Symmetry Theorems for Isospin-Invariant Reactions

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Symmetry theorems, analogous to those well known for angular distributions and correlations, are given for isospin-invariant reactions starting from an initial state of limited complexity. Detailed calculations are carried out when the initial-state isospin does not exceed $\frac{3}{2}$. A statistical generalization is given for averages over experiments starting from different charge states. Some properties of the irreducible tensor operators which arise from recoupling the angular momentum operator to itself are discussed.

1. INTRODUCTION

THE complete analysis of reactions which emit several particles is forbiddingly complicated, even with the help of conservation laws and the Racah algebra. The difficulties are compounded when not all of the emitted particles are detected, since then the conservation laws are nearly useless. In these circumstances, even to determine the effects of isospin invariance on the measured charge distributions is not easy, and to use those distributions to isolate any of the dynamical parameters is much more difficult.

Let $q(\mu_1\cdots\mu_N)$ represent the number of events observed wherein particle 1 has isospin z component μ_1, etc. Identical particles may be labeled according to any criterion not involving their charge state.

$$q(\mu_1\cdots\mu_N)=\sum_\beta|\langle\mu_1\cdots\mu_N,\beta|\psi\rangle|^2. \tag{1.1}$$

ψ stands for the final-state wave function and β for all variables not involving the isospins of the measured particles. The range of the summation depends upon the selection criterion for events to be recorded. In terms of the isospin amplitudes $\langle\alpha,\tau,\mu,\beta|\psi\rangle$, where τ is the resultant isospin of the measured particles, μ its z component, and α stands for all the other invariant isospin quantum numbers of the measured particles, q is given by

$$q(\mu_1\cdots\mu_N)=\sum_\beta|\sum_{\alpha,\tau}\langle\mu_1\cdots\mu_N|\alpha,\tau,\mu\rangle\langle\alpha,\tau,\mu,\beta|\psi\rangle|^2. \tag{1.2}$$

The vector recoupling coefficients in (1.2) are known in principle, but present an increasingly serious obstacle to calculation for large N. In any case, the sum on β precludes the direct use of (1.2) in interpreting charge distribution data unless all final-state particles are observed and all their momenta, spins, etc., are measured.

Probably the greatest advance to data along these lines has been made by Cerulus,[1] who assumes, in addition to isospin conservation, the "statistical hypothesis" that the amplitude $\langle\alpha,\tau,\mu,\beta|\psi\rangle$ does not depend upon α. He then finds ways of carrying out the sum on α in (1.2), to obtain a form which could be written as

$$q(\mu_1\cdots\mu_N)=\sum_\beta|\sum_\tau A(\mu_1\cdots\mu_N,\tau)\langle\tau,\mu,\beta|\psi\rangle|^2, \tag{1.3}$$

with manageable coefficients A. When τ is definite in the final state, (1.3) expresses the charge distributions for each μ in terms of a single parameter, and thereby gives a test of the combined assumptions of isospin independence and the statistical hypothesis. Even where τ is not definite, (1.3) could be utilized by broadening the statistical hypothesis to include τ or β.

We present here a different approach, of overlapping applicability with the statistical one, but assuming only isospin invariance. We give symmetry rules closely related to those familiar for angular distributions[2] and correlations,[3] which claim effectively that an initial state of limited complexity cannot lead to very complex final-state distributions. Some of those isospin relations have in fact been derived before in special cases, usually by direct calculation.[4,5] The most famous example is the requirement that nucleon-deuteron collisions must produce charged and neutral pions in the ratio two-to-one.[5]

In Sec. 2, we construct the symmetric irreducible tensor operators, which play the role usually assigned to the spherical harmonics in angular analyses. In Sec. 3, we carry out the analysis of a charge distribution in terms of its tensor moments. We also derive inequalities on the moments, which follow from the necessity for the charge distribution to be positive. Section 4 presents the symmetry theorems. Tensor moments too high to be constructed from the highest isospins in the initial state must have zero expectation for every part of the final state. Symmetric systems must have vanishing odd moments. Detailed results are given for initial-state isospins up to $\frac{3}{2}$. In Sec. 5, the results of Secs. 3 and 4 are generalized to cover averaged data from experiments involving several charge states. The previous analysis is unchanged, except that the effective S is reduced to half the rank of the highest rank irreducible tensor in the statistical matrix.

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[1] F. Cerulus, Suppl. Nuovo cimento **15**, 402 (1960); and preprint (to be published).

[2] L. Wolfenstein, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, 1956), Vol. 6, p. 43.

