ators. If a_1, a_2, $\cdots$ are the annihilation operators corresponding to a complete orthonormal set of one-particle states, density operators of successively lower orders can be obtained from

$$\Gamma_p=(N-p)^{-1}\sum_i a_i\Gamma_{p+1}a_i^\dagger. \tag{15}$$

It is easy to verify that this relation is independent of the choice of basis, but not so easy to obtain useful information from it.

As one consequence of the theorem proved earlier in this paper, suppose one has a given function $K(x_1,x_2,x_1',x_2')$. A necessary condition for K to be a second-order density matrix for a 3-particle system is that the first-order density matrix obtained from K have the same positive eigenvalues as K itself. From the eigenfunctions, possible wave functions for the system could be constructed according to Eq. (4a). Likewise, knowledge of the eigenvalues of the second-order density matrix of a 4-particle system gives extensive information about the wave function. In the latter case, for example, if the positive eigenvalues were nondegenerate, the wave function would necessarily be of the form

$$\psi=6^{-\frac{1}{2}}\sum_r \lambda_r^{\frac{1}{2}}c_r g_r(1,2)g_r(3,4),$$

where the coefficients c_r have modulus unity. In the other extreme, where the density operator has six eigenfunctions with eigenvalue 1 and all other eigenvalues are zero, the wave function would be

$$\psi=6^{-\frac{1}{2}}\sum_{r,s} c_{rs}g_r(1,2)g_s(3,4),$$

where the c's form a unitary matrix. The extra coefficients c appear in these last expressions because, without knowledge of the wave function, the functions G_r are indeterminate by a phase factor and, in the case of degenerate eigenvalues, by a unitary transformation.

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Nonlinear Interaction of an Electromagnetic Wave with a Plasma Layer in the Presence of a Static Magnetic Field. I. Theory of Harmonic Generation*

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The theory of electromagnetic wave propagation through an anisotropic ionized layer, including the effects of the nonlinear terms in the Boltzmann transport equation, is presented. The method of solution of the nonlinear equations involves an expansion of all of the dependent variables in a Fourier series in time. The differential equations describing wave propagation are then solved, for each frequency in the series, for plane wave propagation, including all of the reflections within the plasma layer. A solution in closed form has been obtained, under small signal conditions, for the field at the kth harmonic in the Fourier series. A discussion of the properties of the wave at the second harmonic frequency as a function of the dc magnetic field strength, the electron density, the electron-neutral particle collision frequency, the field strength of the incident wave, and the thickness of the plasma layer is given.

1. INTRODUCTION

THE propagation characteristics of an electromagnetic wave in the presence of an ionized medium have been discussed by Schlüter,[1,2] Bailey,[3–5] Spitzer,[6] Brown,[7] and many others.[8] These discussions are based upon a set of equations which includes Maxwell's equations and the dynamical equations for an ionized gas. The dynamical equations are obtained, at least implicitly, from the Boltzmann equation, and they are inherently nonlinear equations. However, the usual procedure is to linearize these equations since a general method for obtaining solutions to the nonlinear equations is not available. It is the purpose of this paper to discuss the effects of the nonlinear terms in the equations on electromagnetic wave propagation phenomena.

Relatively little has been reported on solutions to these equations when the nonlinear terms are included. Ginsburg[9] has discussed the mixing of two electromagnetic waves when one of the waves, at frequency ω_1, causes an electron density gradient which varies at the ω_1 rate. The interaction of this electron density variation with a second electromagnetic wave, at frequency ω_2, is

* This research was supported in part by the U. S. Air Force Cambridge Research Laboratories.
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[7] S. C. Brown, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. XXII, p. 531.
[8] For an exhaustive list see "Bibliography on Plasma Physics and Magnetohydrodynamics," Engineering and Physical Sciences Library, University of Maryland, College Park, Maryland, 1959.
[9] V. L. Ginzburg, J. Exptl. Theoret. Phys. U.S.S.R. **35**, 1573 (1958) [translation: Soviet Phys.—JETP **35**(8), 1100 (1959)].

then discussed. Huxley and Ratcliffe[10] have discussed the case of cross-modulation, the so-called Luxemburg effect. This occurs when the modulation on a strong electromagnetic wave produces a variation in the electron-neutral particle collision frequency, because of its electron temperature dependence, at the modulation rate. This modulation is then transferred to a second signal which is traversing the same part of the medium. These papers[9,10,11] discuss the interaction of a strong signal and a weak signal with a plasma when the strong signal produces a variation in one of the plasma parameters. Although such phenomena are related, in a certain sense, to the nonlinear nature of an ionized gas, in effect these discussions relate to the interaction of an electromagnetic wave with a plasma in which one of the plasma parameters varies as a function of time.

Sturrock[11] has given a general approach to the effects of nonlinearities on electron plasma oscillations in the absence of collisions. Dawson[12] has discussed several special cases for a similar situation. However, a hydrodynamical approach is employed in these two papers and they are, therefore, restricted to low-frequency phenomena.

It is the purpose of this paper to discuss the nonlinear interaction of an electromagnetic wave incident on a layer of ionized gas in the presence of a dc magnetic field. The ionized medium is assumed to be of finite thickness in the direction of propagation of the wave and infinite in extent in the other directions. It is assumed that the ordered motion of the plasma caused by the electromagnetic forces predominates over the random motion caused by thermal and pressure effects and, therefore, these terms in the equations are neglected in the following development. However, the effects of electron-neutral particle collisions are included in the theory. An iteration technique is employed in order to solve the nonlinear equations, and this scheme restricts the validity of the solutions to the case of small amplitude signals. A solution, in closed form, is found for the second harmonic power generated within an ionized medium because of the nonlinear interaction of the wave and the plasma.

In subsequent reports on this work, we will discuss the experimental verification of the theoretical predictions. In addition, a measure of the range of validity of the small signal theory, in terms of the magnitude of the incident power, will be given. The equations describing the propagation of the incident wave will be reiterated to obtain the effects of the nonlinear terms on propagation at the fundamental frequency. We will also discuss effects introduced when the incident wave produces ionization within the plasma layer.

