

Self-Diffusion in Liquid He³†

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The Landau model of a Fermi liquid is employed to obtain an expression for the coefficient of self-diffusion, D , in liquid He³. It is found that $D=A/T^2$, where $A \approx 2 \times 10^{-6}$ cm² sec⁻¹ degree², and T is the temperature in degrees Kelvin. As this temperature dependence differs from that experimentally observed (T^{-3}), although the order of magnitude of the theoretical prediction agrees well with experiment, the possible effect of the 1% He⁴ impurity known to be present is investigated; it is found that the diffusion coefficient is essentially unaffected by the impurity (a result also obtained experimentally). It is estimated that the Landau model is applicable only well below 0.05°K. As even the most recent experiments have been carried out only down to 0.03°K, it is concluded that the transport properties of He³ should be investigated experimentally at lower temperatures in order to check the validity of the theory.

A METHOD for treating the transport properties of He³ employing Landau's model of a Fermi liquid¹ has been developed by Abrikosov and Khalatnikov.² Now that experiments on self-diffusion in liquid He³ have been performed³ in the temperature region where this model may be expected to be valid, it seems desirable to examine the theoretical predictions concerning the behavior of the diffusion coefficient. We follow closely the method and the notation of I.

We consider a system of He³ in which there exists a magnetization gradient in the absence of an external field, maintained by unspecified sources of "up" and "down" spins. A steady-state diffusive flow is maintained. In the absence of impurities the Boltzmann equation is

$$\partial n / \partial t + \nabla_r n \cdot \nabla_p \epsilon - \nabla_p n \cdot \nabla_r \epsilon = I(n), \quad (1)$$

where $\epsilon(\mathbf{p}, \boldsymbol{\sigma})$ is the actual energy associated with a quasiparticle of momentum \mathbf{p} and spin $\boldsymbol{\sigma}$ and is given for small deviations from the equilibrium distribution at zero temperature by

$$\epsilon(\mathbf{p}, \boldsymbol{\sigma}) = \epsilon_0(\mathbf{p}, \boldsymbol{\sigma}) + \frac{1}{2} \text{Tr}_{\boldsymbol{\sigma}'} \int f(\mathbf{p}, \mathbf{p}', \boldsymbol{\sigma}, \boldsymbol{\sigma}') \delta n(\mathbf{p}', \boldsymbol{\sigma}') \frac{2}{(2\pi\hbar)^3} d\mathbf{p}'. \quad (2)$$

We may write $f(\mathbf{p}, \mathbf{p}', \boldsymbol{\sigma}, \boldsymbol{\sigma}')$ in the form:

$$f(\mathbf{p}, \mathbf{p}', \boldsymbol{\sigma}, \boldsymbol{\sigma}') = f(\mathbf{p}, \mathbf{p}') + \zeta(\mathbf{p}, \mathbf{p}') \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \quad (3)$$

if we assume that all of the spin dependence of f is of exchange origin. Thus quasi-particle spin is not a good quantum number, but it provides a convenient ap-

proximate description. In (1), n is the actual distribution function for quasi-particles

$$n = n_0 + \delta n, \quad |\delta n| \ll n_0, \quad n_0(\epsilon) = [\exp((\epsilon - \mu)/T) + 1]^{-1}, \quad (4)$$

where μ may be a function of position and is chosen as usual to make the total number of particles come out right. Unless otherwise indicated in specific cases, temperature will be taken in energy units throughout this paper.

The distribution functions and the chemical potentials of the systems of up and down spin particles should be considered separately for the system of interest in the present problem. However, the calculations are greatly simplified if we consider the point where the magnetization $M=0$. Then the two chemical potentials will be equal, although their gradients will be opposite in sign. This results in the relation $\delta n_{\uparrow} = -\delta n_{\downarrow}$, where we have now introduced a spin index. Furthermore, there is a simple relation between the gradient of μ for particles of a specified spin and that of the corresponding concentration, N . With $d\tau = [2/(2\pi\hbar)^3] d\mathbf{p}$, we have

$$\frac{\partial N}{\partial x} = \frac{1}{2} \frac{\partial}{\partial x} \int n(\epsilon) d\tau = \frac{1}{2} \int \frac{\partial n}{\partial \epsilon} \left(\frac{\partial \epsilon}{\partial x} - \frac{\partial \mu}{\partial x} \right) d\tau = -\frac{1}{4} \bar{\zeta} (d\tau/d\epsilon)_{\mu} (\partial N/\partial x) + \frac{1}{2} (d\tau/d\epsilon)_{\mu} (\partial \mu/\partial x), \quad (5)$$