[3] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

[4] For example, Y. Eisenberg *et al.*, Nuovo cimento **9**, 745 (1958).

[5] K. N. Watson, Phys. Rev. **85**, 852 (1952).

Our notation for the angular momentum quantities is that of Rose.[3] In discussing the statistical matrix, we follow the langauge and ideas of Fano.[6] We carry $\hbar$ in Sec. 2, but replace it by one when isospin operators are contemplated.

2. IRREDUCIBLE TENSOR OPERATORS

The symmetric irreducible tensor operators T_{LM} are defined in close analogy with the spherical harmonics, $Y_{LM}(\theta,\varphi)$, whose transformation law under rotation they share.

$$T_{00}=1/(4\pi)^{\frac{1}{2}}, \tag{2.1}$$

$$T_{1,\pm1}=\mp(3/8\pi)^{\frac{1}{2}}(J_x\pm iJ_y), \tag{2.2}$$

$$T_{10}=(3/4\pi)^{\frac{1}{2}}J_z, \tag{2.3}$$

$$T_{LM}=\left[\frac{4\pi}{3}\left(\frac{2L+1}{L}\right)\right]^{\frac{1}{2}} \times\sum_\mu C(L-1,1,L;\mu,M-\mu) \times T_{L-1,\mu}T_{1,M-\mu}. \tag{2.4}$$

The vector coefficients C are defined by the orthogonal transformation

$$C(j_1j_2j;\mu_1\mu_2)=\langle j_1,j_2,\mu_1,\mu_2|j_1,j_2,j,\mu\rangle, \tag{2.5}$$

with the Condon-Shortley phase convention.[3]

The recursion formula (2.4) results in

$$T_{LL}=N_L(J_x+iJ_y)^L, \tag{2.6}$$

$$N_L=\frac{(-\frac{1}{2})^L}{L!}\left[\frac{(2L+1)!}{4\pi}\right]^{\frac{1}{2}}. \tag{2.7}$$

The homogeneous harmonic polynomials $Z_{LM}(x,y,z)$ of degree L, are given by

$$Z_{LM}(x,y,z)=r^LY_{LM}(\theta,\varphi). \tag{2.8}$$

Comparison of (2.6) with

$$Z_{LL}=N_L(x+iy)^L, \tag{2.9}$$

shows that T_{LL} goes into Z_{LL} when x, y, z are substituted for J_x, J_y, J_z, and the commutators are neglected, i.e., $\hbar$ is replaced by zero. The shared transformation law then guarantees that all the T_{LM} go into Z_{LM} in the same sense, and can likewise be written as homogeneous polynomials of degree L in J_x, J_y, J_z. In fact, the known Y_{LM} give by substitution explicit formulas for the T_{LM} when products of components J_k are taken symmetrically. That is, the products are written in all possible orders and averaged with equal weight. Such a form is usually very cumbersome, and better ones can be obtained with the help of the commutation rules

$$[J_x,J_y]=i\hbar J_z, \tag{2.10}$$

etc. (2.10) shows that however the T_{LM} are written, they will be homogeneous polynomials of degree L in J_x, J_y, J_z, $\hbar$.

The matrix elements of the T_{LM} can be calculated from the Wigner-Eckart theorem.

$$\langle j',m'|T_{LM}|j,m\rangle=\delta_{jj'}\delta_{m',m+M}C(j,L,j';m,M) \times\langle j\|T_L\|j\rangle. \tag{2.11}$$

They vanish for $j\neq j'$ because T_{LM} is constructed from the operator J. The reduced matrix element is evaluated in the Appendix by considering the case $m'=j$, $M=L$, to yield

$$\langle j\|T_L\|j\rangle=\frac{\hbar^L}{2^L}\left[\frac{2L+1}{4\pi}\right]^{\frac{1}{2}}\left[\frac{1}{2j+1}\frac{(2j+L+1)!}{(2j-L)!}\right]^{\frac{1}{2}}. \tag{2.12}$$

These operators T_{LM} are of course not the only tensors of rank L which can be constructed from the angular momentum operator, but they are the simplest. T_{LM} corresponds to coupling $(T_1)^L$ to the resultant rank L. Higher powers of T_1 can also be coupled to rank L. These correspond to polynomials of higher degree in J_x, J_y, J_z, $\hbar$. Since they must differ from the T_{LM} only in their reduced matrix elements, they are in fact equal to the T_{LM}, multiplied by polynomials in J^2, $\hbar$. For this reason, they add nothing interesting.

