Electromagnetic Interaction of a Beam of Charged Particles with Plasma

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The plasma-beam instability has been studied by Akhiezer and Fainberg under the assumption that $\theta=0$, where θ is the angle formed by the direction of the beam and the direction of the growing wave resulting from the instability. Under these conditions the interaction is electrostatic, i.e., the wave is longitudinal. In this investigation the above assumption is generalized so as to include the case of $\theta\neq0$ and the effect of electromagnetic interaction. For $\omega\sim\omega_1$, where ω_1 is the Langmuir frequency of the plasma, the interaction is electrostatic for all values of θ and the resulting instability which produces a longitudinal wave increases exponentially in accordance with the term $\exp(3\sqrt{3}\omega_0{}^2kv_0\cos\theta/8)^{\frac{1}{3}}$ (where ω_0 is the Langmuir frequency of the beam). For a frequency range below ω_1 the instability is less pronounced. However, this instability is significant, since the interaction is electromagnetic and the "growing wave" resulting from this interaction is characterized by an electric vector having both transverse and longitudinal components. In investigating the above instabilities, an assumption was made that the density of the incident beam is small and the results cover all values of θ except those in the immediate neighborhood of $\pi/2$. For θ in the neighborhood of $\pi/2$ the assumption is more general and the results apply to any density of the beam.

I. INTRODUCTION

THIS investigation deals with the passage of an electron beam through plasma and particularly with the generalization of the problem formulated by Akhiezer and Fainberg.[1] These authors considered an instability resulting from the growth of a longitudinal wave moving in the same direction as the beam in the absence of an externally applied magnetic field. Under these conditions the electromagnetic interactions are negligible, i.e., the instability is of purely electrostatic origin. A somewhat similar approach based on electrostatic interactions has been used on related problems by Haeff,[2] Pierce,[3] Bohm and Gross,[4] Sumi,[5] and others.

In our formulation we shall take into account the electromagnetic interaction and investigate the behavior of any initial disturbance in the presence of such interaction. It will be shown that the electrostatic interaction is particularly effective for those frequencies that are in the immediate neighborhood of the Langmuir frequency of the plasma. The instability that results from this interaction is relatively intense. An expression will be derived representing the rate of growth of the wave as a function of the angle θ between the direction of the wave and the direction of the beam. For a frequency range below the Langmuir frequency of the plasma the instability is less intense. However this frequency range is of a particular interest, since it is associated with an electromagnetic interaction and gives rise to "growing waves" in which the electric field has both a transverse and longitudinal component. The growing waves resulting from the electromagnetic interaction are polarized.

The plasma-beam instability is closely related to the sporadic radio emissions from the solar corona. These have been discussed in the astrophysical literature with particular reference to the mechanism described by Akhiezer and Fainberg. It should be noted, however, that Akhiezer and Fainberg considered a specific case of an instability, occuring in the absence of an external magnetic field, i.e., the case for which $\theta=0$. Therefore, the only type of waves reported by these investigators are longitudinal waves. In order to enable these waves to propagate in free space, a mechanism for the conversion of the longitudinal into transverse oscillations is necessary. According to Ginzburg and Zhelezniakov,[6] this conversion is effected in a homogeneous plasma by the interaction of the longitudinal waves with the density fluctuation ΔN of the electronic plasma having concentration N. The term expressing the density fluctuation has two components, i.e., $\Delta N=\Delta N'+\Delta N''$. The component $\Delta N'$ designates the fluctuation of N caused by the change in plasma density and $\Delta N''$ represents the fluctuation in which the ionic concentration does not change. The transverse waves produced by this mechanism are unpolarized.

It is noted that some of the solar emissions designated as type I bursts are polarized, and other emissions designated as type II and III bursts are unpolarized. Thus one cannot correlate the type I bursts with the mechanism described by Akhiezer and Fainberg, and it is assumed that this mechanism is responsible for the generation of type II and III bursts only. It is thus generally assumed that these latter bursts occur in those regions of the solar corona in which the external magnetic field is insignificant. In order to explain the occurrence of type I bursts, and particularly their polarization, it has been postulated that these bursts are produced in those regions of the solar corona in which an external magnetic field is present.