where $\bar{\zeta}$ is the angular average of $\zeta(\mathbf{p}, \mathbf{p}')$ with \mathbf{p} and \mathbf{p}' restricted to the Fermi surface. Thus,

$$\frac{\partial \mu}{\partial x} = 2 \left(\frac{d\epsilon}{d\tau} \right)_{\mu} \left[1 + \frac{1}{4} \bar{\zeta} \left(\frac{d\tau}{d\epsilon} \right)_{\mu} \right] \frac{\partial N}{\partial x}. \quad (6)$$

The same factor involving $\bar{\zeta}$ occurs in the expression for the magnetic susceptibility.²

The collision integral, $I(n)$, is given by

$$I(n) = - \int w [n_1 n_2 (1 - n_1') (1 - n_2') - (1 - n_1) (1 - n_2) n_1' n_2'] \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2') \times \delta(\epsilon_1 + \epsilon_2 - \epsilon_1' - \epsilon_2') d\tau_2 d\tau_1' d\mathbf{p}_2', \quad (7)$$

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¹ L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 1058 (1956) [English translation: Soviet Phys.—JETP **3**, 920 (1957)].

² A. A. Abrikosov and I. M. Khalatnikov, *Reports on Progress in Physics* (The Physical Society, London, 1959), **32**, 329, hereafter referred to as I.

³ H. R. Hart and J. C. Wheatley, Phys. Rev. Letters **4**, 3 (1960). More recent experiments [A. C. Anderson, H. R. Hart, and J. C. Wheatley, Phys. Rev. Letters **5**, 133 (1960)], of greater accuracy and carried to lower temperatures (0.03°K), confirm the order of magnitude of D as observed earlier and more definitely establish the temperature dependence as T^{-1} .

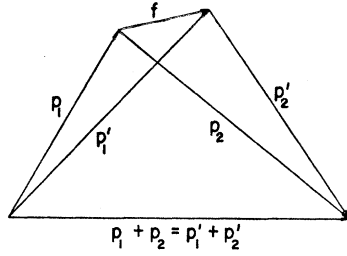


FIG. 1. Relationship of momenta for He^3 - He^3 collision.

where we have written $n(p_{1,\sigma}) = n_1$ (the spin index is suppressed temporarily) for notational convenience. We note here that the operator f appearing in the expression for the energy of a quasi-particle is an important characteristic of a Fermi liquid; Landau⁴ has demonstrated that it is related to the forward scattering amplitude of two quasi-particles.

As collisions conserve momentum, we see that the scattering of one quasi-particle from another of like spin will contribute nothing to the diffusion of spin. Thus if we consider the Boltzmann equation for $n_{1\uparrow}$, we need in the collision integral only those terms in the initial state for which particle 2 has spin down. Because of the identity of particles, we may without loss of generality take in the final state 1' to have spin up, and 2' spin down.

The collision integral vanishes when the equilibrium distribution functions of the true quasi-particle energies are inserted. Therefore, we express the distribution functions appearing in $I(n)$ in terms of $n_0(\epsilon)$:

$$n(\epsilon) = n_0(\epsilon_0) + \delta n \\ = n_0(\epsilon) - \frac{1}{2} \left(\frac{\partial n_0}{\partial \epsilon_0} \right) \text{Tr}_{\sigma'} \int f \delta n' d\tau' + \delta n. \quad (8)$$

We write δn in the form

$$\delta n = - \left(\frac{n_0(1-n_0)}{T} \right) \nu = \left(\frac{\partial n_0}{\partial \epsilon_0} \right) \nu. \quad (9)$$

Then

$$n(\epsilon) = n_0(\epsilon) + \frac{\partial n_0}{\partial \epsilon_0} \left[\nu + \left(\frac{d\tau}{d\epsilon_0} \right) \frac{1}{\epsilon_0 = \mu} \int \zeta(\theta) \nu d\Omega / 4\pi \right] \\ \equiv n_0(\epsilon) + \frac{\partial n_0}{\partial \epsilon_0} \psi. \quad (10)$$

Substituting into (7) and keeping only first-order terms in ψ , we find

$$I(n) = \frac{1}{4} T^{-1} \int w n_{01} n_{02} (1 - n_{01}') (1 - n_{02}') \\ \times (\psi_1 + \psi_2 - \psi_1' - \psi_2') \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2') \\ \times \delta(\epsilon_1 + \epsilon_2 - \epsilon_1' - \epsilon_2') d\tau_2 d\tau_1' d\mathbf{p}_2'. \quad (11)$$

⁴ L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 262 (1958) [translation: Soviet Phys.—JETP 7, 182 (1958)].