The most useful of the T_{LM}, and the only ones of physical interest in the isospin case, are the T_{L0}. It is convenient to renormalize these by writing them in terms of operators Q_L, defined analogously to the Legendre polynomials P_L.

$$Q_L=\left[\frac{4\pi}{2L+1}\right]^{\frac{1}{2}}T_{L0}. \tag{2.13}$$

(2.11) and (2.12) show that Q_L is diagonal in j,m with

$$\langle j,m|Q_L|j,m\rangle=\langle j\|Q_L\|j\rangle C(j,L,j;m,0), \tag{2.13}$$

$$\langle j\|Q_L\|j\rangle=\left(\frac{\hbar}{2}\right)^L\left[\frac{1}{2j+1}\frac{(2j+L+1)!}{(2j-L)!}\right]^{\frac{1}{2}}. \tag{2.14}$$

It gives zero when applied to a wave function with $j<\frac{1}{2}L$.

In practical applications of (2.13), it is very useful to have the Q_L, or equivalently the vector coefficients, explicitly at least for small L. These can be obtained most readily from the recursion formula,[7]

$$Q_{L+2}=\frac{2L+3}{L+2}J_zQ_{L+1}-\left[\frac{L+1}{L+2}J^2-\frac{L(L+1)}{4}\hbar^2\right]Q_L. \tag{2.15}$$

[6] U. Fano, Revs. Modern Phys. **29**, 74 (1957).

[7] A. Meckler, Suppl. Nuovo cimento **12**, 1 (1959). Eq. (2.14) can be seen very simply from the above considerations, since (1) Q_L must go over into P_L with its recursion formula; (2) Q_L is homogeneous of degree L in J_x, J_y, J_z, $\hbar$; (3) Q_L is even or odd in J_z, according to L; (4) Q_{L+1} and Q_{L+2} vanish identically when $j=\frac{1}{2}L$.

The first few examples are

$$\left.\begin{aligned}
&Q_0=1,\\
&Q_1=J_z,\\
&Q_2=\tfrac{3}{2}J_z^2-J^2,\\
&Q_3=\tfrac{5}{2}J_z^3-\tfrac{3}{2}[J^2-\tfrac{1}{3}\hbar^2]J_z,\\
&Q_4=\frac{35}{8}J_z^4-\frac{15}{4}[J^2-\tfrac{5}{6}\hbar^2]J_z^2+\tfrac{3}{8}J^2(J^2-2\hbar^2).
\end{aligned}\right\} \quad (2.16)$$

The algebraic properties of the operators T_{LM} have been exposed in great detail by Meckler.[7]

3. TENSOR ANALYSIS OF RESULTANT CHARGE DISTRIBUTIONS

The channel intensity $g(\tau,\mu)$ is defined by

$$g(\tau,\mu)=\sum_\beta \langle\psi|\Lambda_\beta\Lambda_\tau\Lambda_\mu|\psi\rangle, \quad (3.1)$$

where Λ_τ and Λ_μ are projection operators which select that part of the complete final-state wave function ψ which endows the measured particles with the resultant quantum numbers τ, μ. The sum on β takes care of the selection criteria for variables not involving isospins of the measured particles. Since individual events usually do not have good τ, the channel intensities are directly related to measurement only through the resultant charge distribution $h(\mu)$.

$$h(\mu)=\sum_\tau g(\tau,\mu)=\sum_{\mu_1+\cdots+\mu_N=\mu} q(\mu_1\cdots\mu_N). \quad (3.2)$$

The channel moments $f(\tau,L)$ are defined by

$$f(\tau,L)=\frac{\sum_\beta \langle\psi|\Lambda_\beta Q_L\Lambda_\tau|\psi\rangle}{\langle\tau\|Q_L\|\tau\rangle}, \quad (3.3)$$

for integer $L\leqq 2\tau$. The operator Q_L is understood to be constructed from the resultant isospin operator of the measured particles alone.

Since Q_L and the projection operators are diagonal in τ, μ,

$$f(\tau,L)=\sum_\mu \frac{\langle\tau,\mu|Q_L|\tau,\mu\rangle}{\langle\tau\|Q_L\|\tau\rangle}g(\tau,\mu), \quad (3.4)$$

$$f(\tau,L)=\sum_\mu C(\tau,L,\tau;\mu,0)g(\tau,\mu), \quad (3.5)$$

$$g(\tau,\mu)=\sum_L \frac{2L+1}{2\tau+1}C(\tau,L,\tau;\mu,0)f(\tau,L). \quad (3.6)$$

(3.5) and (3.6) give the transformation between channel moments and channel intensities. (2.16) gives practical formulas for the needed vector coefficients, as in (3.4).

