_l^I is the subtraction constant. By writing ν in (13) as $\nu+i\epsilon$ and substituting (13) into (6), we get the set of nonlinear integral equations for $\nu>0$:

$$\mathrm{Im}A_l^I(\nu)=F_l^I(\nu)\Bigg\{[\mathrm{Im}A_l^I(\nu)]^2+\Bigg[a_l^I+\frac{\nu-\nu_0}{\pi}P\int_{-\infty}^{-1}\frac{d\nu'\,\mathrm{Im}A_l^I(\nu')}{(\nu'-\nu)(\nu'-\nu_0)}+\frac{\nu-\nu_0}{\pi}P\int_0^{\infty}\frac{d\nu'\,\mathrm{Im}A_l^I(\nu')}{(\nu'-\nu)(\nu'-\nu_0)}\Bigg]^2\Bigg\}, \qquad (14)$$

where

$$F_l^I(\nu)=[\nu/(\nu+1)]^{\frac{1}{2}}R_l^I. \qquad (15)$$

Let us denote the inverse scattering amplitude by

$$G_l^I(\nu)=1/A_l^I(\nu). \qquad (16)$$

In virtue of the unitarity condition (5) the scattering amplitude behaves like a constant at infinity. We shall assume that this constant is nonvanishing and therefore $G_l^I(\nu)$ tends to a constant at infinity. The function

$$\Phi_l^I(\nu,z)=\frac{G_l^I(\nu)}{(\nu-\nu_0)(\nu-z)} \qquad (17)$$

is single-valued within the contour Γ described by Fig. 2, and it has poles at the two points z and ν_0. Partial-wave amplitude of order l vanishes at the origin like

$$C_l\nu^l+O(\nu^{l+1}) \qquad (18)$$

and this behavior has to be accounted for in (17). In addition to these singularities in $\Phi_l^I(\nu,z)$, there may also occur poles within the contour Γ at the complex zeros of $A_l^I(\nu)$. Since $G_l^I(\nu)$ shares the same branch points and the same branch cuts as $A_l^I(\nu)$, the function $\Phi_l^I(\nu)$

has cuts in the intervals

$$-\infty<\mathrm{Re}\nu\leqslant -1,\quad \mathrm{Im}\nu=0,\qquad 0\leqslant \mathrm{Re}\nu<\infty,\quad \mathrm{Im}\nu=0. \tag{19}$$

The contour of integration Γ consists of two small circles ρ and c encircling, respectively, the origin and the point -1, of two semicircles of radius R, and two contours connecting the semicircles along the edges of the cuts and separated from these cuts by a distance δ. We shall assume that $A_l^I(\nu)$ has no zero at $\nu=-1$, and that by letting R tend to infinity we make the integrals along the semicircles vanish. Then by an application of the residue theorem in the limit as $\delta\to 0$, $\rho\to 0$, and $c\to 0$, we obtain:

$$\frac{1}{2\pi i}\int_\Gamma \Phi_l^I(\nu,z)d\nu = \lim_{\delta\to 0}\frac{1}{2\pi i}\int_0^\infty d\nu\frac{[G_l^I(\nu+i\delta)-G_l^I(\nu-i\delta)]}{(\nu-\nu_0)(\nu-z)} + \lim_{\delta\to 0}\frac{1}{2\pi i}\int_{-\infty}^{-1}\frac{d\nu[G_l^I(\nu+i\delta)-G_l^I(\nu-i\delta)]}{(\nu-\nu_0)(\nu-z)} - \sum_{n=0}^{l-1}\frac{z^{n-l}}{n!}g_l^{(n)}(0)=\frac{G_l^I(z)}{z-\nu_0}+\frac{G_l^I(\nu_0)}{\nu_0-z}, \tag{20}$$

where the third term arises from an evaluation of the integral around the small circle ρ as $\rho\to 0$:

$$\int_\rho \frac{d\nu\, g_l(\nu)}{(\nu-z)\nu^l}=\sum_{n=0}^{l-1}\frac{z^{n-l}}{n!}g_l^{(n)}(0), \tag{21}$$

and

$$g_l^{(n)}(0)=\left(\frac{\partial^n g_l(\nu)}{\partial\nu^n}\right)_{\nu=0} = \left(\frac{1}{(\nu-\nu_0)(C_l+D_{l+1}\nu+\cdots)}\right)^{(n)}_{\nu=0}. \tag{22}$$