[10] L. G. H. Huxley and J. A. Ratcliffe, Proc. Inst. Elect. Eng. **96.3**, 433 (1949).
[11] P. A. Sturrock, Proc. Roy. Soc. (London) **A242**, 277 (1957).
[12] J. M. Dawson, Phys. Rev. **113**, 383 (1959).

2. AN OUTLINE OF THE THEORY

The theoretical model to be discussed is assumed to have the following properties:

(a) The plasma is electrically neutral in the absence of the externally imposed electromagnetic wave.
(b) The electron density is uniform in the absence of the externally imposed electromagnetic wave.
(c) All inelastic collisions between the plasma constituents are neglected.
(d) Only elastic collisions between electrons and neutral particles are included, and these are assumed to be represented by a constant, ν, which is independent of the electron velocity.
(e) Thermal gradients are neglected.
(f) Pressure gradients are neglected.
(g) The positive ion current is neglected.
(h) The plasma has a finite thickness, d, in the direction of propagation and is infinite in the other directions.
(i) The externally imposed electromagnetic wave is a plane wave propagating normal to the plasma surface.
(j) The externally imposed dc magnetic field is assumed to be uniform throughout the extent of the plasma.

Assumptions (a), (e), (f), (g), and (j) are generally accepted as providing a reasonable model of an ionized gas under steady state conditions. Assumptions (c) and (d) are good approximations in the case of He gas when the electron temperature is above two electron volts, particularly in the case of propagation of a weak electromagnetic wave. Assumptions (b), (h), and (i) are not usually satisfied in a practical situation. However, these conditions can be approached experimentally[13] and have been reasonably approximated in the experimental arrangement to be discussed in a subsequent report.

The following set of equations will be used to describe the interaction of an electromagnetic wave with an anisotropic ionized medium under the above set of assumptions:

$$\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t, \qquad \nabla\cdot\mathbf{B}=0; \tag{1}$$

$$\nabla\times\mathbf{H}=\partial\mathbf{D}/\partial t+en\mathbf{v}, \quad \nabla\cdot\mathbf{D}=e(n-n_{i0}); \tag{2}$$

$$\partial n/\partial t+\nabla\cdot n\mathbf{v}=0; \tag{3}$$

$$\partial\mathbf{v}/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{v}=(e/m)(\mathbf{E}+\mathbf{v}\times\mathbf{B})-\nu\mathbf{v}; \tag{4}$$

and $\mathbf{D}=\epsilon_0\mathbf{E}$ and $\mathbf{B}=\mu_0\mathbf{H}$, where $\mathbf{E}$ and $\mathbf{H}$ represent the electric and magnetic field intensity vectors, respectively; ϵ_0 is the dielectric constant of free space; μ_0 is the permeability of free space; n is the electron density; n_{i0} is the steady-state ion concentration; $\mathbf{v}$ is the electron velocity vector; e is the charge of an electron and is negative; m is the mass of an electron; and ν is the electron-neutral particle collision frequency. Mks units are used throughout this discussion. Equations (1) and

[13] J. Kannelaud and R. Whitmer, J. Appl. Phys. (to be published).

(2) are Maxwell's equations, Eq. (3) expresses the conservation of the number of electrons, and Eq. (4) expresses the conservation of momentum of the electrons. Equations (3) and (4) can be obtained[6] by taking the first and second velocity moments of the Boltzmann equation if it is assumed that the collision integral is independent of the electron velocity. It can be seen from Eqs. (2), (3), and (4) that the terms $n\mathbf{v}$, $(\mathbf{v}\cdot\nabla)\mathbf{v}$, and $\mathbf{v}\times\mathbf{B}$ are quadratic with respect to the physical variables, and these are the nonlinear terms which will be taken into account by the present theory.

Since a method of obtaining a general solution to a set of equations such as Eqs. (1) through (4) is not available, a perturbation technique for obtaining an approximate solution will be employed. It will be assumed that each of the variables $\mathbf{E}$, $\mathbf{B}$, n, and $\mathbf{v}$ can be expanded in a Fourier series of the form

$$\mathbf{f}(r,t)=\mathbf{f}_0+\sum_{h=1}^{\infty}\mathbf{f}_h(r)e^{-ih\omega t}, \tag{5}$$

where the incident wave is assumed to be at frequency ω and $\mathbf{f}_1\gg\mathbf{f}_2\gg\cdots\gg\mathbf{f}_h$ and $\mathbf{f}_0$ is a constant. This assumption is justified when the incident wave is of sufficiently low amplitude so that the nonlinear terms are of second order importance. The procedure then will be to write Eqs. (1)–(4) in terms of the series representations of the dependent variables given in Eq. (5). The nonlinear terms will be written as products of the appropriate series. Then, in each equation, terms multiplied by the same power of $e^{-i\omega t}$ will be equated. This will result in a set of equations, similar to Eqs. (1)–(4), for each power of $e^{-i\omega t}$. Within each of these sets of equations any term which involves a dependent variable with a subscript which is higher than the power of $e^{-i\omega t}$ for that particular set of equations will be neglected. For example, in the set of equations for the $e^{-2i\omega t}$ power there may be terms of the form $\mathbf{f}_0\mathbf{f}_2$, $\mathbf{f}_1\mathbf{f}_1$, $\mathbf{f}_1{}^*\mathbf{f}_3$, etc., where the asterisk indicates the complex conjugate. All terms involving $\mathbf{f}_3$ or higher will be neglected since it is assumed that $\mathbf{f}_3\ll\mathbf{f}_2\ll\mathbf{f}_1$.