This paper may possibly contribute to a modification of this assumption since it is shown that a growing

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[1] A. I. Akhiezer and Ia. B. Fainberg, Zhur. Eksp. Teoret. Fiz. **21**, 1262 (1951).
[2] A. V. Haeff, Phys. Rev. **74**, 1532 (1948); A. V. Haeff, Proc. Inst. Radio Engrs. **37**, 4 (1949).
[3] J. R. Pierce, J. Appl. Phys. **20**, 1060 (1949).
[4] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851, 1864 (1949); **79**, 992 (1950).
[5] Masao Sumi, J. Phys. Soc. Japan **13**, 1476 (1958); **14**, 653 (1959).
[6] V. L. Ginzburg and V. V. Zhelezniakov, Astron. Zhur. **35**, 694 (1958); **36**, 233 (1959).

polarized electromagnetic wave may occur in those regions of solar corona in which an externally applied magnetic field is absent or negligible.

II. GENERAL DISPERSION FORMULA

Consider an extended uniform medium consisting of S beams of charged particles. Each beam is of infinite width and is characterized by a uniform density n_p of particles having mass m_p, charge e_p and velocity $\mathbf{V}_p$ ($p=1, 2, \cdots S$). In the absence of a perturbation these beams neutralize each other, i.e., $\sum n_pe_p=0$. We assume that the medium is subjected to a perturbation caused by an electromagnetic field

$$\mathbf{E}'=\mathbf{E}\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r})];\quad \mathbf{B}'=\mathbf{B}\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r})].$$

If the perturbation is sufficiently small, the velocity of each beam may assume a value $\mathbf{V}_p+\mathbf{V}_p'$, where $\mathbf{V}_p'\ll\mathbf{V}_p$. The electromagnetic field polarizes the medium and the total polarization $\mathbf{P}'$ can be expressed as a sum of terms $\mathbf{P}_p'=\mathbf{P}_p\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r})]$, each of which is associated with a corresponding beam p. Neglecting small terms, we obtain:

$$\frac{D^2\mathbf{P}_p}{Dt^2}=\frac{\omega_p^2}{4\pi}\left[\mathbf{E}'+\frac{1}{c}\mathbf{V}_p\times\mathbf{B}'\right],\tag{1}$$

where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\mathbf{V}_p\,\mathrm{grad},\quad \omega_p^2=\frac{4\pi n_pe_p^2}{m_p}.$$

The interaction of the electromagnetic field with the beams can be described by Maxwell's equations as follows:

$$\mathrm{curl}\mathbf{E}'=-\frac{1}{c}\frac{\partial\mathbf{B}'}{\partial t};\tag{2}$$

$$\mathrm{div}\mathbf{B}'=0;\tag{3}$$

$$\mathrm{curl}\mathbf{B}'=-\frac{1}{c}\frac{\partial\mathbf{E}'}{\partial t}-\frac{4\pi}{c}\sum_p\left[\frac{\partial\mathbf{P}_p'}{\partial t}+\mathrm{curl}(\mathbf{P}_p'\times\mathbf{V}_p)\right];\tag{4}$$

$$\mathrm{div}\mathbf{E}'=-4\pi\textstyle\sum_p\mathrm{div}\mathbf{P}_p'.\tag{5}$$

We choose rectangular coordinates x, y, z, and $k_z=k$; $k_x=k_y=0$; $V_{p,x}=V_{p,1}=V_p\sin\theta_p$; $V_{py}=V_{p,2}=0$; $V_{pz}=V_{p,3}=V_p\cos\theta_p$. Then using the expression (1) and (2) we obtain:

$$P_{p,k}=\kappa_{p,kl}E_l,\quad (k, l=1, 2, 3),\tag{6}$$

where

$$\kappa_{p,11}=\kappa_{p,22}=-\frac{\omega_p^2}{4\pi\omega(\omega-kV_p\cos\theta_p)};$$

$$\kappa_{p,31}=-\frac{\omega_p^2kv_p\sin\theta_p}{4\pi\omega(\omega-kV_p\cos\theta_p)^2};\tag{7}$$