The tensor analysis of the measured distribution function is given by combining (3.2) with (3.6). Since the even and odd parts of $h(\mu)$ receive contributions from only even and only odd L, respectively, the analysis separates into distinct parts.

$$h_\pm(\mu)=\tfrac{1}{2}h(\mu)\pm\tfrac{1}{2}h(-\mu), \quad (3.7)$$

$$h_\pm(\mu)=\sum_{\tau=t}^{T}\sum_{L=0}^{2\tau}{}^{\pm}\frac{2L+1}{2\tau+1}C(\tau,L,\tau;\mu,0)f(\tau,L), \quad (3.8)$$

where $\sum^+$ contains only even L and $\sum^-$ only odd. t and T bound the isospins of the open channels.

In the usual case, (3.8) is insufficient to determine the channel moments unless additional conditions on the moments are assumed. A theory which implies sufficiently many such conditions can be tested by (3.8). In addition, if the moments are at least determined, the inequalities

$$g(\tau,\mu)\geqq 0 \quad (3.9)$$

imply a series of tests on the $f(\tau,L)$. These relations, which do not follow automatically from (3.8), must be applied with the aid of (3.6). A weaker, but sometimes more applicable form of (3.9) is given by

$$0\leqq f(\tau,0\leqq \sum_{\mu=-\tau}^{\tau} h(\mu), \quad (3.10)$$

$$|f(\tau,L)|\leqq f(\tau,0)\ \max_\mu\{|C(\tau,L,\tau;\mu,0)|\}. \quad (3.11)$$

In special cases, the inequalities can sometimes be sharpened.

4. ISOSPIN SYSTEMS OF LIMITED COMPLEXITY

We consider reactions whose final-state isospin, while not necessarily a good quantum number, does not exceed some value S. In isospin-conserving reactions, this comes about either because the initial state is one of limited complexity in the same sense, or because the reaction goes appreciably only through channels of limited isospin. Then, according to the Wigner-Eckart theorem, any tensor of rank greater than $2S$ has zero expectation in the final state. As the projection operator Λ_τ is a scalar, (3.2) gives

$$f(\tau,L)=0, \quad (4.1)$$

when $2S<L\leqq 2\tau$, and Q_L is made from any isospin in the final state.

When the final-state wave function ψ contains only even integer or only odd integer total isospin, and only zero z component, the Wigner-Eckart theorem also gives (4.1) for all odd L. These conditions are satisfied whenever the initial state consists of two particles with $\mu_1=\mu_2=0$. However, the success of the implied test, $h(\mu)=h(-\mu)$, reflects only on charge symmetry of the forces involved. In fact, charge symmetry alone then requires $q(\mu_1\cdots\mu_N)=q(-\mu_1\cdots-\mu_N)$.

Suppose first that an object of definite isospin τ is emitted in the final state, among arbitrarily many others, and that its charge distribution is measured.

If $\tau \geqq S$, there are $2\tau - 2S$ symmetry relations of the form (4.1). They can be put most conveniently with the aid of (2.16) as

$$\langle\mu\rangle_{\mathrm{av}}=0, \quad \text{if} \quad S<\tfrac{1}{2}\leqq\tau, \tag{4.2}$$

$$\langle\mu^2\rangle_{\mathrm{av}}=\tfrac{1}{3}\tau(\tau+1), \quad \text{if} \quad S<1\leqq\tau, \tag{4.3}$$

$$\langle\mu^3\rangle_{\mathrm{av}}=[\tfrac{3}{5}\tau(\tau+1)-\tfrac{1}{5}]\bar{\mu} \quad \text{if} \quad S<\tfrac{3}{2}\leqq\tau, \tag{4.4}$$

etc., where the expectations are defined by

$$\langle\mu^n\rangle_{\mathrm{av}}=\frac{\sum_\mu \mu^n h(\mu)}{\sum_\mu h(\mu)}. \tag{4.5}$$

(4.2) always depends upon charge symmetry alone. (4.3) reduces for pions ($\tau=1$) to the above-mentioned two-to-one rule.

If two or more particles are emitted with definite resultant τ, these rules apply unchanged to their resultant μ. Combining them with the vector coupling formulas can also give some information about the $q(\mu_1\cdots\mu_N)$, but the additional results test only the purity of the resultant isospin, and not the charge invariance. The simplest example is the required absence of events with $\mu_1=\mu_2=0$ when $\tau-\tau_1-\tau_2$ is odd.