Since at low temperatures only quasi-particles with momenta near the Fermi surface will contribute to $I(n)$, w depends—approximately—only on θ , the angle between \mathbf{p}_1 and \mathbf{p}_2 , and on ϕ , the angle between the planes determined by $(\mathbf{p}_1, \mathbf{p}_2)$ and $(\mathbf{p}_1', \mathbf{p}_2')$. Rotating one of these planes about their line of intersection, $\mathbf{p}_1 + \mathbf{p}_2$, until it coincides with the other, we have the diagram of Fig. 1. The angle between each of \mathbf{p}_1 , \mathbf{p}_2 , \mathbf{p}_1' , and \mathbf{p}_2' , and $\mathbf{p}_1 + \mathbf{p}_2$ is approximately $\theta/2$, as $|f| \ll p_0$, the Fermi momentum. We integrate first with respect to \mathbf{p}_2 to eliminate the momentum δ function. Then, transforming to integration over f instead of τ_1' :

$$d\tau_1' \rightarrow p_0 \sin(\theta/2) df_r df_z d\phi \times 2 / (2\pi\hbar)^3, \quad (12)$$

where we have used cylindrical coordinates with $\mathbf{p}_1 + \mathbf{p}_2$ as the polar axis, and r , z , and ϕ take on the usual meanings. We note that this is the same sense in which ϕ was used above. We introduce the dimensionless variables:

$$x = (\epsilon_1' - \mu) / T, \quad y = (\epsilon_2' - \mu) / T, \\ t = (\epsilon_1 - \mu) / T, \quad \kappa = \mu / T \quad (13)$$

and use the approximate relation between energy and momentum,

$$\epsilon - \mu = v_0(p - p_0) = (p - p_0)p_0 / m^* \quad (14)$$

to transform variables again. Then

$$d\tau_1' \int d\tau_2 \delta(\epsilon_1 + \epsilon_2 - \epsilon_1' - \epsilon_2') \\ = \frac{m^{*3} T^2}{8\pi^4 \hbar^6 \cos(\theta/2)} dx dy \left(\frac{d\Omega}{2\pi} \right) \left(\frac{d\phi_2}{2\pi} \right), \quad (15)$$

where $d\Omega = \sin\theta d\theta d\phi$, and $0 \leq \varphi \leq 2\pi$. By extending the range of φ to 2π and restricting the spin of particle 1' to be up we have taken into account all possible results of the scattering of $1\uparrow$ with $2\downarrow$.

On the left hand side of Eq. (1) we insert $n = n_0$, as defined by Eq. (4). Then the kinetic equation becomes

$$\nabla_{\mathbf{p}} n \cdot \nabla_{\mathbf{r}} \epsilon + \nabla_{\mathbf{r}} n \cdot \nabla_{\mathbf{p}} \epsilon \\ = \frac{m^{*3} T^2}{16\pi^4 \hbar^6} \int \frac{w(\theta, \phi)}{4\pi \cos(\theta/2)} d\Omega \int_{-\kappa}^{\infty} dx \int_{-\kappa}^{\infty} dy n_0(t) \\ \times n_0(x+y-t) [1 - n_0(x)] [1 - n_0(y)] \\ \times (\psi_1 + \psi_2 - \psi_1' - \psi_2') \quad (16)$$

with the restriction throughout that $x+y-t \geq (-\kappa)$.