The content of the unitarity condition (6) in the physical region $\nu>0$, can be expressed by

$$F_l^I(\nu)=\lim_{\delta\to 0}\mathrm{Im}G_l^I(\nu-i\delta) = \lim_{\delta\to 0}\frac{1}{2i}[G_l^I(\nu-i\delta)-G_l^I(\nu+i\delta)],\quad (\nu>0) \tag{23}$$

where $F_l^I(\nu)$ is defined by (15). The relation (23) is an alternative way of writing the nonlinear integral equations (14). The unitarity condition (6) is only valid for $\nu>0$. In the unphysical region $\nu<-1$, we define

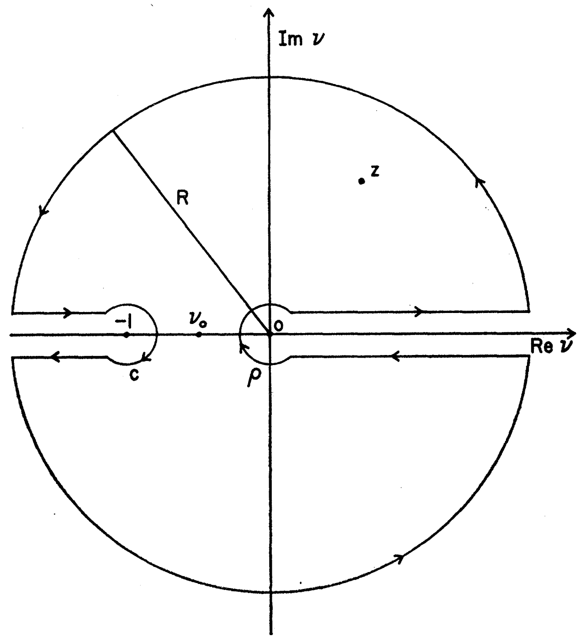

FIG. 2. The contour Γ used in the determination of the pion-pion partial-wave amplitude.

$$K_l^I(\nu)=\lim_{\delta\to 0}\frac{1}{2i}[G_l^I(\nu-i\delta)-G_l^I(\nu+i\delta)],\qquad (\nu<-1). \tag{24}$$

The quantity $K_l^I(\nu)$ for $\nu<-1$ can be calculated in terms of the absorptive amplitude on the physical cut by an application of the crossing symmetry.[5] We shall return to this problem in Sec. 5.

By substituting the relations (23) and (24) into (20), we get

$$-\frac{1}{\pi}\int_0^\infty\frac{d\nu F_l^I(\nu)}{(\nu-z)(\nu-\nu_0)}-\frac{1}{\pi}\int_{-\infty}^{-1}\frac{d\nu K_l^I(\nu)}{(\nu-z)(\nu-\nu_0)} - \sum_{n=0}^{l-1}\frac{z^{n-l}}{n!}g_l^{(n)}(0)=\frac{G_l^I(z)}{z-\nu_0}+\frac{G_l^I(\nu_0)}{\nu_0-z}. \tag{25}$$

We now write $z=\nu+i\epsilon$, where ϵ is a small positive quantity. By using the identity

$$\frac{1}{\nu'-\nu-i\epsilon}=P\frac{1}{\nu'-\nu}+i\pi\delta(\nu'-\nu), \tag{26}$$

we obtain the form of the scattering amplitude $A_l^I(\nu)$:

$$A_l^I(\nu)=\frac{1}{\dfrac{1}{a_l^I}+L_l^I(\nu,\nu_0)+N_l^I(\nu,\nu_0)-(\nu-\nu_0)\sum_{n=0}^{l-1}\dfrac{\nu^{n-l}}{n!}g_l^{(n)}(0)-iT_l^I(\nu)}, \tag{27}$$

where

$$T_l^I(\nu)=R_l^I[\nu/(\nu+1)]^{\frac{1}{2}} \quad \text{for} \quad \nu>0$$
$$=K(\nu) \quad \text{for} \quad \nu<-1, \tag{28}$$

$$L_l^I(\nu,\nu_0)=-\frac{\nu-\nu_0}{\pi}P\int_0^\infty \frac{d\nu' \, F_l^I(\nu')}{(\nu'-\nu)(\nu'-\nu_0)}, \tag{29}$$

and

$$N_l^I(\nu,\nu_0)=-\frac{\nu-\nu_0}{\pi}P\int_1^\infty \frac{d\nu' \, K_l^I(-\nu')}{(\nu'+\nu)(\nu'+\nu_0)}. \tag{30}$$

