Mass and Lifetime of Unstable Particles*

R. JACOB† AND R. G. SACHS‡
University of Wisconsin, Madison, Wisconsin
(Received August 19, 1960)

The relationship between the properties of the propagator of an unstable particle and the observation of mass and lifetime is considered. For illustrative purposes a model of a scalar (or pseudoscalar) particle (θ) weakly coupled to two pions is treated. The propagator is shown to have a simple pole on the second (unphysical) Riemann sheet and it is assumed, as suggested by Peierls, that this is generally the case. By analysis of a prototype experiment in terms of wave packets, it is shown that the measured mass and lifetime are determined by the real and imaginary parts of the pole, respectively. Nonexponential terms occur in the lifetime curve, as is well known. These are shown to be related to the uncertainty in the time of the production or detection event under normal circumstances. This conclusion is similar to those of Lévy and of Schwinger, but more closely related to experimental conditions. In particular it is found that the wave packets introduce a "mass filter" in a somewhat different manner from that suggested by Schwinger.

Under special conditions a $t^{-\frac{3}{2}}$ term may occur in the amplitude but would be unimportant in magnitude for, say, the decay of a strange particle. It is noted that such nonexponential decay curves might occur for certain low-energy nuclear processes.

Consideration is also given to the treatment of two degenerate, unstable particles, such as the neutral K mesons. The general method for handling the problem leads, in the weak-coupling limit, to the same result as the Wigner-Weisskopf method.

1. INTRODUCTION

VARIOUS aspects of the treatment of unstable particles in quantum field theory have recently been discussed by several authors.[1-6] From these discussions, several questions have emerged. Although it is well established on the basis of the uncertainty principle that a measurement of the mass of an unstable particle will not lead to a unique answer, it is nevertheless possible to pose the problem of defining some quantity, to be called the "mass" of the particle, which locates a focus for the mass distribution. It has been suggested by Peierls,[1] by Matthews and Salam,[3] and by others that the definition of this quantity should be related to the spectral function of Lehmann[7] defining the propagator. In particular, Peierls suggested that there is a pole in the lower half plane of the second Riemann sheet of the propagator, and that the real and imaginary parts of the pole serve to define the mass and lifetime of the particle. The existence of the pole in the case of the Lee model of an unstable particle has been clearly demonstrated by Lévy[4] and we shall demonstrate it below in perturbation theory applied to a highly simplified model of the decay mechanism of the θ particle.

When the form of the propagator is given, it is generally assumed that the history of the particle may be described in terms of the time-dependence of the propagator. This leads not only to the characteristic exponential decay of the particle, but also to certain additional terms decreasing as inverse powers of the time for sufficiently large times. In particular, there is associated with the branch point in the propagator, following from the possible decay or dissociation of the particle, a characteristic asymptotic time dependence proportional to $t^{-\frac{3}{2}}$ (for S-wave decay). The question of the measurability of such behavior naturally comes to mind and in this connection both Lévy[5] and Schwinger[6] have considered to some extent the influence of production and observation mechanisms on the asymptotic time dependence. We shall look further, and somewhat more directly, into these matters below. It will be shown that the distribution in time is not given directly by the time-dependence of the propagator but, instead, by a function incorporating the form of the wave packets which serve to express the experimental conditions. Under "normal" conditions, i.e., when the energy spectrum of the production process is reasonably limited, the nonexponential time behavior is governed entirely by the time distribution of the production and detection events. On the other hand, when the production and detection processes encompass a wide energy range, overlapping the branch point in the propagator, the $t^{-\frac{3}{2}}$ term should occur. However, in the event that the instability of the particle is due to a weak interaction, the coefficient of this term is so small that the probability of the event corresponds to the order of magnitude of cross sections for weak-interaction events [such as production of $\Lambda+2\pi$ by a pion-nucleon collision below the threshold for (Λ,K) production]. If a strong interaction is responsible for the instability, a measurement of the chronological behavior is not usually feasible.

For illustrative purposes, attention will be directed to the simple model of θ decay mentioned above al-

* Work supported in part by the U. S. Atomic Energy Commission and in part by the Research Committee of the University of Wisconsin with funds provided by the Wisconsin Alumni Research Foundation.

† Part of this work performed as a National Science Foundation Predoctoral Fellow. Present address: Tufts University, Medford 55, Massachusetts.

‡ Guggenheim Fellow, 1959–60.

[1] R. E. Peierls, *Proceedings of the 1954 Glasgow Conference on Nuclear and Meson Physics* (Pergamon Press, New York, 1955), p. 296.

[2] G. Höhler, Z. Physik **152**, 546 (1958).

[3] P. T. Matthews and A. Salam, Phys. Rev. **112**, 283 (1958); **115**, 1079 (1959).

[4] M. Lévy, Nuovo cimento **13**, 115 (1959).