As $\nabla_{\mathbf{r}} n \cdot \nabla_{\mathbf{p}} \epsilon + \nabla_{\mathbf{p}} n \cdot \nabla_{\mathbf{r}} \epsilon = - (dn_0/d\epsilon_0) \nabla_{\mathbf{r}} \mu \cdot \nabla_{\mathbf{p}} \epsilon$, we see from symmetry considerations that ψ must be of the form $\psi_1 = q(t) \nabla_{\mathbf{p}} \epsilon \cdot \nabla_{\mathbf{r}} \mu$. Call θ_1' the angle between \mathbf{p}_1 and \mathbf{p}_1' and call θ_2' that between \mathbf{p}_1 and \mathbf{p}_2' . Then

$$\psi_2 = -q(x+y-t) \nabla_{\mathbf{p}} \epsilon \cdot \nabla_{\mathbf{r}} \mu \cos\theta, \\ \psi_1' = q(x) \nabla_{\mathbf{p}} \epsilon \cdot \nabla_{\mathbf{r}} \mu \cos\theta_1', \\ \psi_2' = -q(y) \nabla_{\mathbf{p}} \epsilon \cdot \nabla_{\mathbf{r}} \mu \cos\theta_2', \quad (17)$$

where $p = p_1$, $\epsilon = \epsilon_1$ and $\mu = \mu \uparrow$. These last results have been obtained by using the addition theorem for spherical harmonics and integrating over ϕ_2 . We use the same theorem to establish the relations

$$\begin{aligned}\cos\theta_1' &= \cos^2(\theta/2) + \sin^2(\theta/2) \cos\phi, \\ \cos\theta_2' &= \cos^2(\theta/2) - \sin^2(\theta/2) \cos\phi.\end{aligned}\quad (18)$$

Then the kinetic equation becomes an equation for q . It is shown in I that under the assumption $t^2 \ll \pi^2$, q can be taken to be approximately a constant. (We note that this is equivalent to the assumption of a constant relaxation time.) Then

$$q = \frac{16\pi^2 \hbar^6}{m^* \bar{v}^2} \left\{ \left[w(\theta, \phi) \cos - (1 - \cos\phi) \right]_{\text{av}} \right\}^{-1}, \quad (19)$$

where the average is over the angles.

Torrey⁵ has shown by a semi-classical argument that the magnetization satisfies a conservation law:

$$\partial \mathbf{M} / \partial t - \nabla \cdot D \nabla \mathbf{M} = 0, \quad (20)$$

where D is the diffusion coefficient, so that the magnetization current is given by $J_M = -D dM/dx$. This of course implies a similar relation between the particle current and the concentration gradient

$$J_N = -D dN/dx. \quad (21)$$

We have assumed a one-dimensional inhomogeneity in M . Hart⁶ has demonstrated that the above relations are valid even in a quantum fluid.

The Fermi distribution function $n_0(\epsilon)$ corresponds to no current, so the spin up current is obtained from the expression

$$\begin{aligned}J_{N\uparrow} &= \frac{1}{2} \int [n(\epsilon) - n_0(\epsilon)] (\partial \epsilon / \partial x) d\tau \\ &= \frac{1}{2} \int (dn_0/d\epsilon_0) \psi (\partial \epsilon / \partial x) d\tau \\ &= -\frac{1}{2} [\psi (d\tau/d\epsilon) (\partial \epsilon / \partial x)]_{\mu}.\end{aligned}\quad (22)$$

Substituting for ψ in terms of q and using the relation (6) we find that Eqs. (21) and (22) give the diffusion coefficient as

$$D = \frac{1}{3} \left[1 + \frac{1}{4} \bar{\xi} (d\tau/d\epsilon)_{\mu} \right] q v_0^2, \quad (23)$$

where v_0 is the Fermi velocity, $v_0 = p_0/m^*$. This result is in a form which can be compared directly with the kinetic theory expression for D in a Fermi gas. The modification due to the dependence of the quasi-particle energy on the distribution function is exhibited in the factor involving $\bar{\xi}$, which may be evaluated empirically from measurements of the susceptibility.² We may interpret q as the appropriate relaxation time.

⁵ H. C. Torrey, Phys. Rev. **104**, 563 (1956).

⁶ H. R. Hart, Jr., thesis, University of Illinois, Urbana, Illinois (unpublished).