The next step will be to solve the individual sets of equations. Each set of equations consists of four equations in four unknowns and, therefore, each set can be reduced to one equation in one unknown. For example, the single equation for the $e^{-3i\omega t}$ set will appear as

$$\mathbf{L}(\mathbf{f}_3)=G(\mathbf{f}_1\mathbf{f}_2),$$

where $\mathbf{L}$ is a differential operator. Such an equation will be treated as an inhomogeneous equation in which the right-hand side is a known function which has been determined from solutions to the lower order equations. This equation can be solved directly. In addition, the equations can be reiterated at any step to provide successively more accurate approximations to the solutions obtained up to that point. A discussion of this will appear in a subsequent report.

3. THE GENERAL EQUATIONS

If the series form for the dependent variables in Eq. (5) is substituted into Eqs. (1)–(4), these equations can be combined to form a single equation for $\mathbf{E}_h$, the electric field for the hth harmonic, in terms of lower order quantities. This equation is

$$\begin{aligned}&[\nu-ih\omega+(\mathbf{v}_0\cdot\nabla)-\boldsymbol{\omega}_c\times]\\&\quad\times[\nabla\times\nabla\times\mathbf{E}_h+(ih\omega/c)^2\mathbf{E}_h-(ih\omega/c^2)(\nabla\cdot\mathbf{E}_h)\mathbf{v}_0]\\&\qquad-(ih\omega/c^2)\omega_p{}^2\mathbf{E}_h-(\omega_p{}^2/c^2)\mathbf{v}_0\times(\nabla\times\mathbf{E}_h)=\mathbf{G}_h,\end{aligned} \tag{6}$$

where

$$\boldsymbol{\omega}_c=-(e/m)\mathbf{B}_0,\quad \omega_p{}^2=n_0e^2/m\epsilon_0, \tag{7}$$

and $\mathbf{G}_h$ is of the form

$$\mathbf{G}_h=\tfrac{1}{2}(1-\delta_{h1})\sum_{s=1}^{h-1}\mathbf{A}_s\mathbf{D}_{h-s}+\tfrac{1}{2}\sum_{s=h}^{\infty}(\mathbf{A}_{s+h}\mathbf{D}_s{}^*+\mathbf{A}_s{}^*\mathbf{D}_{s+h}), \tag{8}$$

where $\mathbf{A}$ and $\mathbf{D}$ represent typical variables such as $\mathbf{v}$, $\mathbf{B}$, or n, and δ_{h1} equals zero for $h\neq1$ and one for $h=1$. The factor "one-half" enters since when a term such as "$\mathbf{v}\times\mathbf{B}$" is written in Eq. (4) the implied meaning is "Re$\mathbf{v}\times$Re$\mathbf{B}$." Because of the iteration procedure adopted here, in which it is assumed that each succeeding term in the series of Eq. (5) is much smaller than the preceding term, all terms of order greater than h will be dropped in Eq. (8). This simply implies that these terms are assumed to be of second-order importance when compared with the remaining terms. Equation (6) then reduces to

$$\begin{aligned}&[\nu-ih\omega+(\mathbf{v}_0\cdot\nabla)-\boldsymbol{\omega}_c\times]\\&\quad\times[\nabla\times\nabla\times\mathbf{E}_h+(ih\omega/c)^2\mathbf{E}_h-(ih\omega/c^2)(\nabla\cdot\mathbf{E}_h)\mathbf{v}_0]\\&\qquad-(ih\omega/c^2)\omega_p{}^2\mathbf{E}_h-(\omega_p{}^2/c^2)\mathbf{v}_0\times(\nabla\times\mathbf{E}_h)\\&\cong\frac{ih\omega\omega_p{}^2(1-\delta_{h1})}{2c^2}\\&\quad\times\sum_{s=1}^{h-1}\{(\mathbf{v}_s\times\mathbf{B}_{h-s})-(m/e)(\mathbf{v}_s\cdot\nabla)\mathbf{v}_{h-s}\\&\qquad+(m/en_0)[\nu-ih\omega+(\mathbf{v}_0\cdot\nabla)-\boldsymbol{\omega}_c\times]n_s\mathbf{v}_{h-s}\},\end{aligned} \tag{9}$$

and this equation, with the appropriate boundary conditions at the surfaces of the layer, is to be solved.

The method for obtaining plane wave solutions to Eq. (9) is straightforward. Setting $h=1$ and assuming

$$\mathbf{E}_1\propto e^{\pm i\mathbf{k}_1\cdot\mathbf{r}}, \tag{10}$$

where $\mathbf{k}_1$ is the propagation vector of the wave at frequency ω, the right-hand side of Eq. (9) reduces to zero. The equation for $\mathbf{E}_1$ is linear and homogeneous, and the solution for the jth component of $\mathbf{E}_1$ is

$$E_{1j}=E_{1j}{}^{\pm}e^{\pm i\mathbf{k}_1\cdot\mathbf{r}},\quad j=x,y,z, \tag{11}$$

where the k_{1j} are well known[14] and the $E_{1j}^{\pm}$ are determined by the boundary conditions. Setting $h=2$ yields an inhomogeneous equation in which the right-hand side can be determined from the solution for $\mathbf{E}_1$. The E_{2j} are then given by

$$E_{2j} \propto (\mathbf{E}_{2c}^{\pm})_j e^{\pm i\mathbf{k}_2\cdot\mathbf{r}} + (\mathbf{E}_{2p}^{\pm})_j e^{\pm i2\mathbf{k}_1\cdot\mathbf{r}}, \qquad (12)$$

where the $\mathbf{E}_{2c}^{\pm}$ are the complementary solutions and the $\mathbf{E}_{2p}^{\pm}$ are the particular solutions to the inhomogeneous equation and $\mathbf{k}_2$ is the propagation vector of the wave at frequency 2ω. The $\mathbf{E}_{2p}^{\pm}$ are completely determined from $\mathbf{E}_1$, and the $\mathbf{E}_{2c}^{\pm}$ can be determined from the boundary conditions. By induction the general solution to Eq. (9) can then be written as

$$\mathbf{E}_h = (\mathbf{E}_{hc}^{\pm})e^{\pm i\mathbf{k}_h\cdot\mathbf{r}} + \sum_m (\mathbf{E}_{hp}^{\pm})_m \prod_q e^{\pm i(\mathbf{k}_q)_m\cdot\mathbf{r}}, \qquad (13)$$

where the partitions of h should be arranged in an arbitrary order and so numbered.[15] m then indicates the number of a particular partition and the sum is taken over all the partitions of h. q signifies all the integers in a particular partition and the $\mathbf{k}_q$ should be taken with all $\pm$ combinations. The $\mathbf{E}_{hc}^{\pm}$ are the complementary solutions and the $\mathbf{E}_{hp}^{\pm}$ are the particular solutions to Eq. (9), and $\mathbf{k}_h$ is the propagation constant obtained from the solution to the homogeneous equation for $\mathbf{E}_h$.