In the low-energy region $0<\nu<3$, we have $R_l^I=1$ and the integral (29) becomes $L(\nu,\nu_0)=h(\nu)-h(\nu_0)$, where for $\nu>0$ or $\nu<-1$ the $h(\nu)$ is determined by

$$h(\nu)=\frac{2}{\pi}\left(\frac{\nu}{\nu+1}\right)^{\frac{1}{2}} \ln[(|\nu|)^{\frac{1}{2}}+(|\nu+1|)^{\frac{1}{2}}], \tag{31}$$

and, for $-1<\nu<0$,

$$h(\nu)=\frac{2}{\pi}\left(\frac{-\nu}{\nu+1}\right)^{\frac{1}{2}} \tan^{-1}\left(\frac{1+\nu}{-\nu}\right)^{\frac{1}{2}}. \tag{32}$$

It is easily checked that for $\nu>0$ the partial-wave amplitude $A_l^I(\nu)$ in (27) satisfies the unitarity condition (6). We also find that

$$\text{Im}[A_l^I(\nu)]^{-1}=-[\nu/(\nu+1)]^{\frac{1}{2}}, \quad (0<\nu<3) \tag{33}$$

which in view of (5) agrees with our expectations.

If $K(\nu)$ is known explicitly, then (27) provides us with a solution of the partial-wave amplitude for all angular-momentum states and for both elastic and inelastic scattering. In order to obtain the general solution to $A_l^I(\nu)$, we must supplement (27) with the residues associated with any complex poles of the inverse partial-wave amplitude. These residues would generate additional parameters in the problem.[12]

4. RESONANCE BEHAVIOR OF PION-PION SCATTERING AMPLITUDE

It is now possible to study the resonance behavior of the pion-pion scattering system by considering the phase shifts obtained from the implicit solution (27). In order to provide a theoretical description of the low-energy behavior of the pion-pion phase shifts, we shall restrict ourselves to the elastic scattering region $0<\nu<3$ in which the phase shifts are real. In this low-energy approximation there will not occur any complex zeros in $A_l^I(\nu)$. In virtue of (5), we obtain

$$[\nu/(\nu+1)]^{\frac{1}{2}} \cot\delta_l^I=\text{Re}[A_l^I(\nu)]^{-1}. \tag{34}$$

With the aid of (27) and (34), we obtain for the phase shift δ_l^I in the physical region

$$\left(\frac{\nu^{2l+1}}{\nu+1}\right)^{\frac{1}{2}} \cot\delta_l^I=\frac{\nu^l}{a_l^I}+\nu^l[h(\nu)-h(\nu_0)]+\nu^l N_l^I(\nu,\nu_0)$$
$$-(\nu-\nu_0)\sum_{n=0}^{-1}\frac{\nu^n}{n!}g_l^{(n)}(0), \tag{35}$$

where $h(\nu)$ is given by (31).

The S-wave phase shift δ_0^I is determined by

$$\left(\frac{\nu}{\nu+1}\right)^{\frac{1}{2}} \cot\delta_0^I=\frac{1}{a_0^I}+h(\nu)-h(\nu_0)+N_0^I(\nu,\nu_0), \tag{36}$$

where in the case of S waves we choose $\nu_0=-\frac{2}{3}$, and $h(\nu_0)=(2/\pi)\sqrt{2}\tan^{-1}(1/\sqrt{2})$. In the approximation where $K_0^I(\nu)$ is neglected, we obtain

$$\left(\frac{\nu}{\nu+1}\right)^{\frac{1}{2}} \cot\delta_0^I=\frac{1}{a_0^I}+h(\nu)-\frac{2}{\pi}\sqrt{2}\tan^{-1}(1/\sqrt{2}). \tag{37}$$

If the D-wave and higher waves are small, then $A_0^0 \approx -5\lambda$ and $A_0^2\approx -2\lambda$. The result (37) for the S-wave coincides with the corresponding one derived by Chew and Mandelstam.[5]

The P-wave phase shift in the physical region will be determined by

$$\left(\frac{\nu^3}{\nu+1}\right)^{\frac{1}{2}} \cot\delta_1=\frac{\nu}{a_1}+\nu[h(\nu)-h(\nu_0)]$$
$$+\nu N_1(\nu,\nu_0)-\xi_1(\nu-\nu_0). \tag{38}$$

The P state can develop a resonance of appropriate width and position if we adopt suitable values for the two parameters a_1 and ξ_1.