We now evaluate q and thereby D . Due to the relationship between f and the forward scattering amplitude of two quasi-particles⁴ we may use $w \approx (2\pi/\hbar) f^2$. In terms of Fig. 1, forward scattering corresponds to having the angle between \mathbf{p}_1' and \mathbf{p}_2' equal to θ and also having φ equal to zero. That the first of these conditions is always approximately satisfied is obvious from the diagram; using the forward scattering amplitude to evaluate w then corresponds to equating that value—taken at $\varphi=0$ —with an average over φ . If f for $p = p' = p_0$ is expanded in a series of Legendre polynomials in the cosine of the angle, χ , between \mathbf{p} and \mathbf{p}' , the first two terms can be obtained² in terms of the velocity of sound in the liquid and the effective mass m^* , defined as the ratio of the Fermi velocity to the Fermi momentum. To this order, for He³:

$$f(\chi) = [(d\tau/d\epsilon)_{\epsilon=\mu}]^{-1} (6.4 + 3.0 \cos\chi). \quad (24)^7$$

This expression should be modified to include the effect of the limitation of collisions to those between particles of anti-parallel spin. For most transport properties the occurrence of both parallel and anti-parallel spin collisions tends to average out the effect of the spin-dependent part of f , and it is the resultant f which is determined by the empirical values of m^* and the velocity of sound in the liquid. We must therefore increase f by a factor of the order of $1 - \frac{1}{4} \bar{\xi} / \bar{f}$. From empirical data on the magnetic susceptibility² and the specific heat⁷ we estimate this factor to be 1.12.

Using this expression for f to calculate the appropriate angular average of w , we obtain $q = 2.9 \times 10^{-13} T^{-2}$ sec. The values of the other quantities needed for the calculation of D are: 0.21 for the factor involving $\bar{\xi}$ and $v_0 = 8.27 \times 10^8$ cm/sec. Then we find

$$D = A/T^2, \quad (25)$$

where $A \approx 1.5 \times 10^{-6}$ cm² sec⁻¹ degrees², and T is in °K. Because of the approximations made, the value of A may be trusted only to within about a factor of five.

This result for D agrees in order of magnitude with the empirical data obtained in the estimated region of validity of the model³, namely below 0.1°K. (D is observed to be about 1.5×10^{-3} cm²/sec at 0.03°K.)³ However, the experimental temperature dependence seems to be T^{-3} rather than T^{-2} .

A small amount of He⁴ was known to be present in the samples tested. At low temperatures one would expect scattering of He³ atoms from He⁴ to become increasingly important due to the lack of an exclusion principle for He⁴, which limits the fraction of atoms able to participate significantly in the scattering processes. We therefore investigate the behavior of the self-diffusion coefficient

⁷ Our expression differs from that of I because we have used a more recent value for m^*/m , obtained from low temperature specific heat measurements. D. F. Brewer, J. G. Gaunt, and A. K. Sreedhar, Phys. Rev. **115**, 836 (1959) give $m^*/m = 2.0 \pm 0.05$. The value of the velocity of sound in He³ at $T=0$ has also been revised slightly from 183 to 183.9 m/sec. [H. L. Laquer, S. G. Sydorik, and T. R. Roberts, Phys. Rev. **113**, 417 (1959).]

cient under the assumption that He^3-He^3 scattering is the dominant process.⁸

IMPURITY SCATTERING

Zinov'eva and Peshkov⁹ have found that one cannot maintain an arbitrarily high concentration of He^4 in He^3 at sufficiently low temperatures (below 0.88°K). A phase separation occurs; for temperatures below 0.67°K the He^3 -rich (upper) phase is in the normal rather than superfluid state. We postulate that in this phase the He^4 concentration is reduced with temperature in just such a way that condensation never occurs. We will return to this point later; for the moment we consider that the He^4 concentration is sufficiently low that no phase separation occurs over the temperature range of interest (i.e., that of the experiments³). The estimated actual impurity concentration of 1% at least approximately satisfies this condition.

If He^3-He^4 scattering is the dominant process inhibiting diffusion, then the important term in the collision integral of the Boltzmann equation for the He^3 distribution is

$$I(n) = - \int w [nN(1-n') - (1-n)n'N'] \times \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \delta(\epsilon + E - \epsilon' - E') d\tau' d\mathbf{q} d\mathbf{q}', \quad (26)$$

where N = the distribution function for He^4 , n = the distribution function for He^3 , \mathbf{p} = the initial momentum of the He^3 particle, \mathbf{q} = the initial momentum of the He^4 particle, ϵ = the initial energy of the He^3 particle, and E = the initial energy of the He^4 particle. Primes indicate the corresponding quantities in the final state.