The inequalities (3.9) are automatic for unique τ, and test nothing.

Where more than one τ channel is open, the results are too complicated to write out generally, but the method remains simple. We give complete details for the most useful cases, $S\leqq 1$.

$S=0$, $T>0$

For $S=0$, charge symmetry alone gives

$$h_-(\mu)=0. \tag{4.6}$$

Since all the moments vanish except for $L=0$, the even system (3.8) can be solved sequentially, starting from $\mu=T$. All the vector coefficients are equal to one when $|\mu|\leqq\tau$, zero otherwise.

$$f(\tau,0)=(2\tau+1)[h_+(\tau)-h_+(\tau+1)]. \tag{4.7}$$

The otherwise meaningless symbol $h(T+1)$ is to be read as zero. If $t>\frac{1}{2}$, the even equations (3.8) also give the symmetry equations,

$$h_+(\mu)=h_+(\mu+1) \tag{4.8}$$

for $0\leqq\mu<t$. Whatever the value of t, (3.9) is nontrivial and gives

$$h_+(\mu)\geqq h_+(\mu+1) \tag{4.9}$$

for $\mu\geqq 0$. (4.6) and (4.9) are the only general tests obtained when $S=0$ and $t<1$.

When two particles are detected with integer τ_1 and τ_2, the information that an event with $\mu_1=\mu_2=0$ must have $\tau-\tau_1-\tau_2$ even can be used to put a lower bound on the sum of $g(\tau,0)$ for such τ. This bound is not implied by (4.9).

$S=\frac{1}{2}$, $T>\frac{1}{2}$

The even part of the analysis is identical to that for $S=0$. The same procedure is applied to the odd moments.

$$C(\tau,1,\tau;\mu,0)=\frac{\mu}{[\tau(\tau+1)]^{\frac{1}{2}}}, \tag{4.10}$$

$$f(\tau,1)=\frac{2\tau+1}{3}\left[\left(\frac{\tau+1}{\tau}\right)^{\frac{1}{2}}h_-(\tau) -\left(\frac{\tau}{\tau+1}\right)^{\frac{1}{2}}h_-(\tau+1)\right]. \tag{4.11}$$

If $t>1$, the symmetry equations are

$$h_-(\mu)=\frac{\mu}{\mu+1}h_-(\mu+1) \tag{4.12}$$

for $0<\mu<t$. The inequalities become

$$\left|h_-(\mu)-\frac{\mu}{\mu+1}h_-(\mu+1)\right|\leqq h_+(\mu)-h_+(\mu+1) \tag{4.13}$$

for $0<\mu$.

$S=1$, $T>1$

The odd part of the analysis is identical to that for $S=\frac{1}{2}$, except that (4.13) no longer applies. In fact, the inequality (3.9) can be tested only if the even part of the analysis can be carried out. (3.10), (3.11) give the rather weak inequality

$$\left|\frac{\tau+1}{\tau}h_-(\tau)-h_-(\tau+1)\right|\leqq\frac{3}{2\tau+1}\sum_{\mu=-\tau}^{\tau}h(\mu). \tag{4.14}$$

The even system (3.8) now fails to separate in the unknowns $f(\tau,0)$ and $f(\tau,2)$. Nevertheless, these moments can be found if they are not too numerous. Counting out the unknown moments shows that the excess number of equations, E, is given by

$$E=2t-I(T+\tfrac{3}{2}), \tag{4.15}$$

where $I(x)$ is the greatest integer which does not exceed x. Thus there are E symmetry equations in the even (3.8) if $E\geqq 0$, and in that case the inequalities (3.10) can also be tested.

Even when $E<0$, some tests can be performed if $t>1$. From (2.16) and (3.8),

$$h_+(\mu)=F_0+F_2\mu^2, \tag{4.16}$$

when $\mu\leqq t$, where

$$F_0=\sum_{\tau=t}^{T}\frac{1}{2\tau+1}\left[f(\tau,0)-\frac{5}{2}\frac{\tau(\tau+1)}{\langle\tau\|Q_2\|\tau\rangle}f(\tau,2)\right], \tag{4.17}$$

$$F_2=\tfrac{3}{2}\sum_{\tau=t}^{T}\frac{5}{2\tau+1}\frac{f(\tau,2)}{\langle\tau\|Q_2\|\tau\rangle}. \tag{4.18}$$

(4.16) gives $I(t+1)$ equations for F_0 and F_2, so that there are $I(t-1)$ symmetry equations. In special cases, useful inequalities can be found by combining (3.10), (3.11), (4.11), with (4.17) and (4.18). Where F_0 and F_2 are determined, but not the $f(\tau,0)$ and $f(\tau,2)$, these inequalities represent the only use which we have made of $h_+(\mu)$ for $\mu>t$.