In the physical region $\nu>0$, the P-wave amplitude obtained from (27) can be written

$$\frac{1}{\nu}A_1(\nu)$$
$$=\frac{\Gamma}{\bar{\nu}_R-\nu[1-h(\nu)\Gamma-N_1(\nu,\nu_0)\Gamma]-i\theta(\nu)[\nu^3/(\nu+1)]^{\frac{1}{2}}\Gamma}, \tag{39}$$

where

$$\Gamma=\frac{\alpha}{\alpha[h(\nu_0)-1/a_1]-1}, \quad \bar{\nu}_R/\Gamma=\sigma/\alpha. \tag{40}$$

We have put $\xi_1=-1/\alpha$ and $\nu_0=-\sigma$, and $\bar{\nu}_R$ denotes the approximate position of the resonance,[13] while Γ is the width associated with this resonance. If we let a_1 approach infinity, and neglect the contribution of $N_1(\nu,\nu_0)$, then a pole is generated in $(1/\nu)A_1(\nu)$ at $\nu_0=-\sigma$, and we obtain the phenomenological one-pole

---

[12] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[13] The exact location of the resonance will be determined by the formula

$$\nu_R=\frac{\bar{\nu}_R}{1-[h(\nu_R)+N_1(\nu_R,\nu_0)]\Gamma}.$$

approximation to the P-wave amplitude used by Frazer and Fulco to study the isotopic-vector form factors associated with the nucleon.[4,10] Frazer and Fulco obtain the one-pole approximation to the P-wave amplitude from the Chew-Mandelstam integral equations by replacing the left-hand branch cut by a pole at the point ν_0.

5. THE ITERATION METHOD

We shall now develop an iteration procedure in order to estimate the influence of the unphysical region on the behavior of the P-wave pion-pion resonance.

The absorptive amplitude in the unphysical region is not known explicitly, but must be calculated by means of the "crossing relations" and the absorptive amplitude on the physical cut. Using the crossing symmetry, Chew and Mandelstam have derived the following absorptive amplitude in the unphysical region:

$$\mathrm{Im}A_l^I(\omega)=-\frac{1}{\omega}\int_0^{\omega-1}d\nu'\,P_l\left(1-2\frac{\nu'+1}{\omega}\right)\times\sum_{I'=0,1,2}\alpha_{II'}A_a^{I'}\left(\nu',1-2\frac{\omega-1}{\nu'}\right),\quad(41)$$

where we have put $-\nu=\omega>1$, and $A_a^{I'}$ denotes the complete absorptive amplitude on the right-hand cut. The crossing matrix is given by

$$\alpha_{II'}=\begin{bmatrix}2/3 & 2 & 10/3\\ 2/3 & 1 & -5/3\\ 2/3 & -1 & 1/3\end{bmatrix}.\quad(42)$$

By resolving $A_a^{I'}$ into partial waves, and truncating the expansion after the P wave, we get

$$\mathrm{Im}A_l^I(\omega)=-\frac{1}{\omega}\int_0^{\omega-1}d\nu'\,P_l\left(1-2\frac{\nu'+1}{\omega}\right)\times\Bigg\{\alpha_{I0}\,\mathrm{Im}A_0^0(\nu')+\alpha_{I2}\,\mathrm{Im}A_0^2(\nu')+3\left(1-2\frac{\omega-1}{\nu'}\right)\alpha_{I1}\,\mathrm{Im}A_1^1(\nu')\Bigg\}.\quad(43)$$

The right-hand side of (43) is determined in terms of the absorptive amplitude on the physical cut given by

$$\mathrm{Im}A_l^I(\nu)=[(\nu+1)/\nu]^{\frac{1}{2}}\sin^2\delta_l^I.\quad(44)$$

We are interested in P-wave dominant solutions corresponding to a low-energy resonance in the P state. In this case the third term in (43) will give the main contribution to $\mathrm{Im}A_1(\omega)$. Let us denote by $I(\omega)$ the absorptive P-wave amplitude on the unphysical cut, and by $E(\omega)$ the real part of the amplitude on the same cut. Then $K_1(\omega)$ is given by

$$K_1(\omega)=\frac{I(\omega)}{E(\omega)^2+I(\omega)^2}.\quad(45)$$

When $K(\nu)=0$ the P-wave absorptive amplitude on the physical cut is

$$\mathrm{Im}A_1(\nu)=\frac{[\nu/(\nu+1)]^{\frac{1}{2}}}{[h(\nu)-h(\nu_0)+(\nu-\nu_0)/\alpha\nu]^2+\nu/(\nu+1)},\quad(46)$$