We can define ψ as before, and similarly Φ

$$\delta N = -\Phi N_0/T. \quad (27)$$

However, at low temperatures we may neglect Φ with respect to ψ .⁸ Furthermore, since condensation is prevented, we take as an approximation to N a Boltzmann distribution. We have

$$I(n) \approx T^{-1} \int w n_0 N_0 (1 - n_0') (\psi - \psi') \times \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \delta(\epsilon + E - \epsilon' - E') d\tau d\mathbf{q} d\mathbf{q}'. \quad (28)$$

As before, we integrate over \mathbf{q}' to eliminate the momentum δ function and we have the diagram of Fig. 2 corresponding to that of Fig. 1 for pure He^3 scattering. As $q \ll p_0$, we would not expect w to depend strongly on

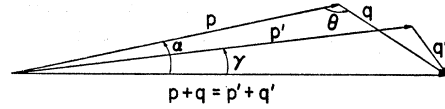


FIG. 2. Relationship of momenta for He^3-He^4 collision. The size of q and q' relative to that of p and p' has of necessity been greatly increased in order to clarify the diagram.

q . Indeed we assume it to be a constant. We define \mathbf{f} as before and introduce new variables,

$$x = (\epsilon' - \mu)/T, \quad y = E'/T, \quad t = (\epsilon - \mu)/T, \quad \kappa = \mu/T. \quad (29)$$

We next integrate over the magnitude of \mathbf{q}

$$\int q^2 \delta(\epsilon + E - \epsilon' - E') dq = M^* q, \quad (30)$$

where,

$$q = [2M^*(x + y - x)T]^{\frac{1}{2}}. \quad (31)$$

We consider the integral over \mathbf{p}' with q fixed by transforming to the variables f_r , f_z , and ϕ .

$$p_0 + (\epsilon' - \mu)/v_0 = p' \approx p + f_z \cos \alpha + f_r \sin \alpha \approx p + f_z, \quad (32)$$

$$(2M^*E')^{\frac{1}{2}} = q' \approx q + f_z \cos(\pi - \theta - \alpha) + f_r \sin(\pi - \theta - \alpha) \approx q - f_z \cos \theta + f_r \sin \theta,$$

where α is the angle between \mathbf{p} and $\mathbf{p} + \mathbf{q}$.

$$d\mathbf{p}' = p_0 \sin \gamma df_r df_z d\phi, \quad (33)$$

where γ is the angle between \mathbf{p}' and $\mathbf{p} + \mathbf{q}$. But we have the approximate relation

$$p_0 \sin \gamma \approx p_0 \sin \alpha + f_z, \quad (34)$$

and we can drop the f_z as it will tend to zero upon integrating by symmetry considerations. Furthermore,

$$p_0 \sin \alpha \approx q \sin \theta. \quad (35)$$

We transform to integration over x and y instead of f_r and f_z and insert the explicit expression for N_0 :

$$N_0 = N (2\pi T M^*)^{-\frac{3}{2}} e^{-(x+y-t)}. \quad (36)$$

Then

$$I(n) = T^{-1} \int w [n_0(1 - n_0')] (\psi - \psi') N \pi^{-\frac{3}{2}} (2\pi \hbar)^{-3} \times (T/p_0) m^* M^* (x + y - t) y^{-\frac{1}{2}} dx dy d\Omega d\phi'. \quad (37)$$

We consider the rate of flow of particles of one spin only; spin conservation in collisions then introduces a factor of $\frac{1}{2}$ into $I(n)$. As before we take $\psi = q(t) \nabla_{\mathbf{p}} \epsilon \cdot \nabla_{\mathbf{p}}$.

$$1 - n_0(t) = \frac{1}{2} \pi^{-\frac{3}{2}} (2\pi \hbar)^{-3} (T/p_0) m^* M^* N \times \int d\Omega \int_{-\infty}^{\infty} dx \int_0^{\infty} dy w [1 - n_0(x)] \times [q(t) - q(x) \cos \theta'] (x + y - t) y^{-\frac{1}{2}} e^{-(x+y-t)}, \quad (38)$$

⁸ A discussion of various other transport properties of weak solutions of He^4 in liquid He^3 with similar assumptions is given by V. N. Zharkov and V. P. Silin, J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 143 (1959) [English translation: Soviet Phys.—JETP **10**, 102 (1960)].

⁹ K. N. Zinov'eva and V. P. Peshkov, J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 33 (1959) [English translation: Soviet Phys.—JETP **10**, 22 (1960)].

with the restriction $x+y \geq l$. θ' is the angle between \mathbf{p} and \mathbf{p}' .