[5] M. Lévy, Nuovo cimento **14**, 619 (1960).

[6] J. Schwinger, Ann. Phys. **9**, 169 (1960).

[7] H. Lehmann, Nuovo cimento **11**, 342 (1954).

though the results are clearly much more general. The properties of the θ particle are of particular interest in this connection because a direct measurement of the difference in masses between the two particle species, θ_1 and θ_2, is feasible. Just how the mass difference between two unstable particles can be sharply defined is exactly the question concerning a sharp definition of mass that was raised above. It will be shown that the quantity being measured is the shift in the real part of the pole in the propagator, as expected.

2. STRUCTURE OF THE PROPAGATOR

The model to be used as the basis for our discussion of the structure of the propagator is that of a scalar (or pseudoscalar) particle, which we call the θ particle, subject to a direct weak interaction of strength g converting it into two pions. This is the only interaction that will be included in the considerations of this Section. The influence of a mass degeneracy, such as occurs for the physical θ particles, will be treated in Sec. 5.

The propagator of the bare θ particle is

$$\Delta_F(k^2)=\lim_{z\to k^2+i\epsilon}[z-M_0^2]^{-1}, \tag{1}$$

where M_0 is the bare θ mass and k is the four-momentum with metric chosen so that k^2 is positive for timelike k. The propagator corrected for the coupling to pions is denoted by $\Delta_F'(k^2)$ and

$$\Delta_F'(k^2)=\lim_{z\to k^2+i\epsilon}[z-M_0^2-\Pi^*(z)]^{-1}, \tag{2}$$

where $\Pi^*(k^2)$ gives, in Dyson's notation, the proper self-energy contribution to the propagator. To lowest order in g^2 (the pion bubble diagram), straightforward calculation of Π^* yields

$$\Pi^*(k^2)=\delta M^2+g^2[1-(4m^2/k^2)]^{\frac{1}{2}}\ln\frac{[1-(4m^2/k^2)]^{\frac{1}{2}}+1}{[1-(4m^2/k^2)]^{\frac{1}{2}}-1},$$

for $k^2<4m^2$. δM^2 is a logarithmically divergent real constant which may be incorporated into the mass renormalization in the usual way. If

$$M^2=M_0^2+\delta M^2,$$

we may write Eq. (2) in the form

$$\Delta_F'(k^2)=\lim_{z\to k^2+i\epsilon}F(z), \tag{3}$$

where

$$F(z)=[z-M^2-f(z)]^{-1} \tag{4}$$

and $f(z)$ is defined in the complex plane cut along the real axis from $z=4m^2$ to ∞ by

$$f(z)=g^2[1-(4m^2/z)]^{\frac{1}{2}}\ln\frac{[1-(4m^2/z)]^{\frac{1}{2}}+1}{[1-(4m^2/z)]^{\frac{1}{2}}-1}. \tag{5}$$

The boundary values of $f(z)$ on the cut from the upper and lower half-planes are $(x>4m^2)$

$$f(x\pm i\epsilon)=g^2[1-(4m^2/x)]^{\frac{1}{2}}\times\left\{\ln\frac{1+[1-(4m^2/x)]^{\frac{1}{2}}}{1-[1-(4m^2/x)]^{\frac{1}{2}}}\mp i\pi\right\}. \tag{6}$$

To generalize the discussion, we write

$$f(x\pm i\epsilon)=u(x)\mp iv(x) \tag{7}$$

and note that in the special case described by Eq. (6),

$$u(x)=g^2[1-(4m^2/x)]^{\frac{1}{2}}\ln\frac{1+[1-(4m^2/x)]^{\frac{1}{2}}}{1-[1-(4m^2/x)]^{\frac{1}{2}}}, \tag{8a}$$

and

$$v(x)=\pi g^2[1-(4m^2/x)]^{\frac{1}{2}}. \tag{8b}$$

The spectral function $\rho(x)$ for the propagator is defined by[7]

$$F(z)=\int_{b^2}^{\infty}dx'\frac{\rho(x')}{z-x'}, \tag{9}$$

where $F(z)$ is given by Eq. (4) and b^2 is the branch point for $f(z)$;

$$b^2=4m^2 \tag{10}$$

in the special case. Since

$$F(x+i\epsilon)-F(x-i\epsilon)=-2\pi i\rho(x), \tag{11}$$

according to Eq. (9), we find by means of Eq. (4) and Eq. (7) that $\rho(x)$ is given by

$$2\pi i\rho(x)=2iv(x)\{[x-M^2-u(x)]^2+v^2(x)\}^{-1}. \tag{12}$$

Using, for example, the expressions Eq. (8) for $u(x)$ and $v(x)$, we may define $\rho(z)$ for complex z. The roots z_0 and z_0^* of

$$z_0-M^2-u(z_0)+iv(z_0)=0 \tag{13a}$$

and

$$z_0^*-M^2-u(z_0^*)-iv(z_0^*)=0 \tag{13b}$$

then define poles in ρ, provided that they exist. In fact, for small coupling constant g, u and v are small compared to M^2; hence $z_0\approx M^2$ and the positions of the poles to first order in g^2 are

$$z_0=M^2+u(M^2)-iv(M^2), \tag{14}$$

and its conjugate complex, z_0^*.