where we have chosen a_1 equal to infinity. In the unphysical region it is convenient to introduce the variable

$$x=(-\nu)^{-\frac{1}{2}}=\omega^{-\frac{1}{2}},\quad(47)$$

which will run from 0 to 1. When $K(\nu)$ is equal to zero, the real part of the P-wave amplitude in the unphysical region is given by

$$\bar{E}(x)=\frac{1}{\bar{h}(x)-h(-\sigma)+(1-\sigma x^2)/\alpha},\quad(48)$$

where, for $\nu<-1$,

$$h(x)=-\frac{2}{\pi}\frac{1}{(1-x^2)^{\frac{1}{2}}}\ln\frac{1+(1-x^2)^{\frac{1}{2}}}{x},\quad(49)$$

and we have introduced the notation $\bar{E}(x)=E(-1/x^2)$. When $K(\nu)$ is zero, we obtain from (43) and (46) in the case of a P-wave dominant solution

$$\bar{I}(x)=3x^2\int_0^{1/x^2-1}\frac{d\nu'[1-2x^2(\nu'+1)]\left(2\frac{1/x^2-1}{\nu'}-1\right)\left(\frac{\nu'}{\nu'+1}\right)^{\frac{1}{2}}}{[h(\nu')-h(\nu_0)+(\nu'-\nu_0)/\alpha\nu']^2+\nu'/(\nu'+1)}.\quad(50)$$

This leads to the following expression for $\bar{K}_1(x)$ to a first iteration:

$$\bar{K}_1^{(1)}(x)=\frac{[\bar{h}(x)-h(-\sigma)+(1-x^2\sigma)/\alpha]^2\bar{I}(x)}{1+[\bar{h}(x)-h(-\sigma)+(1-x^2\sigma)/\alpha]^2\bar{I}^2(x)}.\quad(51)$$

If the procedure converges, then we can continue the iteration by substituting $\bar{K}_1^{(1)}(x)$ into $\bar{I}(x)$ and $\bar{E}(x)$ and thus obtain $\bar{K}_1^{(2)}(x)$ to a second iteration.

6. CALCULATIONS AND RESULTS

Let us put the parameter $a_1=A_1(\nu_0)$ equal to infinity and neglect the contribution of $N_1(\nu,\nu_0)$ in (38). This yields the P-wave phase shift

$$\left(\frac{\nu^3}{\nu+1}\right)^{\frac{1}{2}}\cot\delta_1=\nu[h(\nu)-h(\nu_0)]-\xi_1(\nu-\nu_0).\quad(52)$$

By choosing $\nu_0=-652$ and $\nu_R=1.5$, we obtain the approximate resonance value $\bar{\nu}_R=1.2$ and the resonance width $\Gamma=0.4$.[14] With these values of the constants ν_R and ν_0 the phase shift δ_1 passes through 90° when $\xi_1=-0.00457$ or $\alpha=218$.

We are faced with the problem of estimating the amount by which the resonance position is shifted when the contribution of the left-hand cut is included in (52). In order to estimate this shift, we must compute the integral $\bar{I}(x)$ in (50), the function $\bar{K}_1^{(1)}(x)$ and, finally, the integral

$$N_1^{(1)}(1.5,-652)=-(1.5+652)\frac{2}{\pi}P\int_x^1 \frac{dx\, x\bar{K}_1^{(1)}(x)}{(1+x^2 1.5)(1-x^2 652)}. \quad (53)$$

We have cut off the integrals at small values of x in virtue of the limited range of the physical region in which our equations remain valid. Both integrals are convergent as $x\to 0$ (i.e., as $\omega\to\infty$), and the cutoff only represents our present lack of knowledge of high-energy processes. Chew and Mandelstam[8] have estimated that on the right-hand cut the elastic approximation should be adequate for $\nu\leqslant 10$, and the crossing relation (41) indicates that an average $\bar{\nu}$ on the right gives $\bar{\nu}=\frac{1}{2}(\omega-1)$. Therefore, we expect a failure of our equations on the unphysical cut at $\omega\sim 20$ corresponding to $x\sim\frac{1}{5}$.