Higher S

The odd moments for $S=\frac{3}{2}$ are treated similarly to the even ones for $S=1$, except that now (2.16) gives

$$C(\tau,3,\tau;\mu,0)=\frac{5\mu^3-[3\tau(\tau+1)-1]\mu}{2\langle\tau\|Q_3\|\tau\rangle}. \tag{4.19}$$

$$h_-(\mu)=F_1\mu+F_3\mu^3, \tag{4.20}$$

for $\mu\leqq t$.

$$F_1=\sum_\tau\frac{1}{2\tau+1}\left[\frac{3f(\tau,1)}{[\tau(\tau+1)]^{\frac{1}{2}}}-\frac{5[3\tau(\tau+1)-1]f(\tau,3)}{2\langle\tau\|Q_3\|\tau\rangle}\right], \tag{4.21}$$

$$F_3=\tfrac{5}{2}\sum_\tau\frac{5}{2\tau+1}\frac{f(\tau,3)}{\langle\tau\|Q_3\|\tau\rangle}. \tag{4.22}$$

The general analysis of the even moments for $S=\frac{3}{2}$ and all cases for higher S is more complicated and less useful except in the event of large t.

5. STATISTICALLY SIMPLE ISOSPIN SYSTEMS

When the maximum isospin S of the initial state is too high to make the symmetry theorems useful, it may still be possible to test isospin invariance by analyzing a suitable average of the results of experiments with different charge states. This average is analogous to angular measurements with incompletely polarized collision partners. It will be analyzed using the statistical, or density, matrix technic. The main result may be anticipated: the distributions in an equally weighted average over all charge states of one of the collision partners may be treated as if that partner had zero isospin.

The statistical charge distribution $Q(\mu_1\cdots\mu_N)$ is defined as the weighted average of the $q(\mu_1\cdots\mu_N)$ for the individual experiments. The statistical total charge distribution is then

$$H(\mu)=\sum_{\substack{\mu_1+\cdots\\+\mu_N=\mu}}Q(\mu_1\cdots\mu_N). \tag{5.1}$$

We now proceed to restate the previous considerations in terms of the statistical matrix ρ of the final state, instead of the wave function ψ. The statistical channel amplitudes and moments are defined as

$$G(\tau,\mu)=\sum_\beta \mathrm{Tr}\{\rho\Lambda_\beta\Lambda_\tau\Lambda_\mu\}, \tag{5.3}$$

$$F(\tau,L)=\frac{\sum_\beta \mathrm{Tr}\{\rho\Lambda_\beta\Lambda_\tau Q_L\}}{\langle\tau\|Q_L\|\tau\rangle}, \tag{5.4}$$

where τ, μ, and Q_L refer to the isospins of the measured particles alone. The sum on β is the same as that in Sec. 3, and Λ_β represents the selection criteria on other variables than the isospins of the measured particles.

$$H(\mu)=\sum_\beta \mathrm{Tr}\{\rho\Lambda_\beta\Lambda_\mu\}=\sum_\tau G(\tau,\mu). \tag{5.5}$$

Since Q_L is diagonal in τ and μ, it follows that

$$\Lambda_\tau Q_L=\sum_\mu\langle\tau,\mu|Q_L|\tau,\mu\rangle\Lambda_\tau\Lambda_\mu. \tag{5.6}$$

$$F(\tau,L)=\sum_\mu C(\tau,L,\tau;\mu,0)G(\tau,\mu), \tag{5.7}$$

$$G(\tau,\mu)=\sum_L\frac{2L+1}{2\tau+1}C(\tau,L,\tau;\mu,0)F(\tau,L). \tag{5.8}$$