These poles in ρ may also be interpreted, following Peierls' suggestion,[1] as poles on the second Riemann sheet of the function $F(z)$. Since

$$\Delta_F'(x)=F(x+i\epsilon)=[x-M^2-u(x)+iv(x)]^{-1},$$

the continuation of the propagator into the lower half plane has just the pole z_0 given by Eq. (13). At the same time, the distribution $\rho(x)$ has the form assumed by Matthews and Salam.[3] The question of whether the mass and lifetime as determined by experiment are related directly to z_0 (Peierls) or to moments of ρ

(Matthews and Salam) is concerned with the nature of the measurements, which is taken up in Secs. 3 and 4.

3. ANALYSIS OF THE PROTOTYPE EXPERIMENT

The prototype experiment for measuring the mass and lifetime of an unstable particle can be described as follows: Two particles such as the pion and proton collide to produce the unstable particle (θ), possibly in association with another (such as a hyperon which is taken to be stable for the purpose of this discussion). The θ either decays or interacts with an additional incident particle after some time has elapsed. The corresponding Feynman diagram is indicated in Fig. 1. We take all particles to be scalar particles for the sake of simplicity. The only essential feature of this diagram is that there are two vertices connected by a single internal line which describes the virtual θ particle. We denote by p_i the *total* 4-momentum of the external lines coming into the right-hand vertex and by p_e the *total* emitted 4-momentum. Similarly p_i' and p_e' refer to the external lines at the left-hand vertex. Note that the fourth component of p_i, say, includes the internal energy of the particles incident upon the one vertex. All other internal variables are denoted collectively by η_i, η_e, η_i', and η_e', respectively.

The result to be expected from an observation of the above type may be obtained from the corresponding element of the S matrix. However under normal conditions the incoming and outgoing particles are not in plane wave states but in wave packets; hence we wish to determine the S-matrix element between states described by appropriate packets. The amplitude of the packet describing the collection of external lines entering the right-hand vertex will be denoted by ψ_i, and the amplitudes associated with the other sets of external lines will be ψ_e, ψ_i', and ψ_e', respectively. The packet ψ_i must describe, at early times, localized fields which are progressing toward a common meeting point $\mathbf{r}$. Let us assume that, in the absence of interaction, the packets would meet at this point at time t. Then the 4-vector $x=(\mathbf{r},t)$ denotes the space-time location of the center of mass of the incoming system at the instant of collision. We denote the Fourier amplitude of the packet at this instant by $\psi_i(p_i,\eta_i)\exp(-i\mathbf{p}_i\cdot\mathbf{r})$. At an arbitrary time T, the Fourier amplitude of the noninteracting packet would be $\psi_i(p_i,\eta_i)\exp[ip_i\cdot(X-x)]$, where $X=(0,T)$. In a similar fashion, we may write $\psi_e(p_e,\eta_e)\times\exp[ip_e\cdot(X-x)]$ for the packet of outgoing waves which extrapolates back to the same space-time point x of the initial collision. Finally, if x' is the space-time point of decay or final collision, the packets at the left-hand vertex have the amplitudes[8] $\psi_i'(p_i',\eta_i')\times\exp[ip_i'\cdot(X-x')]$ and $\psi_e'(p_e',\eta_e')\exp[ip_e'\cdot(X-x')]$.

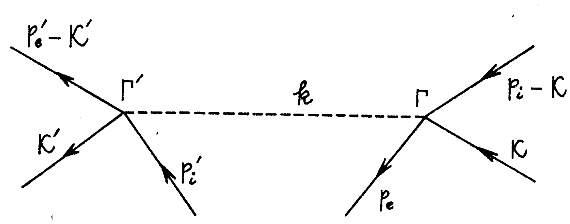

FIG. 1. Protype of production and detection events. The dashed line represents the unstable (θ) particle.

The S-matrix element between the states described by the packets is

$$S=\int dp\int d\eta\int d^4k\,\psi_e'^*\psi_e^*\Gamma'\delta(p_e'-p_i'-k)$$
$$\times\Delta_F'(k^2)\delta(p_i-p_e-k)\Gamma\psi_i\psi_i'\exp[-i(p_e'-p_i')\cdot(X-x')-i(p_e-p_i)\cdot(X-x)],\quad(15)$$

with

$$dp=d^4p_e'd^4p_i'd^4p_ed^4p_i$$

and

$$d\eta=d\eta_e'd\eta_i'd\eta_ed\eta_i.$$

The vertex functions Γ' and Γ may be taken to be constant since the vertex interactions take place over a region which is extremely small compared to the distance traveled by the unstable particle.