In order to simplify, somewhat, the calculations to follow, two additional assumptions will be made. First, it will be assumed that $\mathbf{v}_0=0$. Since it can be shown that $\mathbf{v}_0$ enters into the equations in such a manner that $\mathbf{v}_0$ is compared with c, this assumption is easily satisfied in most experimental arrangements. Secondly, the following discussion will be restricted to the case of propagation in the x direction where the uniform dc magnetic field is assumed to be in the z direction. This case has been chosen because of the ease in which it can be realized experimentally. In this situation it is known[14] that

$$\mathbf{E}_1 = (\hat{x}_0 E_{1x}^{\pm} + \hat{y}_0 E_{1y}^{\pm})e^{\pm ik_{1x}x} + \hat{z}_0 E_{1z}^{\pm} e^{\pm ik_{1z}x}, \qquad (14)$$

where k_{1z} is independent of the dc magnetic field and

$$E_{1x}^{\pm} = \gamma_1 E_{1y}^{\pm}, \qquad (15)$$

where

$$\gamma_1 = \frac{\omega_c[c^2k_{1x}^2 - \omega^2]}{\omega[\nu\omega + i(\omega_p^2 - \omega^2)]}. \qquad (16)$$

Later in the discussion it will be shown that the effects of the nonlinear terms are predominant in the region $\omega \approx \omega_c$. Since the z component of the incident field does not depend upon B_0, and E_z never enters into the equations in such a way as to be multiplied by B_0, the z component can only affect the harmonics as a second order effect. Consequently, it will be assumed that $E_z=0$ because of the simplification in the resulting equations. Hereafter, k_{1x} will be written k_1. (These conclusions concerning the importance of E_{1z} have been verified experimentally, as will be discussed in a subsequent report.)

From the homogeneous part of Eq. (9) one obtains, for the general case,

$$k_h = -\frac{ih\omega}{c} \times \left[-1 + \frac{\omega_p^2/h\omega[h^2\omega^2 - \omega_p^2 + ih\omega\nu]}{h\omega(h^2\omega^2 - \omega_p^2 - \omega_c^2 - \nu^2) + i\nu(2h^2\omega^2 - \omega_p^2)}\right]^{\frac{1}{2}}, \qquad (17)$$

and

$$\gamma_h = \frac{\omega_c[c^2k_h^2 - \omega^2]}{h\omega[h\omega\nu + i(\omega_p^2 - h^2\omega^2)]}. \qquad (18)$$

Before proceeding to the solutions of Eq. (9), it is of interest to examine the nonlinear terms. The nonlinearities enter into the equation in three terms, $\mathbf{v}\times\mathbf{B}$, $(\mathbf{v}\cdot\nabla)\mathbf{v}$, and $n\mathbf{v}$. The effects of these terms can be foreseen intuitively through the examination of the case $h=2$. For this case

$$\mathbf{v}_1\times\mathbf{B}_1 = \frac{\pm ik_1}{\omega^2\mu_0 e n_0} E_{1y}^{\pm}[\hat{x}_0(k_1^2 - \omega^2/c^2)E_{1y}^{\pm} + \hat{y}_0(\omega^2/c^2)E_{1x}^{\pm}], \qquad (19)$$

$$(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1 = \frac{\pm ik_1}{\mu_0^2c^2e^2n_0^2} E_{1x}^{\pm}[\hat{x}_0(-\omega^2/c^2)E_{1x}^{\pm} + \hat{y}_0(k_1^2 - \omega^2/c^2)E_{1y}^{\pm}], \qquad (20)$$

$$n_1\mathbf{v}_1 = \frac{\pm k_1}{\omega\mu_0^2c^2e^2n_0} E_{1x}^{\pm}[\hat{x}_0(-\omega^2/c^2)E_{1x}^{\pm} + \hat{y}_0(k^2 - \omega^2/c^2)E_{1y}^{\pm}], \qquad (21)$$

since the terms involving $E_1^+E_1^-$ cancel. The right-hand side of Eq. (9) then becomes

$$\mathbf{G}_2 = \frac{\omega\omega_p^2(\pm ik_1)(E_{1y}^{\pm})^2}{c^2\mu_0 e n_0}\left\{\left[\frac{\gamma_1\omega_c}{\omega\omega_p^2} - \frac{i}{\omega^2}\right] \times [\hat{x}_0(k_1^2 - \omega^2/c^2) + \hat{y}_0(\omega^2/c^2)\gamma_1] + \gamma_1\left[\frac{(\nu - 2i\omega)}{\omega\omega_p^2} - \frac{i}{\omega_p^2}\right] \times [-\hat{x}_0(\omega^2/c^2) + \hat{y}_0(k_1^2 - \omega^2/c^2)]\right\}, \qquad (22)$$

where the terms multiplying $\gamma_1\omega_c$ and $\nu - 2i\omega$ come from $n_1\mathbf{v}_1$, the term multiplying i/ω^2 from $\mathbf{v}_1\times\mathbf{B}_1$, and the term multiplying i/ω_p^2 from $(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1$. First, it is apparent that the nonlinear terms may be of the same order of magnitude, depending upon the values of the plasma parameters, and therefore all three terms must be retained in Eq. (9). Furthermore, if $\omega_c=0$, which implies that the longitudinal component to the electric field at frequency ω is zero, all the nonlinear terms are zero except for the $\mathbf{v}_1\times\mathbf{B}_1$ term, and this term has only a longitudinal component. Therefore, the wave at the second harmonic frequency is longitudinal and there is no power flow associated with this wave. Hence, the

[14] R. F. Whitmer, Microwave J. 2, 3, 47 (1959).