For twice the maximum isospin appearing in ψ, we wish to substitute the rank of the highest rank irreducible tensor appearing in ρ. We therefore express ρ in the form

$$\rho=\sum_{\gamma,L,M}B(\gamma,L,M)S_{LM}{}^\gamma. \tag{5.9}$$

Each set $S_{LM}{}^\gamma$ forms an irreducible tensor of rank L, that is it transforms under rotation according to

$$RS_{LM}{}^\gamma R^{-1}=\sum_{N=-L}^{L}S_{LN}{}^\gamma D_{NM}{}^L(R), \tag{5.10}$$

where R is the rotation operator and D^L the corresponding irreducible representation of the rotation group.[3] The $S_{LM}{}^\gamma$ include, but are not exhausted by, the T_{LM}, since the latter are diagonal in the total isospin. The coefficients $B(\gamma,L,M)$ may be functions of operators which commute with all the isospins. Then the part of ρ which has rank L is given by

$$\rho(L,M)=\sum_\gamma B(\gamma,L,M)S_{LM}{}^\gamma. \tag{5.11}$$

It can in principle be calculated, by using the orthogonality of the $D_{NM}{}^L$, as

$$\rho(L,M)=\frac{2L+1}{8\pi^2}\int D_{MM}{}^L(R)^*(R\rho R^{-1})d\Omega, \tag{5.12}$$

where the integral is carried over all rotations.

The orthogonality of the $D_{NM}{}^L$ also shows that

$$\mathrm{Tr}\{\rho Q_L\}=\mathrm{Tr}\{\rho(L,0)Q_L\}, \tag{5.13}$$

since the trace is invariant under rotation and may be replaced by its average over all rotations. (5.13) is then the statistical analog of the Wigner-Eckart theorem, and implies the analog of (4.1), that

$$F(\tau,L)=0, \tag{5.14}$$

when $2S<L\leqq 2\tau$, where now $2S$ is defined to be the maximum L for which $\rho(L,0)$ is different from zero. The scalar factors Λ_β, Λ_τ in (5.4) do not disturb the applicability of (5.13).

There may of course be isolated L less than $2S$ for which $\rho(L,0)$ vanishes. For instance, it will appear below that $\rho(L,0)$ vanishes for odd L in statistical systems having up-down symmetry. However, $\rho(L,0)$ might also vanish "accidentally" for some L, and to this extent (5.14) is broader than its mechanical analog (4.1), where that cannot happen.

(5.14) completes the statistical generalization of the symmetry theorems, which now apply directly under the substitution of F, G, H for f, g, h. It remains only to determine S in the initial state of isospin-conserving reactions, since for such reactions the vanishing of $\rho_i(L,0)$ in the initial state implies the same in the final state. If the initial state consists of only two objects the analysis can be carried out, provided that their charge states are uncorrelated with other variables. The importance of the proviso is that then the initial charge distribution takes the form

$$Q_i(m_1m_2)=\mathrm{Tr}\{\rho_i\Lambda_{m_1}\Lambda_{m_2}\textstyle\sum_\beta A_\beta\Lambda_\beta\}, \tag{5.15}$$

where the A_β are the weights of the nonisospin states. Since $\sum_\beta A_\beta\Lambda_\beta$ is a scalar, it may be ignored for our purposes.

Consider first the statistical matrix ρ_s for a single particle of isospin s.

$$Q(m)=\mathrm{Tr}\{\rho_s\Lambda_m\}=\textstyle\sum_{L,M}\mathrm{Tr}\{\rho_s(L,M)\Lambda_m\}. \tag{5.16}$$

Since Λ_m is diagonal in m, (5.16) receives contributions only from $M=0$.

$$Q(m)=\textstyle\sum_L\langle s\|\rho_s(L)\|s\rangle C(s,L,s;m,0), \tag{5.17}$$

$$\langle s\|\rho_s(L)\|s\rangle=\frac{2L+1}{2S+1}\textstyle\sum_m C(s,L,s;m,0)Q(m). \tag{5.18}$$

From (5.18) it follows that $S\leqq s$, and that $\rho_s(L,0)$ vanishes for odd L if $Q(m)=Q(-m)$. $\rho_s(L,0)$ may also vanish accidentally for other $L<2S$, a possibility which does not exist if $Q(m)$ represents a pure state, since the coupling coefficients never vanish for $m\neq 0$. In particular, if all the $Q(m)$ are equal, so that the particle is "unpolarized," $S=0$.