The essential features of the packets are described by the function

$$\varphi(k^2,\mathbf{k})=\Gamma'\Gamma\int d\eta\int d^4p_e'\,\psi_e'^*(p_e',\eta_e')\psi_i'(p_e'-k,\eta_i')$$
$$\times\int d^4p_i\,\psi_e^*(p_i-k,\eta_e)\psi_i(p_i,\eta_i).\quad(16)$$

In terms of φ, Eq. (15) becomes

$$S=\int d^4k\;\varphi(k^2,\mathbf{k})\Delta_F'(k^2)e^{ik\cdot\xi}\quad(17)$$

with

$$\xi=x'-x.\quad(18)$$

We shall write

$$\xi\equiv(\boldsymbol{\rho},\tau),\quad(19)$$

and note that τ is positive since we intend that the process progress from right to left in the diagram of Fig. 1.

We separate the space and time dependence of S by writing

$$S=\int d^3k\exp(i\mathbf{k}\cdot\boldsymbol{\rho})I(\mathbf{k},\tau),\quad(20)$$

where

$$I(\mathbf{k},\tau)=\int dk_0\;\varphi(k^2,\mathbf{k})\Delta_F'(k^2)e^{-ik_0\tau}.\quad(21)$$

The significant question now concerns the behavior of S as a function of ξ or, more specifically, the behavior of I as a function of τ. The probability for an event in

[8] When dealing with a decay process having no particles incident on this vertex, we simply set $\psi_i'=\delta(p_i')$.

which incoming packets collide at x to produce a virtual θ which then decays or interacts with a packet at x' to yield the specified outgoing state is given by $|S|^2$. Hence it is appropriate to designate by $S(\xi)$ the amplitude for such an event.

An evaluation of $I(\mathbf{k},\tau)$ requires some knowledge of the functions $\varphi(k^2,\mathbf{k})$. We assume that the energy spectrum of all incident particles is limited to a finite range of energy. Consequently the function φ is distinct from zero only for k^2 within certain bounds which we take to be M_1^2 and M_2^2. The crucial property of φ will be the manner in which it vanishes at M_1^2 and M_2^2. We take

$$\varphi(k^2,\mathbf{k})=(k^2-M_1^2)^n(k^2-M_2^2)^n\Phi(k^2,\mathbf{k})/n!,\qquad M_1^2<k^2<M_2^2, \tag{22}$$

$\varphi(k^2,\mathbf{k})=0$, otherwise.

The symmetrical behavior at M_1^2 and M_2^2 is chosen only to simplify the algebra. $\Phi(k^2,\mathbf{k})$ is taken to be analytic in k^2 over the domain of interest.

Setting

$$z=k^2=k_0^2-\mathbf{k}^2, \tag{23}$$

we may now rewrite Eq. (21) as

$$I=\frac{1}{2}\int_{M_1^2+i\epsilon}^{M_2^2+i\epsilon} dz(z+\mathbf{k}^2)^{-\frac{1}{2}}\varphi(z,\mathbf{k})F(z)\times\exp[-i(z+\mathbf{k}^2)^{\frac{1}{2}}\tau], \tag{24}$$

where $F(z)$ is the function appearing in the expression Eq. (3) for $\Delta_F'(k^2)$. The branch point of $F(z)$ occurs at b^2 and we must distinguish the case for which b^2 lies between M_1^2 and M_2^2 from the more usual situation in which M_1^2 and M_2^2 lie close together and well above b^2. Unless otherwise specified, it is assumed that $M_1^2<\mathrm{Re}z_0<M_2^2$.

For the usual case the path of integration is that shown as P, between M_1^2 and M_2^2, in Fig. 2. It is assumed that the upper limit, M_2^2, on the wave packet spectrum lies below any other branch points occurring in $F(z)$. Otherwise the physical mode of dissociation characterizing these branch points would also occur in the reaction. The path of integration is now deformed as indicated in Fig. 2.