[15] E.g., the partitions of 4 are (1,3); (1,1,2); (1,1,1,1); (2,2).

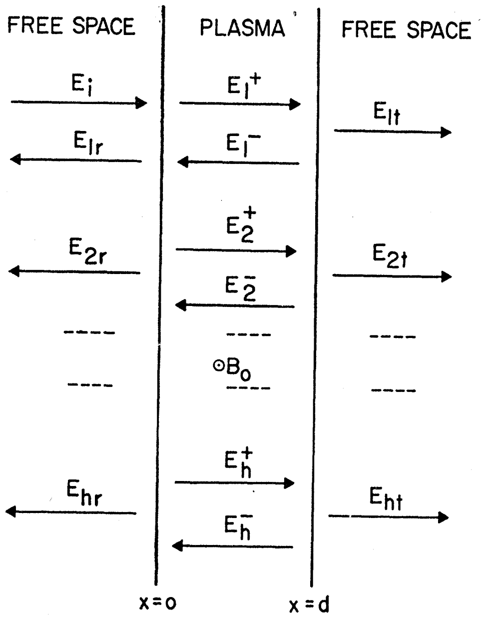

FIG. 1. The plasma layer and the reflected and transmitted waves.

propagation of energy is affected by the nonlinearities only to a third-order approximation when $\omega_c=0$. The effect of the dc magnetic field is to couple the longitudinal wave to the incident transverse wave, giving rise to second-order effects. In addition, since k_1 and γ_1 have resonances in the region $\omega\approx\omega_c$, it would be expected that the nonlinearities are most effective within this frequency range. It can be shown that the coupling of $\mathbf{E}_s^+$ with $\mathbf{E}_s^-$ does not affect any of the even harmonics but does play a role in the generation of the odd harmonics.

In order to proceed to the solution of Eq. (9) it is necessary to introduce the boundary conditions. The plasma will be assumed to exist between $x=0$ and $x=d$, with free space on either side of the plasma. The geometry of the system is as shown in Fig. 1. Since E_x and E_y are related it is sufficient to employ the condition that the tangential components of E and H are continuous across the boundaries in order to completely determine the required fields. Following Stratton,[16] $(\mathbf{E}_c^\pm)_y$ is given by

$$(B_{\alpha\beta})\begin{bmatrix} E_{hr} \\ E_{ht} \\ (\mathbf{E}_{hc}^+)_y \\ (\mathbf{E}_{hc}^-)_y \end{bmatrix} = \begin{bmatrix} p_{hr} \\ p_{ht} \\ p_{hc}^+ \\ p_{hc}^- \end{bmatrix}, \tag{23}$$

where

$$B_{\alpha\beta}=\begin{bmatrix} 1 & 0 & -1 & -1 \\ -k_0 & 0 & -k_h & k_h \\ 0 & e^{ik_0d} & -e^{ik_hd} & -e^{-ik_hd} \\ 0 & k_0e^{ik_0d} & -k_he^{ik_hd} & k_he^{-ik_hd} \end{bmatrix}, \tag{24}$$

[16] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 511.

and

$$\begin{aligned} p_{hr}&=-\delta_{h1}E_0+\textstyle\sum_m [(\mathbf{E}_{np}^+)_m+(\mathbf{E}_{hp}^-)_m]_y, \\ p_{ht}&=-k_0E_0\delta_{h1}+\textstyle\sum_m(k_q)_m[(\mathbf{E}_{hp}^+)_m-(\mathbf{E}_{hp}^-)_m]_y, \\ p_{hc}^+&=\textstyle\sum_m [(\mathbf{E}_{hp}^+)_m \prod_q e^{i(k_q)md} \\ &\qquad +(\mathbf{E}_{hp}^-)_m \prod_q e^{-i(k_q)md}]_y, \\ p_{hc}^-&=\textstyle\sum_m(k_q)_m[(\mathbf{E}_{hp}^+)_m \prod_q e^{i(k_q)md} \\ &\qquad -(\mathbf{E}_{hp}^-)_m \prod_q e^{-i(k_q)md}]_y, \end{aligned} \tag{25}$$

where it has been assumed that

$$\mathbf{E}_i=\hat{y}_0E_0e^{-i\omega t+ik_0x}. \tag{26}$$

In principle, then, one can determine completely the field both inside the plasma layer and in free space from the above equations. Therefore, each term in the solution for the hth harmonic, Eq. (13), has been specified with the exception of $\mathbf{E}_{hp}$. $\mathbf{E}_{hp}$ is, in each case, known from Eq. (9) and Eq. (13); however, the form of $\mathbf{E}_{hp}$ is much too complex to write down for the hth harmonic and, therefore, it will be derived for the specific cases to be discussed in this paper.