The numerical integrations were carried out on the Martin Company IBM 709. $\bar{I}(x)$ is zero at $x=1$ and passes through one zero as x decreases in value. Thereafter, $\bar{I}(x)$ tends as a smoothly varying function to a positive constant. Our equations are valid within the range $(7/32)<x<1$ and integration yields for this interval

$$N_1^{(1)}(1.5,-652)=-8.528\times 10^{-2}. \quad (54)$$

In the larger interval $(1/16)<x<1$, we obtain the value

$$N_1^{(1)}(1.5,-652)=5.167\times 10^{-2}. \quad (55)$$

From the result (54), we find that the resonance position $\nu_R=1.5$ is shifted by an amount

$$\delta\nu_R=-6.4\times 10^{-2}, \quad (56)$$

while the second result (55) yields the shift

$$\delta\nu_R=3.9\times 10^{-2}. \quad (57)$$

Inspection shows that the value of the integral in (53) decreases steadily as $x\to 0$.

We have assumed that the inverse partial-wave amplitude does not possess any complex zeros. One may enquire about the possibility of such complex zeros developing within the region in which our iteration scheme is valid. Expressed as a function of complex ν in the cut-plane, the approximate P-wave amplitude can be written

$$\frac{1}{\nu}A_1(\nu)=\frac{\alpha}{\nu-\nu_0}\Bigg/\left[1-\frac{\alpha\nu}{\pi}\int_0^\infty \frac{d\nu'[\nu'/(\nu'+1)]^{\frac{1}{2}}}{(\nu'-\nu)(\nu'-\nu_0)} -\frac{\alpha\nu}{\pi}\int_1^\infty \frac{d\nu'\,K(-\nu')}{(\nu'+\nu)(\nu'+\nu_0)}\right], \quad (58)$$

where the parameter a_1 has been chosen equal to infinity. For $\nu=\mathrm{Re}\nu+i\,\mathrm{Im}\nu$, the imaginary part of the denominator is given by

$$-\frac{\alpha\,\mathrm{Im}\nu}{\pi}\left[\int_0^\infty \frac{d\nu'[\nu'^3/(\nu'+1)]^{\frac{1}{2}}}{[(\nu'-\mathrm{Re}\nu)^2+\mathrm{Im}\nu^2](\nu'-\nu_0)} +\int_1^\infty \frac{d\nu'\,\nu'K(-\nu')}{[(\nu'+\mathrm{Re}\nu)^2+\mathrm{Im}\nu^2](\nu'+\nu_0)}\right], \quad (59)$$

and therefore vanishes only when $\mathrm{Im}\nu=0$ ($\alpha\neq 0$) if the expression within the brackets has a single sign. It follows immediately that in the zeroth approximation when $K(\nu)=0$, $\nu A_1^{-1}(\nu)$ does not possess any complex zeros. Inspection shows that in the first iteration the expression in brackets is positive definite ($\nu_0=-652$) for values of the cutoff well beyond the limits of validity of the low-energy approximation. Therefore, no complex zeros occur in the iteration of $\nu A_1^{-1}(\nu)$ within the energy range determined by our approximations.

7. CONCLUSIONS

These results confirm that the unphysical cut has little influence on the low-energy resonance behavior of the pion-pion system. Since (53) is small to a first iteration, it is reasonable to expect that the iteration procedure converges, and that the higher iterated contributions can be neglected. In view of these results and the fact that we have chosen a_1 equal to infinity in our calculations, we can deduce that replacing the unphysical branch cut by a pole leads to a good approximation to the P-wave amplitude. This in turn confirms that the pion form factor adopted by Frazer and Fulco in their analysis of the nucleon electromagnetic structure[4,10] is a physically acceptable solution. However, we are not forced to set $a_1\equiv A_1(\nu_0)$ equal to infinity in our calculations. We can instead adjust the parameter a_1 to the experimental resonance data with the knowledge that the left-hand cut can be neglected.

In a subsequent paper the problem of coupled S and P waves and dominant S waves in pion-pion scattering will be treated by our iteration techniques.

ACKNOWLEDGMENTS

I am grateful to Professor G. F. Chew, Professor W. R. Frazer, and Professor M. L. Goldberger for helpful discussion. I am also grateful to Dr. P. Schwed, Dr. W. Rarita, and Professor T. Fulton for helpful discussions and criticism. My thanks are due to Mr. T. Englar and Mr. H. Brown for coding the problem and carrying out the machine computations on the Martin Company IBM 709.

[14] These are the values of ν_R, Γ, and ν_0 used by Frazer and Fulco in their study of nucleon structure.[10] These values of the constants may be changed in order to obtain a better fit to the nucleon structure data, but a reasonable change of these constants will not alter our conclusions regarding the influence of the left-hand cut on the P-wave resonance. Since the position of the pole depends almost exponentially on Γ, we find that increasing Γ to 0.5 gives $\nu_0\sim -150$.