The continuation of $F(z)$ to the next Riemann sheet

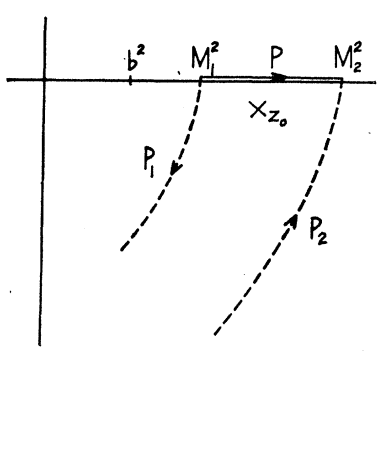

FIG. 2. Path of the integration in the "usual" case, $b^2<M_1^2<M_2^2$.

on passing through the cut is denoted by $F_{\mathrm{II}}(z)$,

$$F_{\mathrm{II}}(z)=[z-M^2-u(z)+iv(z)]^{-1}, \tag{25}$$

according to Eqs. (4) and (7). It is assumed that $u(z)$ and $v(z)$ are analytic within the domain of deformation. $F_{\mathrm{II}}(z)$ has a pole at z_0 given by Eq. (13a); hence Eq. (24) becomes[9]

$$I=-\pi i(z_0+\mathbf{k}^2)^{-\frac{1}{2}}\varphi(z_0,\mathbf{k})\exp[-i(z_0+\mathbf{k}^2)^{\frac{1}{2}}\tau]+J_2(\tau,\mathbf{k})+J_1(\tau,\mathbf{k}), \tag{26}$$

where

$$J_j(\tau,\mathbf{k})=\frac{1}{2}\int_{P_j} dz(z+\mathbf{k}^2)^{-\frac{1}{2}}\varphi(z,k)F_{\mathrm{II}}(z)\times\exp[-i(z+\mathbf{k}^2)^{\frac{1}{2}}\tau]. \tag{27}$$

The paths P_j in Fig. 2 are defined by allowing ω to range from 0 to ∞ for $j=1$ and from ∞ to 0 for $j=2$ in the expression

$$z=[-i\omega+(M_j^2+\mathbf{k}^2)^{\frac{1}{2}}]^2-\mathbf{k}^2. \tag{28}$$

Therefore

$$J_j(\tau,\mathbf{k})=(-1)^j i\exp[-i(M_j^2+\mathbf{k}^2)^{\frac{1}{2}}\tau]\times\int_0^\infty d\omega\;\varphi(z(\omega),\mathbf{k})F_{\mathrm{II}}(z(\omega))e^{-\omega\tau}. \tag{29}$$

An asymptotic expansion of $J(\tau,\mathbf{k})$ for large τ may be obtained by substituting $y=\omega\tau$ for the integration variable and expanding the integrand in powers of τ^{-1}. Because φ has the form indicated by Eq. (22), the leading term in the expansion is found to be

$$J_j(\tau,\mathbf{k})\approx-(-1)^{j(n+1)}(i\tau)^{-(n+1)}\times(M_2^2-M_1^2)^n2^n(M_j^2+\mathbf{k}^2)^{n/2}\Phi(M_j^2,\mathbf{k})\times\Delta_F'(M_j^2)\exp[-i(M_j^2+\mathbf{k}^2)^{\frac{1}{2}}\tau], \tag{30}$$

when use is made of Eq. (3). The higher order terms in τ^{-1} involve derivatives of $F_{\mathrm{II}}(M_j^2)$ and it can be seen from the form of $F_{\mathrm{II}}(z)$, Eq. (25), that these are of the order of $[2M_j/(M_j^2-M^2)]\Delta_F'(M_j^2)\approx(\Delta M)^{-1}\Delta_F'$, where $\Delta M=M-M_j$. Therefore the critical time interval in the asymptotic expansion is of the order of $(\Delta M)^{-1}$, namely, of the order of the uncertainty in the definition of the time of arrival of the wave packets. It will be very short compared to the lifetime under normal conditions.[10]

We turn now to the less usual case in which the incident spectrum overlaps the branch point: $M_1^2<b^2<M_2^2$. Then the path of integration for Eq. (24) is shown as P in Fig. 3, and the path is deformed as indicated.

The result differs from Eq. (26) only in the contributions of the paths P_b' and P_b, the former lying on the

[9] The propagator has been renormalized (wave function renormalization) so that $F(z)=(z-z_0)^{-1}$ in the neighborhood of the pole.

[10] It follows that the asymptotic form of the integrals J_j may be used for values of τ smaller than or comparable to the lifetime, as well as for much longer time intervals.

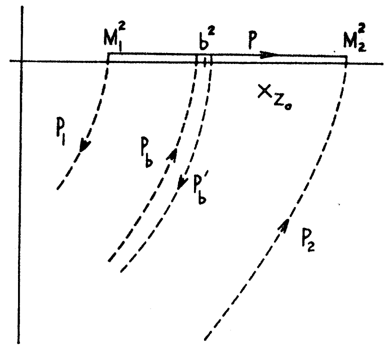

FIG. 3. Path of integration for $M_1^2 < b^2 < M_2^2$.