4. THE SECOND HARMONIC

In this section the solution for the second harmonic wave will be discussed in detail. The wave at the second harmonic frequency can be determined from Eq. (9) and Eq. (13) by setting $h=2$. The nonlinear terms for this case have already been written in Eq. (22). $(\mathbf{E}_{2p}^\pm)_y$ becomes

$$(\mathbf{E}_{2p}^\pm)_y=(eD/mc\omega)(S^\pm/a)E_0^2, \tag{27}$$

where

$$\begin{aligned} a&=[(1+b_1)^2-e_1^2(1-b_1)^2]^2, \\ S^\pm&=[2(1\pm b_1)e_1^{\delta_\pm}]^2, \end{aligned} \tag{28}$$

where $\delta_\pm=(1\mp 1)/2$ and

$$D=ib_1\frac{gp-mo}{gr-qo}, \tag{29}$$

where

$$\begin{aligned} b_1&=ck_1/\omega, \\ e_1&=e^{ik_1d}, \\ g&=2i(4-\omega_p^2/\omega^2)-4\nu/\omega, \\ m&=\gamma_1^2(\omega^2/\omega_p^2)(3i-\nu/\omega)+(1-b_1^2)(i-\gamma_1\omega_c\omega/\omega_p^2), \\ o&=4\omega_c/\omega, \\ p&=\gamma_1^2\omega_c\omega/\omega_p^2+\gamma_1[(1-b_1^2)(\omega^2/\omega_p^2)(3i-\nu/\omega)-i], \\ q&=-4(\omega_c/\omega)(1-b_1^2), \\ r&=2i[4(1-b_1^2)-\omega_p^2/\omega^2]-(4\nu/\omega)(1-b_1^2), \end{aligned} \tag{30}$$

and Eq. (23) was used to obtain $E_{1y}^\pm$ in terms of E_0. Now Eq. (23) can be used to determine $(\mathbf{E}_{2c}^\pm)_y$.

Since the most accurately measured property of the second harmonic wave is the power, at 2ω, which is radiated from the plasma layer, this quantity will be calculated directly. If P_2 is the power per unit area, at frequency 2ω, in free space, then

$$P_2=\tfrac{1}{2}\,\mathrm{Re}(\mathbf{E}_{2t}\times\mathbf{H}_{2t}^*),\tag{31}$$

where $\mathbf{E}_{2t}$ can be determined from Eqs. (23) and (27). Equation (31) can be written as

$$P_2=(1/2\mu_0c)[(\mathrm{Re}E_{2t})^2+(\mathrm{Im}E_{2t})^2],\tag{32}$$

where

$$E_{2t}=eE_0{}^2D\mathfrak{D}/(mc\omega aA),\tag{33}$$

and

$$\mathfrak{D}=\begin{vmatrix} 1 & -1 & -e_2 & (S^+)+e_1{}^2(S^-) \\ -1 & -b_2 & b_2e_2 & b_1[(S^+)-e_1{}^2(S^-)] \\ 0 & -e_2 & 1 & -[e_1{}^2(S^+)+(S^-)] \\ 0 & b_2e_2 & -b_2 & -b_1[e_1{}^2(S^+)-(S^-)] \end{vmatrix},\tag{34}$$

$$\begin{aligned} b_2&=ck_2/2\omega,\\ e_2&=e^{ik_2d},\\ A&=(1+b_2)^2-e_2{}^2(1-b_2)^2.\end{aligned}\tag{35}$$

Equation (32) then can be written as

$$P_2=(2\mu_0e^2/m^2c)(P_0/\omega)^2Q(\omega_p/\omega,\nu/\omega,\omega_c/\omega,\omega d/c),\tag{36}$$

where P_0 is the incident power and Q is a dimensionless function of the plasma parameters.

Before discussing P_2, the following characteristics of the field at 2ω can be noted. First, there are two waves at 2ω, one with a propagation constant of k_2 (corresponding to the complementary solution) and one with a propagation constant $2k_1$ (corresponding to the particular solution). Furthermore, $E_{2cy}=\gamma_2E_{2cx}$ where γ_2 can be obtained from γ_1 by replacing k_1 by k_2 and ω by 2ω. In addition, when $\omega\approx\omega_c$, k_1, and hence $2k_1$, exhibits a resonant characteristic whereas k_2 does not. Therefore, it may be expected that the wave corresponding to the particular solution will contain the major portion of the power in the second harmonic wave.

From Eq. (33) and Eq. (36),

$$Q=DD^*\mathfrak{D}\mathfrak{D}^*/(AA^*aa^*),\tag{37}$$

where Q is the quantity to be investigated in order to predict the dependence of the power radiated, at the second harmonic frequency, on the plasma parameters. The value of Q for various extremes of the parameters can be seen immediately. For example, if the ambient electron density goes to zero, then $\omega_p=0$, $k_1=\omega_c$, and $\gamma_1=0$. Therefore, $D=0$ and, consequently, $Q=0$, as would be expected. Similarly, if $\omega_c=0$, $\gamma_1=0$, $D=0$, and therefore $Q=0$. It can be shown that in this case a purely longitudinal wave exists within the plasma at the second harmonic frequency, and since no power is transmitted by this wave the output power, P_2, is zero. If d goes to zero then it can be seen that $D=0$ and therefore $Q=0$.

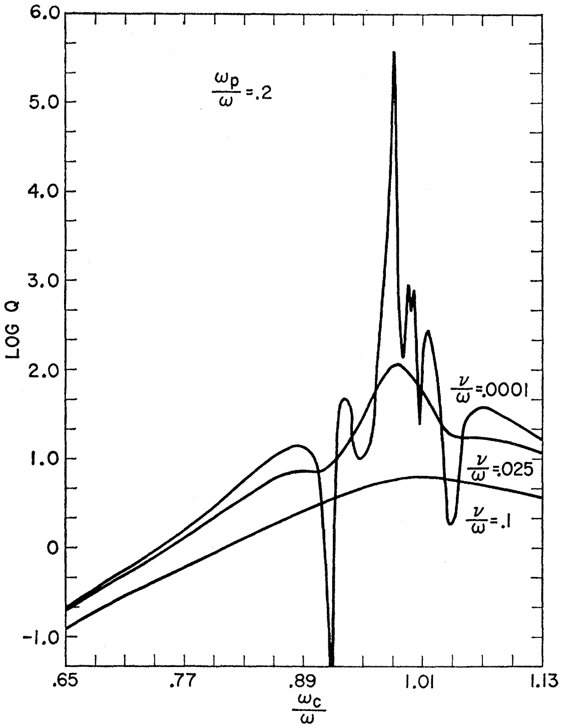

FIG. 3. Second harmonic power versus ω_c/ω (near resonance) for $\omega_p/\omega=0.2$.