In an experiment where the charge distributions of the collision partners are independent,

$$\rho_i=\rho_1\times\rho_2. \tag{5.19}$$

This is in fact the most likely statistical situation, since it includes all cases where one of the collision partners has a definite charge.

$$\rho_i(L,0)=\sum_{L_1,L_2}\rho_1(L_1,0)\rho_2(L_2,0)C(L_1,L_2,L;0,0), \tag{5.20}$$

$$S=S_1+S_2. \tag{5.21}$$

If $\rho_1(L_1,0)$ and $\rho_2(L_2,0)$ vanish for all odd L_1, L_2, then $\rho_i(L,0)$ vanishes for all odd L. If $S_2=0$, the accidental zeroes of ρ_i coincide with those of ρ_1. To find the accidental zeroes of ρ_i when neither S_1 nor S_2 vanishes, it is necessary to find the reduced matrix elements of $\rho_i(L,0)$, as implied by (5.20) and the vector recoupling laws.

$$\langle s'\|\rho_i(L)\|s\rangle=[(2s+1)(2L+1)(2s_1+1)(2s_2+1)]^{\frac{1}{2}}\times\sum_{L_1,L_2}\langle s_1\|\rho_1(L_1)\|s_1\rangle\langle s_2\|\rho_2(L_2)\|s_2\rangle\times\begin{Bmatrix} s_1 & s_2 & s \\ L_1 & L_2 & L \\ s_1 & s_2 & s'\end{Bmatrix}. \tag{5.22}$$

The last factor is the Wigner 9-j symbol.[8] To ensure that $F(\tau,L)=0$, (5.22) must vanish for all pairs s, s'.

Finally, where $Q(m_1,m_2)$ is not a product of independent distributions, the only useful result is the obvious one that only even L enter the analysis if $Q(m_1,m_2)=Q(-m_1,-m_2)$. There might in principle be accidental zeroes, but these would not be useful with $S=s_1+s_2$, since the statistical equations (3.8) would remain underdetermined. Furthermore, to find the accidental zeroes requires a calculation substantially more complicated than that involved in (5.22).

6. DISCUSSION

The symmetry theorems presented here are obviously far from giving all that can be learned from charge distributions, as they deal only with the total charge distribution $h(\mu)$ and discard most of the information contained in the joint distribution $q(\mu_1\cdots\mu_N)$. Additional information about channel intensities and moments, pertaining to the other invariant isospin quantities α, could be obtained by constructing tensor operators from such vectors as $J_1+J_2-J_3$. This would make fuller use of the data in events where at least three particles are observed, at the price of a somewhat more complicated analysis. However, the example of the two-particle event, where there are no other invariants α, shows that even then much of the data would not be used. Constructing other tensors from the same operators does nothing to remedy the lack, since they are only invariant multiples of the T_{LM}. It therefore appears that the above-noted use of isospin

[8] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), p. 100.

differences presents the only prospect of finding additional symmetry theorems.

It is not at all surprising that the isospin invariance should give such limited general results, since much of the information contained in the joint distribution function may be regarded as dealing with "internal" variables which are not affected by rotation of the system as a whole. For this reason, it may be anticipated that the use of difference vectors will not produce very much more, in, e.g., a three-particle analysis, than would be learned by analyzing each pair by the methods given here. Some weak additional limitations arising from the vector coupling coefficients when the last particle is added will probably emerge.

It is interesting to see that the ideas of Sec. 5, applied to single-particle angular distributions, give the symmetry theorems for those problems[2] almost without writing any equations. For instance, suppose particles of spin s impinge on unpolarized targets, and the maximum partial wave that reacts is $\mathcal{L}$. Then, since the statistical matrix is quadratic in the wave function, the statistical matrix has no L greater than $2\mathcal{L}+2s$. Consequently, the final-state expectation of Y_{LM} vanishes for $L>2\mathcal{L}+2s$. Some generalizations for incompletely polarized initial states, analogous to those in the isospin case, can likewise be given.

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APPENDIX

$$\langle j, j|T_{LL}|j, j-L\rangle$$
$$=N_L\langle j, j|(J_x+iJ_y)^L|j, j-L\rangle$$
$$=N_L\hbar^L \prod_{\mu=j-L}^{j-1} [(j-\mu)(j+\mu+1)]^{\frac{1}{2}}$$
$$=\frac{1}{L!}\left(-\frac{\hbar}{2}\right)^L\left[\frac{(2L+1)!}{4\pi}\right]^{\frac{1}{2}}\left[\frac{(2j)!L!}{(2j-L)!}\right]^{\frac{1}{2}}, \quad \text{(A.1)}$$

$$C(j, L, j; j-L, L)=(-1)^L\left[\frac{(2j+1)!(2L)!}{(2j+L+1)!L!}\right]^{\frac{1}{2}}. \quad \text{(A.2)}$$

(A.1) is divided by (A.2) to give (2.12).