second Riemann sheet and the latter on the physical sheet. Thus, the addition to Eq. (26) is

$$J_b = \frac{1}{2}\int_{P_b'} dz(z+\mathbf{k}^2)^{-\frac{1}{2}}\varphi(z,\mathbf{k})[F_{\text{II}}(z)-F(z)] \times \exp[-i(z+\mathbf{k}^2)^{\frac{1}{2}}\tau], \quad (31)$$

where the path P_b' is now defined by

$$z = [-i\omega + (b^2+\mathbf{k}^2)^{\frac{1}{2}}]^2 - \mathbf{k}^2. \quad (32)$$

From Eq. (11) we find

$$J_b = -2\pi \exp[-i(b^2+\mathbf{k}^2)^{\frac{1}{2}}\tau] \times \int_0^\infty d\omega\ \varphi(z(\omega),\mathbf{k})\rho(z(\omega))e^{-\omega\tau}. \quad (33)$$

The asymptotic behavior of the integral is controlled by the manner in which $\rho(x)$ goes to zero at $x=b^2$ rather than by the form of the wave packet (φ is assumed to be regular at b^2) as it was in the usual case. The behavior of $\rho(x)$ depends in turn, on the behavior of $v(x)$ near $x=b^2$, as can be seen from Eq. (12). Following the suggestion of Eq. (8b), we assume that for the general case of an S-wave threshold

$$v(z) = (z-b^2)^{\frac{1}{2}}\gamma^2(z), \quad (34)$$

where the "reduced width," γ^2, is a regular function of z. Then the integral is asymptotically

$$J_b(\tau,\mathbf{k}) \approx (-2\pi i)^{\frac{1}{2}}\tau^{-\frac{3}{2}} \frac{(b^2+\mathbf{k}^2)^{\frac{1}{4}}\gamma^2(b^2)}{[b^2-M^2-u(b^2)]^2} \times \varphi(b^2,\mathbf{k}) \exp[-i(b^2+\mathbf{k}^2)^{\frac{1}{2}}\tau]. \quad (35)$$

In this case the parameter in the asymptotic expansion is τQ, where $Q \approx M-b$ is the energy release on decay of the unstable particle.

The asymptotic expression for I in the event that the mass spectrum overlies the branch point is then

$$I \approx -\pi i\varphi(z_0,\mathbf{k})(z_0+\mathbf{k}^2)^{-\frac{1}{2}} \exp[-i(z_0+\mathbf{k}^2)^{\frac{1}{2}}\tau] + J_2(\tau,\mathbf{k}) + J_1(\tau,\mathbf{k}) + J_b(\tau,\mathbf{k}). \quad (36)$$

4. INTERPRETATION OF THE ANALYSIS

The expressions Eq. (26) or Eq. (36) for $I(\tau,\mathbf{k})$ are now to be inserted into Eq. (20) to obtain the S-matrix element. The matrix element is then made up of two distinctly different types of terms, those having an exponential τ-dependence and those depending on some inverse power of τ. The exponential terms are governed by the location of the pole z_0. In particular, the decay rate of the amplitude for fixed momentum k is $2\,\text{Im}(z_0+\mathbf{k}^2)^{\frac{1}{2}}$, as would be expected. Correspondingly, $\text{Re}(z_0+\mathbf{k}^2)^{\frac{1}{2}}$ determines the phase of the decaying amplitude so that $\text{Re}z_0$ may be defined as the (unique) mass of the unstable particle.

The terms J_1 and J_2 proportional to $\tau^{-(n+1)}$ have a very simple explanation. They arise as a consequence of the way in which the mass spectrum is cut off, as shown in Eq. (22). But the amplitude of the time distribution associated with this mass spectrum is also proportional to $\tau^{-(n+1)}$ for large τ. Hence these terms in I are just a manifestation of the uncertainty in the time at which the interactions take place. They can, of course, be modified by changing the form of the wave packets describing the reacting particles.

The magnitude of these terms relative to the exponential term is roughly

$$(\tau\Delta M)^{-(n+1)}, \quad (37)$$

where ΔM is, as before, the order of magnitude of the uncertainty in the experimental mass spectrum. For τ of the order of the θ_1 lifetime this ratio is $10^{-11(n+1)}$ if the mass is determined to 0.1%. In the probability distribution, $|S|^2$, the interference term will vanish after averaging over a very small time interval as a consequence of the time dependence of the relative phases of the exponential and J_j terms. Hence a measure of the contribution of J_j to the observation is the square of Eq. (37), which is very small.

The J_b terms in Eq. (36) appear only when the experimental mass spectrum overlaps the branch point. This is a consequence of the natural cutoff in the spectrum introduced by the threshold for dissociation of the unstable particle. Its form is governed by the behavior of the propagator; hence it is not very sensitive to experimental conditions other than the important condition on the location of the mass spectrum.[11] For just this reason, the possibility of actually detecting the term is of some interest.