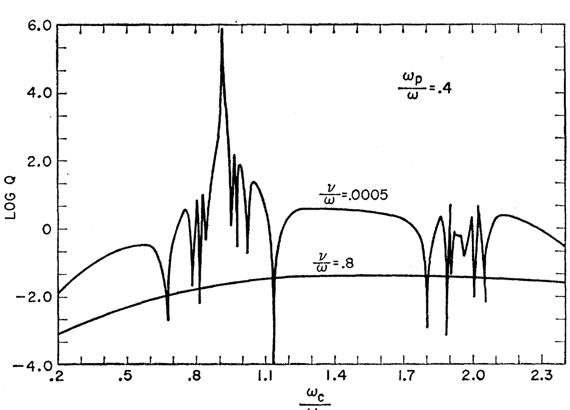

FIG. 2. Second harmonic power versus ω_c/ω for $\omega_p/\omega=0.4$.

It is of more interest to examine the resonance properties of Q. However, due to the complexity of this function it is not easy to perform such an examination. Therefore, curves of the output power, at the second harmonic frequency, as a function of the plasma parameters, obtained from computer plots of Eq. (36), will be presented. A typical case, $\omega d/c=18.63$, is used.

From Eq. (36)

$$Q=\frac{m^2c}{2\mu_0e^2}\frac{P_2}{(P_0/\omega)^2},\tag{38}$$

where P_2 and P_0 are in watts/m². $\mathrm{Log}_{10}Q$ versus ω_c/ω is plotted in Fig. 2 for $\omega_p/\omega=0.4$ and for two representative values of ν/ω. For the low value of ν/ω it can be seen that there is one major peak in Q near $\omega_c/\omega=0.915$ and that there are several smaller resonances in Q on either side of this value of ω_c/ω. In addition, there is a series of

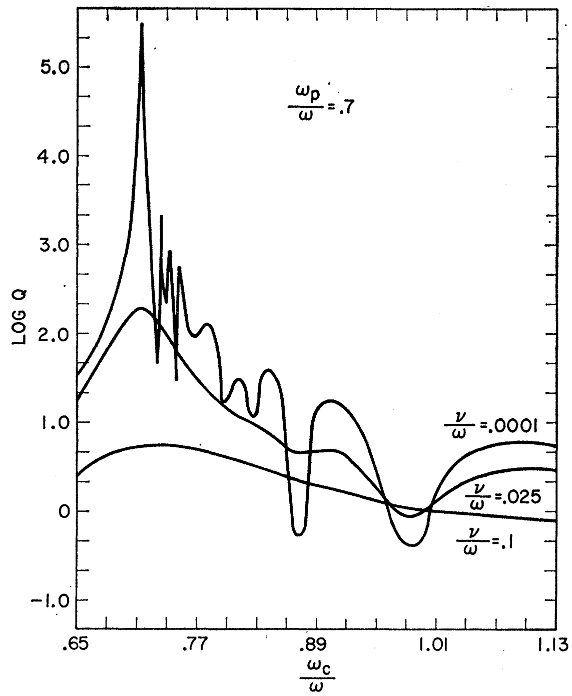

FIG. 4. Second harmonic power versus ω_c/ω (near resonance) for $\omega_p/\omega=0.7$.

resonances near $\omega_c/\omega=2\times0.915$, but these peaks are much smaller than the major peak. For the large value of ν/ω, all of the resonance effects are smoothed over and the amplitude of Q is much lower than in the preceding case. In Figs. 3 and 4, $\log_{10}Q$ versus ω_c/ω in the region of the major peak is plotted for several values of ν/ω, for $\omega_p/\omega=0.2$ and 0.7, respectively. From these curves it can be seen that the amplitude of Q decreases with increasing ν/ω and that the width of the resonance curve increases with increasing ν/ω. For low values of ν/ω there are many minor resonances and these are smoothed over as ν/ω increases. The position of the major resonance is given by

$$(\omega_c/\omega)_{Q_{\max}}=[1-(\omega_p/\omega)^2]^{\frac{1}{2}}+(\nu/\omega)^2, \qquad (39)$$

where $Q_{\max}$ indicates the value of P when ω_c/ω is adjusted for maximum second harmonic power. This indicates that for small values of ω_p/ω and ν/ω the resonance occurs for $\omega_c/\omega\approx1$; and as ω_p/ω is increased the position of the resonance shifts to lower values of ω_c/ω, whereas the shift is to higher values of ω_c/ω as ν/ω increases. The minor resonances, which appear only for $\nu/\omega<0.03$, are caused by reflections within the plasma layer. These disappear for high values of ν/ω due to the increased attenuation within the plasma. The position of the maxima and the minima corresponding to these boundary resonances is given approximately by

$$\omega_c/\omega=\left\{[1-(\omega_p/\omega)^2]\left[\frac{1-(\omega_p/\omega)^2-\beta^2}{1-\beta^2}\right]\right\}^{\frac{1}{2}}, \qquad (40)$$

where

$$\beta=\frac{\pi}{a}\left[\frac{4m+1}{4}\right], \quad m=1,2,3,\cdots \qquad (41)$$

for the maxima and β satisfies the relationship

$$\tan(a\beta)=-a\beta\left|\frac{1-\beta^2}{1+\beta^2}\right| \qquad (42)$$

for the minima. In these equations

$$\beta=(c/\omega)\,\mathrm{Re}k_1, \qquad (43)$$

so that these resonances due to the boundary effects are caused by standing waves within the plasma produced by the wave propagating at frequency ω with a propagation constant k_1. All of the resonances predicted by Eq. (40) do not appear in Figs. 3 and 4. However, for very low values of ν/ω (i.e., $\nu/\omega<0.0001$) they do exist.