$$\kappa_{p,12}=\kappa_{p,13}=\kappa_{p,21}=\kappa_{p,32}=\kappa_{p,23}=0;$$

$$\kappa_{p,33}=-\frac{\omega_p^2}{4\pi(\omega-kV_p\cos\theta_p)^2}.$$

Substituting $\mathbf{P}_p$ as expressed in (6) and (7) in (4) and $\mathbf{B}$ as expressed in (2) in (4), we obtain the following relationship for E_j:

$$a_{ij}E_j=\textstyle\sum_p\epsilon_{p,ij}E_j,\tag{8}$$

where

$$a_{11}=a_{22}=c^2k^2/\omega^2;$$
$$a_{12}=a_{13}=a_{21}=a_{23}=a_{31}=a_{32}=a_{33}=0;\tag{9}$$

$$\epsilon_{p,11}=1-\frac{\omega_p^2}{\omega^2}-\frac{\omega_p^2k^2V_p^2\sin^2\theta_p}{\omega^2(\omega-kV_p\cos\theta_p)^2};$$

$$\epsilon_{p,13}=\epsilon_{p,31}=-\frac{\omega_p^2kV_p\sin\theta_p}{\omega(\omega-kV_p\cos\theta_p)^2};\tag{10}$$

$$\epsilon_{p,33}=1-\frac{\omega_p^2}{(\omega-kV_p\cos\theta_p)^2};\quad \epsilon_{p,22}=1-\frac{\omega_p^2}{\omega^2};$$

$$\epsilon_{12}=\epsilon_{21}=\epsilon_{23}=\epsilon_{32}=0.$$

The relationship (8) is satisfied if

$$|a_{ij}-\textstyle\sum_p\epsilon_{p,ij}|=0.\tag{11}$$

The expression (11) represents the dispersion relation.

III. PASSAGE OF A BEAM THROUGH A HIGH-TEMPERATURE PLASMA

We shall apply the expression (11) to a medium comprising an electron beam having velocity $\mathbf{V}=\mathbf{v}_0$, density n_0, and passing through a plasma having density n_1, and characterized by a Maxwellian velocity distribution

$$f(\mathbf{V})=n_1\left(\frac{3}{2\pi s^2}\right)^{\frac{3}{2}}\exp(-3s^2/2V^2),\tag{12}$$

where s is the mean thermal velocity. We shall designate by θ the angle formed by v_0 and the z axis.

We assume that both the beam and the plasma are charge equilibrated, i.e., the electrons are compensated by a positive charge that is continuously distributed throughout the space. The velocity distribution $f(\mathbf{V})$ represents a limiting case of an assembly of discrete beams. In order to form the assembly we divide the velocity space into equal volume elements $\Delta\mathbf{V}_i$. A vector $\mathbf{V}_i$ connecting the origin of coordinates to any point within the cell $\Delta\mathbf{V}_i$ represents the velocity of an electron beam having density $f(\mathbf{V}_i)\Delta\mathbf{V}_i$, and the density of the entire assembly is $n_0=\sum f(\mathbf{V}_i)\Delta\mathbf{V}_i$. The continuous distribution represents a limit as each volume element $\Delta\mathbf{V}_i$ tends to zero.

The dispersion relation for the medium is represented as

$$|a_{ij}-K_{ij}|=0,\tag{13}$$

where

$$K_{11}=1-\frac{\omega_0^2}{\omega^2}-\frac{\omega_0^2k^2v_0^2\sin^2\theta}{\omega^2(\omega-kv_0\cos\theta)^2}-4\pi\kappa_t;$$

$$K_{22}=1-\frac{\omega_0^2}{\omega^2}-4\pi\kappa_t;\quad K_{33}=1-\frac{\omega_0^2}{(1-kv_0\cos\theta)^2}-4\pi\kappa_l; \quad (14)$$

$$K_{13}=K_{31}=-\frac{\omega_0^2kv_0\sin\theta}{\omega(\omega-kv_0\cos\theta)^2};\quad K_{12}=K_{21}=K_{23}=K_{32}=0;$$

$$\omega_0=(4\pi n_0e^2/m)^{\frac{1}{2}};\quad \omega_1=(4\pi n_1e^2/m)^{\frac{1}{2}}.$$