The $\tau^{-\frac{3}{2}}$ dependence may be understood in the following way. If a particle is produced at a point x having a velocity between v and $v+dv$, it will appear after a time

[11] Note that the condition can be satisfied by considering the interaction of a pion, say, with a nucleus producing a Λ particle and a virtual θ. The propagator discussed here would then provide a description of the virtual θ_1-component of the θ field. This would interact with matter as a (virtual) $\bar{\theta}$, producing another Λ although the energy might be below the threshold for production of a real θ or even below the 2π threshold, which defines the branch point.

τ within a spherical shell of radius $v\tau$ centered on x. The thickness of the shell will be τdv. The probability that it will be found within a small element of volume within the shell is inversely proportional to the volume $4\pi\tau^3v^2dv$ of the shell. Hence the probability amplitude is proportional to $\tau^{-\frac{3}{2}}$. Therefore the entire amplitude S, given by Eq. (20), will be proportional to $\tau^{-\frac{3}{2}}$ if the location of the center of mass of the packet is specified to be within a small volume centered at $\mathbf{r}'$. The additional factor $\tau^{-\frac{3}{2}}$ appearing explicitly in J_b results from the fact that *two* particles are involved.[12] The contribution at the branch point corresponds to production of two real pions at $\mathbf{r}$ and both must arrive within the element of volume at $\mathbf{r}'$ in order to produce the desired reaction.

The order of magnitude of the amplitude relative to the exponential term is

$$(\tau Q)^{-\frac{5}{2}}(\lambda/Q), \tag{38}$$

where λ is the decay rate of the unstable particle

$$\lambda \approx v(M^2)/M, \tag{39}$$

and $Q=M-b$ is the energy release on decay. Since $\lambda \ll Q(\lambda/Q\approx 10^{-14}$ for the θ particle) the ratio Eq. (38) is extremely small when $\tau\approx\lambda^{-1}$. Since again, the quantity to be observed is proportional to the square of Eq. (38), the probability of observing the effect in competition with the exponential decay is very small indeed (10^{-68}).

In general, the effect in question has a probability of the order of $(\lambda/Q)^5$. For it to be observable, the width λ must be of the order of Q. Even for $\lambda/Q=10^{-1}$, the detection of the deviation from an exponential decay would be difficult. Resonances are known for which the width is of the same order as the Q value, for example, the $(\frac{3}{2},\frac{3}{2})$ nucleon isobar. The difficulty in these cases is that the time scale is too short to permit a detailed measurement of the shape of the decay curve. For the sake of discussion we may assume that such a measurement requires a lifetime greater than 10^{-14} sec, or $\lambda<0.1$ ev. Then a reaction with $Q<1$ ev would be required.

It may be worthwhile to note that for a number of nuclei there are slow neutron resonances which seem to satisfy these conditions. The decay curve of the corresponding compound nucleus produced in the proper fashion should be nonexponential in character.

Except under very unusual circumstances it is clear that the chronological history of the particle has the expected form of an exponential decay to a very good approximation. The mass of the particle, which determines the phase of its amplitude, is $\mathrm{Re}z_0$ and the lifetime is $(2\,\mathrm{Im}z_0)^{-1}$.

[12] This point was brought out by discussion with Professor R. Haag.

5. THE CASE OF DEGENERACY

In connection with the θ_1,θ_2 problem it is of some interest to consider the case of a mass degeneracy between two or more particles. The degeneracy in question is removed by the weak interaction, so that the mass operator Π^* is in general a matrix connecting the degenerate states. For example, if the mass operator of the θ particle is calculated it will be found to have terms connecting it to the $\bar{\theta}$. The θ_1 and θ_2 particles are to be defined in such a way that they are uncoupled. If the weak interactions are subject to a strong invariance condition, such as CP invariance, it is trivial to define the decoupled θ_1 and θ_2 as eigenstates of the appropriate operator. Then the treatment of Sec. 2 would yield the propagator of the θ_1 particle, that being the one capable of decaying into two pions. The θ_2 would be stable in this model, and its propagator would be that of a particle of fixed mass.

In the absence of a simple invariance condition, the problem is somewhat more complicated. It's solution has been given[13] in terms of the standard Wigner-Weisskopf perturbation theory, and it will be of interest to demonstrate that the same result is obtained by the present methods.

In an arbitrary representation of the particle states, the quantities M^2 and $f(z)$ are matrices. Hence we define the matrix function

$$G(z)=z\mathbf{1}-M^2-f(z), \tag{40}$$

where $\mathbf{1}$ is the unit matrix. In the θ_1, θ_2 case, G will be a 2×2 matrix but we need not limit attention to this example. Note that although M_0^2 is a multiple of the unit matrix, M^2 is not. The propagator is now also a matrix defined by Eq. (3) in terms of

$$F(z)=G^{-1}(z). \tag{41}$$

Similarly a spectral matrix $\rho(x)$ may be obtained from Eq. (11).