If we estimate $e^{\theta'} + 1 \approx 2$, we can again consider q to be a constant and

$$q \approx \frac{\pi^{\frac{1}{2}}(2\pi\hbar)^{\frac{3}{2}}p_0}{2m^*M^*NTw} \left(\frac{1}{1 - \cos\theta'} \right)_{\text{av}}. \quad (39)$$

We note that θ' is not strictly only a function of θ and ϕ , but by using relations between α , γ , and θ , we may treat it so approximately. As θ' is always small for the collisions of importance, we cannot take $1 - \cos\theta'$ to be of the order of 1. Instead we estimate this quantity by taking $p = p_0$, $q^2/2M^* = T$, $p' = (p_0^2 + q^2)^{\frac{1}{2}}$, $\theta = \pi/2$, so $q' = 0$. This gives $1 - \cos\theta'$ as approximately the ratio $T/E_F \approx 2 \times 10^{-2}$, where E_F is the Fermi energy. Thus the exclusion principle for He³ plus the low momenta of most of the He⁴ particles restricts collisions to those involving small-angle scattering. The corresponding reduction factor in the transition probability is essentially the same as that due to the extra exclusion principle restriction in He³-He³ scattering, so that we expect impurity scattering to be unimportant in limiting self-diffusion.

We will take w to be the same as that used for He³-He³ collisions with one He³ atom at rest and the other with the Fermi velocity—that is, $w = 4\langle w(\pi/3, \phi) \rangle_{\text{av}}$. (This should not include the anti-parallel spin correction factor.) To justify this, we note that exchange effects do not contribute a large amount to the transition probability in pure He³.

Then by a calculation similar to that corresponding to pure He³ we find

$$D_I = (p_0/\hbar)^{\frac{2}{3}} \pi^{\frac{1}{3}} \hbar^{\frac{2}{3}} (m^*M^*N)^{-1} \times \left(\frac{1}{2}\right) 10^2 T^{-1} [4\langle w(\pi/3, \phi) \rangle_{\text{av}}]^{-1}. \quad (40)$$

We take N to be $10^{20}/\text{cm}^3$, i.e., we take about a 1% solution. Then

$$D_I = B/T, \quad (41)$$

where $B \approx 10^{-3} \text{ cm}^2 \text{ sec}^{-1} \text{ degree}$, and T is in $^{\circ}\text{K}$.

Thus for small impurity concentrations our assumptions concerning domination of impurity scattering is wholly invalid. Experimental results on purified samples (Anderson *et al.*³) seem to verify the lack of dependence of the diffusion coefficient on He⁴ impurity concentration.

Equations (25) and (41) would appear to indicate that at sufficiently low temperatures impurity scattering would become the dominant process. However, we must take into account the phase separation process.

If we assume that condensation would occur for a concentration of 1% He⁴ at 0.1 $^{\circ}\text{K}$, then Eq. (41) must be modified to account for the temperature variation of impurity concentration:

$$D_I = BT^{-1}(0.1/T)^{\frac{1}{2}}. \quad (42)$$

We must still explain the discrepancy between the temperature dependence of D as predicted above and that observed experimentally. We note that the estimate for the upper bound on the temperature region in which the theory may be expected to be accurate was obtained by locating the temperature below which the paramagnetic susceptibility is very nearly constant, as it must be for a degenerate Fermi liquid—the case the Landau theory treats. However, the behavior of the susceptibility is governed by the exchange interaction, whereas the transport properties are determined largely by the spin-independent part of the interaction. We can estimate a mean time of flight from the magnitude of the transition probability, $w(\theta, \phi)$. Using this to estimate the uncertainty in the quasi-particle energies, we find that the temperature at which this uncertainty is of the order of the thermal energy, T , is about 0.05 $^{\circ}\text{K}$. For the theory to be valid, the quasi-particle energies should be well defined, so we expect to find verification of the theoretical predictions only well below 0.05 $^{\circ}\text{K}$. The temperatures reached so far are at best borderline by this criterion. Also, the temperature region investigated was sufficiently small so that only the local slope was observed; deviations toward faster temperature dependence at the low-temperature end might well have been too small to be noticed. One can readily plot a curve as given by the above theory which will join smoothly with that obtained experimentally by Anderson *et al.*,³ with a reasonable interpolation between, for instance, 0.01 $^{\circ}\text{K}$ and 0.03 $^{\circ}\text{K}$. Thus we believe that future experiments both with other transport properties than self-diffusion and at lower temperatures will be needed to check the theoretical predictions.

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