In the above expressions the terms κ_t and κ_l are defined as follows:

$$\kappa_t=\frac{\omega_1^2}{4\pi\omega}\int_\Gamma\frac{f(\mathbf{V})d\mathbf{V}}{\omega-\mathbf{kV}}; \quad (15)$$

$$\kappa_l=\frac{\omega_1^2}{4\pi}\int_\Gamma\frac{f(\mathbf{V})d\mathbf{V}}{(\omega-\mathbf{kV})^2}; \quad (16)$$

We shall consider an initial value problem in which the perturbation applied to the medium at $t=0$ is expressed as a function of space coordinates. The corresponding oscillations of the electromagnetic field are defined by the expression (13) in which k is real and ω may be complex. Since the wave motion is of the type $\exp[i(\omega t-\mathbf{kr})]$, we would obtain a wave having an amplitude increasing with time if $\mathrm{Im}(\omega)<0$. We are primarily interested in ascertaining whether or not the medium is unstable against small perturbations. Therefore, our problem consists in determining the sign of $\mathrm{Im}(\omega)$. In the expressions (15) and (16) the contour of integration Γ is along the entire real axis from $-\infty$ to $+\infty$, unless the pole of the integrand has a nonpositive imaginary part in which case Γ dips down below the pole. Assuming $ks\ll\omega$ and using an asymptotic series expansion, we can represent the expressions (15) and (16) as follows[1]:

$$\kappa_t=\frac{1}{4\pi}\left\{\frac{\omega_1^2}{\omega^2}+\frac{\omega_1^2(ks)^2}{3\omega^4}+\frac{3\omega_1^2(ks)^4}{3^2\omega^6}+\frac{15\omega_1^2(ks)^4}{3^3\omega^8}+\cdots+\frac{\sqrt{3}\pi^{\frac{1}{2}}}{\sqrt{2}}i\omega_1\exp(-3\omega^2/2k^2s^2)\right\}; \quad (17)$$

$$\kappa_l=\frac{1}{4\pi}\left\{\frac{\omega_1^2}{\omega^2}+\frac{\omega_1^2(ks)^2}{\omega^4}+\frac{5\omega_1^2(ks)^4}{3\omega^6}+\cdots+\frac{3\sqrt{3}\pi^{\frac{1}{2}}}{2}i\frac{\omega_1^2\omega}{k^3s^3}\times\exp(-3\omega^2/2k^2s^2)\right\}. \quad (18)$$

It is noted that the dispersion relation (13) can be represented in the form of two independent equations as follows:

$$(a_{22}-K_{22})E_y=0; \quad (19)$$

$$\begin{vmatrix}a_{11}-K_{11} & -K_{13}\\ -K_{31} & -K_{33}\end{vmatrix}\begin{vmatrix}E_x\\E_z\end{vmatrix}=0. \quad (20)$$

The expression (19) represents a purely transverse wave in which the electric vector is perpendicular to $\mathbf{k}$ and $\mathbf{v}_0$. The form of this expression is independent of $\mathbf{v}_0$ and, therefore, the propagation of this wave is not influenced by the motion of the electron beam. There is no instability since this wave is characterized by a small attenuation that is inherent in the propagation of a transverse wave in a plasma.[7]

The expression (20) represents a wave in which the electrical vector has a transverse component E_x and a longitudinal component E_z. For $\theta=0$, i.e., when the wave is propagated along the direction of the beam, these two components are separated in the form of the following two waves:

$$\left(\frac{c^2k^2}{\omega^2}-1+\frac{\omega_0^2}{\omega^2}+4\pi\kappa_t\right)E_x=0; \quad (21)$$

$$\left(1-\frac{\omega_0^2}{(\omega-kv_0)^2}-4\pi\kappa_l\right)E_z=0. \quad (22)$$

Equation (21) represents a purely transverse wave similar to the one denoted by (19). Equation (22) represents a purely longitudinal wave that has been found by Akhiezer and Fainberg to be unstable against a small perturbation.

We shall investigate a more general case in which the direction of the propagation of the wave is inclined with respect to the velocity of the beam, i.e., $\theta\neq0$. More specifically, we shall determine the possible occurrence and the nature of the instability for this more general case and shall examine the dependence between the rate of growth of this instability and the angle θ.