As before, the extension of the matrix $F(z)$ to the second sheet, i.e., the continuation $F_{\mathrm{II}}(z)$ of $F(x+i\epsilon)$ into the lower half plane, is expected to have poles at points z_α. We address ourselves to the problem of locating the poles and determining an appropriate representation of the matrices.

Our assumption is that in the immediate neighborhood of z_α, F_{II} has the form

$$F_{\mathrm{II}}(z)=\Omega_\alpha(z-z_\alpha)^{-1}+Q_\alpha(z), \tag{42}$$

where Ω_α is a constant matrix and $Q_\alpha(z)$ is regular at $z=z_\alpha$. The poles z_α are roots of the equation

$$\det G_{\mathrm{II}}(z_\alpha)=0. \tag{43}$$

To show this, Eqs. (41) and (42) are combined to give

$$(z-z_\alpha)^{-1}G_{\mathrm{II}}(z)\Omega_\alpha+G_{\mathrm{II}}(z)Q_\alpha(z)=1 \tag{44a}$$

[13] T. D. Lee, R. Oehme, and C. N. Yang, Phys. Rev. **106**, 340 (1957).

and

$$(z-z_\alpha)^{-1}\Omega_\alpha G_{\mathrm{II}}(z)+Q_\alpha(z)G_{\mathrm{II}}(z)=1 \qquad (44b)$$

for $z\neq z_\alpha$. In order that these equations be valid for z in the neighborhood of z_α, we must have

$$G_{\mathrm{II}}(z_\alpha)\Omega_\alpha=\Omega_\alpha G_{\mathrm{II}}(z_\alpha)=0. \qquad (45)$$

It follows that

$$\Omega_\alpha=\omega_\alpha\times\nu_\alpha, \qquad (46)$$

where ω_α and ν_α are appropriately normalized column and row vectors given by the solutions of the homogeneous linear equations

$$G_{\mathrm{II}}(z_\alpha)\omega_\alpha=0 \qquad (47a)$$

and

$$\nu_\alpha G_{\mathrm{II}}(z_\alpha)=0. \qquad (47b)$$

These solutions exist by virtue of Eq. (43).

We may now write

$$F_{\mathrm{II}}(z)=\sum_\alpha \omega_\alpha\times\nu_\alpha(z-z_\alpha)^{-1}+Q(z), \qquad (48)$$

where the sum is taken over all roots of Eq. (43) and $Q(z)$ is regular in the domain under consideration.

From Eqs. (40) and (47) we find that the ω_α are solutions of

$$[M^2+f(z_\alpha)]\omega_\alpha=z_\alpha\omega_\alpha. \qquad (50)$$

In the special case of weak coupling, $f(z_\alpha)$ may again be replaced by $f(M^2)$, whence it follows that the z_α are the characteristic values of the matrix $M^2+f(M^2)$. The real part of each of these characteristic values defines the mass of a particle and the reciprocal of the imaginary part defines twice the mean life of the same particle. This result is identical with that obtained by means of the Wigner-Weisskopf method.[13]

6. CONCLUSIONS

Our results are in complete accord with Peierls' suggestion[1] that the mass and lifetime of an unstable particle are determined by the pole in the spectral function of the propagator which lies in the lower half plane. An analysis of a typical experimental situation in terms of wave packets shows that the amplitude of the particle state is essentially the Fourier transform of the propagator, as generally assumed, but that there are corrections due to the experimental limitations on the energy. These corrections are not described by Schwinger's[6] "mass filter" acting on the spectral function. Instead, the filter effect of the wave packets acts directly on the propagator. The resulting corrections are a direct manifestation of the uncertainty in the definition of the time associated with the spread in energy of the packets in the usual case of an event having a rather well-defined energy. This corresponds to the points made by Lévy and Schwinger that the nonexponential terms in the decay amplitude are strongly dependent on experimental conditions. However, if the spectrum is broad enough to overlap a branch point (for an S-wave threshold) the terms proportional to $\tau^{-\frac{3}{2}}$ will occur and will not be otherwise sensitive to the experimental conditions. But the magnitude of the effect is such that it may not be possible to observe the deviations from a purely exponential decay with presently available techniques.

It has been assumed throughout this discussion that the analytic structure of the propagator has the general form suggested by the model. No attempt has been made to justify this assumption on general theoretical grounds.[14] However it is clear that if the propagator has this general form, in particular if the pole occurs on the second sheet, then the characteristic behavior of an unstable particle will be observed. Although it has not been proved, it seems unlikely that this behavior will occur under distinctly different circumstances.

ACKNOWLEDGMENT

Discussions with R. F. Sawyer have served to clarify a number of points in this work.

[14] In this connection see J. Gunson and J. G. Taylor, Phys. Rev. **119**, 1121 (1960).