It is probably of more practical interest to examine $Q_{\max}$ as a function of the parameters. In Fig. 5, $\log_{10}Q_{\max}$ versus ν/ω is plotted, and Fig. 6 shows $\log_{10}Q_{\max}$ versus ω_p/ω. It can be seen from these two figures that the maximum second harmonic power is obtained when ν/ω approaches zero. Of course, the width of the resonance line approaches zero under this condition so that a finite ν/ω, that is $\nu/\omega>10^{-5}$, is desirable. From Fig. 6 it can be seen that for $0.1<\omega_p/\omega<0.9$, $Q_{\max}$ is a constant and that outside this range $Q_{\max}$ decreases from this constant value. The relationship between $Q_{\max}$ and the

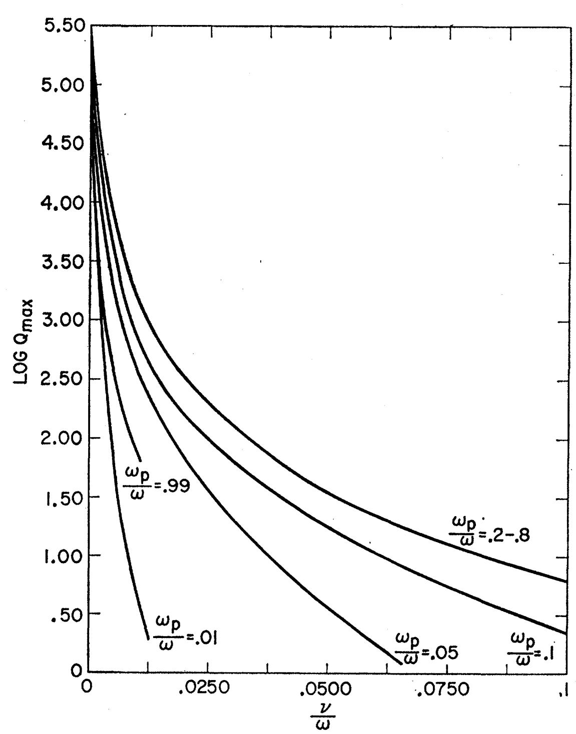

FIG. 5. Second harmonic power at resonance versus ν/ω.

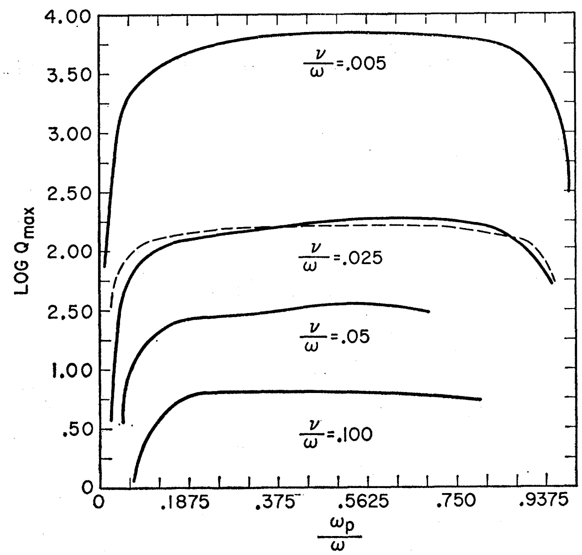

FIG. 6. Second harmonic power at resonance versus ω_p/ω (dashed curve indicates approximate formula).

plasma parameters is given approximately by

$$\log_{10}Q = -1.8 + 25\log_{10}\{1-(1.6)^{10}[(\omega_p/\omega)-0.5]^{10}\} - 2.5\log_{10}\nu/\omega, \quad (44)$$

and this empirical relationship is plotted as a dashed line, for a single value of ν/ω, in Fig. 6. The relationship between $Q_{\max}$ and $\omega d/c$ is relatively simple. For low values of ν/ω and ω_p/ω reflections between the boundaries cause $Q_{\max}$ to oscillate as $\omega d/c$ is varied through multiples of quarter wavelengths within the plasma. However, as the attenuation within the plasma increases, that is for higher values of ν/ω or ω_p/ω, then $Q_{\max}$ is relatively insensitive to changes in $\omega d/c$.

Finally, Eq. (36) shows that P_2 has a square-law dependence on P_0 for constant values of ω and the plasma parameters. In addition, it indicates that P_2 varies inversely as the square of the driving frequency, providing the parameters ω_p, ν, and d are adjusted so that ω_p/ω, ν/ω, and $\omega d/c$ are held constant as ω is varied.

5. SUMMARY

The Boltzmann transport equation, coupled with Maxwell's equations, has been solved, under a small signal plane wave assumption, including the effects of the nonlinear terms in the equations. Using a Fourier series expansion in time for all of the dependent variables, a solution to the equations has been obtained, in closed form, for the wave at the hth harmonic of the Fourier series, including the effects of all of the reflections within the plasma layer. The theory predicts that a major peak in the power at the second harmonic frequency exists near $\omega_c/\omega = 1$ for low values of ω_p/ω and ν/ω. As ω_p/ω increases, the peak shifts to lower values of ω_c/ω, and to higher values of ω_c/ω as ν/ω increases. Minor resonances occur on either side of the major peak because of the standing waves within the plasma layer. An additional peak, of reduced magnitude, exists near $\omega_c/\omega = 2$. The second harmonic power varies directly as the square of the input power and inversely as the square of the frequency of the input wave.

The peak value of the second harmonic power, $Q_{\max}$, has been examined as a function of the plasma parameters. The theory indicates that $Q_{\max}$ is essentially constant in the range $0.1 < \omega_p/\omega < 0.9$ and that $Q_{\max}$ decreases rapidly outside this range. Also, $Q_{\max}$ is inversely proportional to ν/ω and relatively insensitive to the thickness of the plasma layer. An approximate equation relating $Q_{\max}$ to the plasma parameters is given.

In subsequent reports on this work, a discussion of the experimental verification of this theory, the effects of the nonlinear terms on propagation at frequency ω, the limits of the small signal theory, and the effects of ionization produced by the incident wave will be given.

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