Our investigation will be directed to the dispersion form (20). We shall first consider the passage of the beam through a cold plasma and subsequently consider the effect of the temperature.

IV. INSTABILITY FOR ZERO TEMPERATURE

Assuming $s=0$, the dispersion relation (20) can be expressed in the form

$$F(\omega,\theta)\equiv(\omega^2-\omega_0^2-\omega_1^2-c^2k^2)(\omega^2-\omega_1^2)(\omega-kv_0\cos\theta)^2-(\omega^2-\omega_0^2-\omega_1^2-c^2k^2)\omega_0^2\omega^2-\omega_0^2k^2v_0^2\sin^2\theta(\omega^2-\omega_1^2)=0. \quad (23)$$

The expression (23) represents a relationship between the frequency ω and the wave number k of an electromagnetic wave for which the wave vector $\mathbf{k}$ is in-

[7] See for instance: B. N. Gershman, Zhur. Eksp. Teoret. Fiz. **24**, 453 (1953).

clined at angle θ with respect to the velocity $\mathbf{v}_0$ of the beam. Our problem consists in ascertaining whether the waves propagated in the medium are of a "growing" or "decaying" type. This can be determined by solving (23) with respect to ω for real values of k.

We shall consider the behavior of the medium for a beam of very low intensity, i.e., when $\omega_0 \ll \omega_1$. If $\omega_0=0$, Eq. (23) describes three types of oscillations as follows: (A) $\omega_{\omega_0}=\omega_1^2+c^2k^2$, (B) $\omega_{\omega_0=0}=\omega_1^2$, and (C) $\omega_{\omega_0=0}=kv_0\cos\theta$. For small values of ω_0 these solutions may be expressed in the form $\omega=\omega_{\omega_0=0}+\alpha$, where $|\alpha|\ll\omega_{\omega_0=0}=0$. The term α shall be designated as the "frequency increment." This term is of particular interest since the stability of the medium depends upon the character of α. Thus if $\mathrm{Im}(\alpha)<0$, the solutions are unstable and the magnitude of $\mathrm{Im}(\alpha)$ indicates the rate of growth of the wave that results from the instability.

For small values of ω_0 the solutions of the type (A) and (B) do not give rise to any instability. Our particular attention will be devoted to the solution of the type (C), which is expressed as

$$\omega=kv_0\cos\theta+\alpha, \tag{24}$$

where

$$|\alpha|\ll kv_0\cos\theta. \tag{25}$$

The term $kv_0\cos\theta$ shall be designated as the "characteristic frequency" of the beam.

In investigating the dispersion relation (23) we shall differentiate between a "weak interaction" and a "strong interaction." The nature of the interaction depends upon the magnitude of the term

$$L=\lim_{\omega_0\to 0}(\omega_0/|\alpha|).$$

For a strong interaction we have $L=0$, and the rate of growth of the instability as expressed by α is, by an order of magnitude, higher than in the case of a weak interaction for which $L\neq 0$.

Strong Interaction

If we neglect terms involving ω_0^4/α^4, Eq. (23) represents two waves: a transverse wave having the form $(a_{11}-K_{11})E_x=0$ which is stable, and a longitudinal wave expressed as

$$\left(1-\frac{\omega_0^2}{(\omega-kv_0\cos\theta)^2}-\frac{\omega_1^2}{\omega^2}\right)E_z=0. \tag{26}$$

The behavior of the longitudinal wave (26) is particularly significant when the characteristic frequency of the beam is equal to the Langmuir frequency of the plasma, i.e., when

$$kv_0\cos\theta=\omega_1. \tag{27}$$

The frequency of the wave is then determined by the expressions (24), (25) and (27). Substituting these expressions in (26) and neglecting small terms, we obtain

$$\alpha^3-\tfrac{1}{2}\omega_0^2kv_0\cos\theta=0, \tag{28}$$

and consequently

$$\alpha=\left(\frac{\omega_0^2kv_0\cos\theta}{2}\right)^{\frac{1}{3}}\left(\frac{-1\mp i\sqrt{3}}{2}\right). \tag{29}$$

Therefore, the wave grows exponentially in accordance with the term $\exp[(3\sqrt{3}\omega_0^2kv_0\cos\theta/8)^{\frac{1}{3}}t]$.

Note that $\alpha=O(\omega_0^{\frac{2}{3}})$, from which it is evident that $L=0$ is in agreement with our assumption.

Weak Interaction

For the case of a weak interaction the separation into transverse and longitudinal waves is not possible. The instability is described by Eq. (23) and gives rise to growing electromagnetic waves in which the electric vector has a component parallel to $\mathbf{k}$. Substituting (24) in (23), taking into account the inequality (25) and an additional inequality

$$2|\alpha|kv_0\cos\theta\ll\omega_1^2, \tag{30}$$

we obtain

$$\alpha^2=\omega_0^2k^2v_0^2\frac{k^2v_0^2\cos^2\theta-c^2k^2\cos^2\theta-\omega_1^2}{(k^2v_0^2\cos^2\theta-c^2k^2-\omega_1^2)(k^2v_0^2\cos^2\theta-\omega_1^2)}. \tag{31}$$

The numerator in this expression and the first factor in the denominator are negative. Therefore, the occurrence of an instability depends on the sign of the term $(k^2v_0^2\cos^2\theta-\omega_1^2)$. We have an instability if the characteristic frequency is below the Langmuir frequency of the stationary plasma, i.e., if

$$kv_0\cos\theta<\omega_1. \tag{32}$$

The inequality (32) is a necessary but not sufficient condition for an instability. Additional restrictions as expressed by (25) and (30) have to be imposed. Furthermore $L\neq 0$ and, consequently, $\alpha=O(\omega_0)$.

It is noted that the value of the characteristic frequency given by (27) is not included in the region of weak interaction, since, by substituting (27) in (31), we obtain $\alpha=\infty$ and the formula (31) is not applicable. As shown by (32), the characteristic frequencies at which the weak interaction is effective have an upper bound.

Variation of the Rate of Growth with the Angle θ

We shall now summarize some of our results, pointing out the regions of applicability and the significance of the frequency decrement α as expressed by (29) and (31).

We considered two frequency ranges: a relatively high-frequency range in the neighborhood of ω_1 for which the rate of growth of the instability is relatively large ("strong interaction"), and a lower frequency range below ω_1 for which the rate of growth of the instability is smaller. These frequency ranges shall be designated as (*a*) and (*b*), respectively.

For the frequency range (*a*), the wave that results from the instability is purely longitudinal and its frequency decrement is expressed by (29). The rate of growth of the wave depends upon the angle θ, i.e., it decreases when θ increases from 0 to $\pi/2$. The electromagnetic interactions do not influence the behavior of the wave.

For the frequency range (*b*), the instability is caused by an interaction of the electromagnetic field with the plasma-beam system. The wave resulting from this instability is significant since the electric vector has both longitudinal and transverse components. The rate of growth of this wave decreases when θ increases from 0 to $\pi/2$. This can be shown by differentiating (31) with respect to θ. We obtain $d(\alpha^2)/d\theta>0$ for $\omega\sim kv_0\cos\theta<\omega_1$, and since α^2 is negative in this range, the term $|\alpha|$ decreases with the increase of θ.

If we take a finite but small value of ω_0 and assume that $\theta\rightarrow\pi/2$, the expression (31) becomes inapplicable. We shall now remove the restriction that $\omega_0\ll\omega_1$ and show that under these more general conditions there is an instability for $\theta=\pi/2$ and for $\theta\sim\pi/2$.

For $\theta=\pi/2$, the dispersion equation (23) contains only even powers of ω. Therefore, this equation can be expressed as $F(\omega,\pi/2)\equiv G(\omega^2)=0$. Substituting in (23) $\omega^2=0$; and $\omega^2=-\omega_0^2$, we obtain

$$G(0)=\omega_0^2\omega_1^2v_0^2k^2; \tag{33}$$

$$G(-\omega_0^2)=-(2\omega_0^2+\omega_1^2)\{[\omega_0^4+\omega_0^2(\omega_0^2+c^2k^2+\omega_1^2)]-\omega_0^2v_0^2k^2\}-\omega_1^4k^2v_0^2. \tag{34}$$

It is apparent that $G(0)>0$ and $G(-\omega_0^2)<0$. Consequently $G(\omega^2)$ has a negative root between $\omega^2=0$ and $\omega^2=-\omega_0^2$ for all values of $k>0$. Thus the equation $F(\omega,\pi/2)=0$ has a root that is purely imaginary and, therefore, there is an instability. This instability, which corresponds to $\theta=\pi/2$, is related to the instabilities occurring in media having certain anisotropic velocity distributions discussed by Weibel,[8] Fried,[9] and Harris.[10]

It can be shown that when θ differs from $\pi/2$ by a small amount, there appears a real component in the expression for ω (in addition to the imaginary component already present when $\theta=\pi/2$). In order to show this, one writes the expression $F(\omega,\theta)$ so that the even and odd powers of ω are separated and finds that the assumption: "ω is pure imaginary" leads to a contradiction.

[8] E. S. Weibel, Phys. Rev. Letters 2, 83 (1959).
[9] B. Fried, Phys. Fluids 2, 337 (1959).
[10] E. G. Harris, paper presented at the International Plasma Physics Institute, Seattle, Washington, August 31–September 5, 1959 (unpublished).

V. TEMPERATURE EFFECTS

Assuming $s\neq0$, the dispersion equation can be represented in the following form:

$$\left(1-\frac{\omega_0^2}{\omega^2}\frac{3\omega_1^2}{3\omega^2-k^2s^2}-\frac{\omega_0^2k^2v_0^2\sin^2\theta}{\omega^2(\omega-kv_0\cos\theta)^2}-\frac{c^2k^2}{\omega^2}-i\delta_1\right)\times\left(1-\frac{\omega_0^2}{(\omega-kv_0\cos\theta)^2}-\frac{\omega_1^2}{\omega^2-k^2s^2}-i\delta_2\right)+\frac{\omega_0^4k^2v_0^2\sin^2\theta}{\omega^2(\omega-kv_0\cos\theta)^4}=0, \tag{35}$$

where

$$\delta_1=\frac{\pi\sqrt{3}}{\sqrt{2}}\frac{\omega_1^2}{ks\omega}\exp(-3\omega^2/2k^2s^2); \tag{36}$$

$$\delta_2=\frac{3\pi\sqrt{3}}{\sqrt{2}}\frac{\omega_1^2\omega}{k^3s^3}\exp(-3\omega^2/2k^2s^2). \tag{37}$$

The expression (35) is derived from (20) using (14), (17), and (18). Following the procedure used by Akhiezer and Fainberg, we have neglected in the development (17) and (18) the effect of terms of the order $(ks/\omega)^4$. The following expression for α^2 is obtained from (35):

$$\alpha^2=A/B,$$

where

$$\begin{aligned}A&=\omega_0^2(c^2k^2+\omega_0^2-k^2v_0^2)+\omega_0^2k^2v_0^2\cos^2\theta\left(\frac{3\omega_1^2}{3k^2v_0^2\cos^2\theta-k^2s^2}-i\delta_1\right)+\omega_0^2k^2v_0^2\sin^2\theta\left(\frac{\omega_1^2}{k^2v_0^2\cos^2\theta-k^2s^2}-i\delta_2\right);\\ B&=-c^2k^2-\omega_0^2-k^2v_0^2\cos^2\theta-k^2v_0^2\cos^2\theta\left(\frac{3\omega_1^2}{3k^2v_0^2\cos^2\theta-k^2s^2}-i\delta_1\right)\times\left(1-\frac{\omega_1^2}{k^2v_0^2\cos^2\theta-k^2s^2}-i\delta_2\right)\\&\quad+(c^2k^2+\omega_0^2-k^2v_0^2\cos^2\theta)\left(\frac{\omega_1^2}{k^2v_0^2\cos^2\theta}-i\delta_2\right).\end{aligned} \tag{38}$$

If we assume that $\delta_1=\delta_2=0$, then the expressions (35) and (38) will be based on the hydrodynamical representation of the medium. If s is sufficiently small, the hydrodynamical representation does not introduce any substantial modification in the frequency range for which an instability sets in.