permitted by the uniform convergence of the sum in h and z for small enough h. This gives

$$A(h)=\frac{1}{2}\int_L dz\, a(z)\frac{1-h^2}{(1-2zh+h^2)^{\frac{3}{2}}}, \qquad (A.5)$$

where we choose L so as to avoid singularities arising from the denominator. Using standard techniques of continuation,[8] we see that $A(h)$ can be continued along any finite path starting from $h=0$ and not passing through any of the points $\{\alpha+(\alpha^2-1)^{\frac{1}{2}}\}$, or $h=1$, or returning to the origin. The initial coincidence of singularities for $h=\{\alpha-(\alpha^2-1)^{\frac{1}{2}}\}$ is harmless as the arc L is not pinched. It is also easy to see that $h=1$ is not actually a singularity of $A(h)$, as the radius of convergence of $\sum a_n h^n$ is $R_a>1$.

The main theorem follows from the above lemma if we express $f(z)$ in the form

$$f(z)=\frac{1}{(2\pi i)^2}\int_C dh\int_{C'} dh'\frac{A(h)B(h')}{[(hh')^2-2zhh'+1]^{\frac{1}{2}}}, \qquad (A.6)$$

which follows on substituting the expressions

$$a_n=\frac{1}{2\pi i}\int_C \frac{dh}{h^{(n+1)}}A(h),$$

and likewise for b_n, into (A.2) and summing over n. C is a circle centered on the origin and of radius any value in between $R_a-\epsilon$ and $1+\epsilon$ ($\epsilon>0$), if we keep z initially on the interval $[-1, +1]$. C' is defined in a similar manner. The integration in (A.6) is over the distinguished surface $C\times C'$ of a circular bicylinder. This can always be deformed into a general bicylinder when we attempt to continue in z, so as to avoid singularities of the integrand, unless it is pinched by a triple coincidence of the singularities

$$h=\alpha+(\alpha^2-1)^{\frac{1}{2}},\quad h'=\beta+(\beta^2-1)^{\frac{1}{2}},\quad (hh')^2-2zhh'+1=0,$$

which gives

$$z=\alpha\beta+(\alpha^2-1)^{\frac{1}{2}}(\beta^2-1)^{\frac{1}{2}}. \qquad (A.7)$$

(A.6) thus gives the continuation of $f(z)$ required for the theorem. The singularities of $[(hh')^2-2zhh'+1]^{-\frac{1}{2}}$ are in general branch points at $hh'=z\pm(z^2-1)^{\frac{1}{2}}$ joined by a cut. This cut has usually to be deformed when continuing in z and gives no trouble unless we attempt to continue back to $z=\pm1$ on another sheet of the function, as the ends of the cut coincide to form a simple pole at these points. Thus we cannot exclude the possibility of singularities at $z=\pm1$ on other sheets of its Riemann surface, as well as those of the type arising from $h=\alpha-(\alpha^2-1)^{\frac{1}{2}}$.

[8] These are fully described in (4) and by J. C. Polkinghorne and G. R. Screaton, Nuovo cimento **15**, 289 (1960).

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Structure of the S Matrix in the Presence of a Bound State*

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It is shown that the phase factor associated with the "orthogonality phase shift" due to a bound state should be factored out of the S matrix. A crucial test of this statement is found in a study of the final state interaction of an inelastic process which ends in a channel involving the bound state. If we assume that the sum of the Born series for the S matrix gives a right answer after we separate the effect of the bound state in terms of the orthogonality phase shift, an agreement with Watson's result obtains only when the S matrix has the factored structure.

1. INTRODUCTION

Recently it has been stated by Nishijima,[1] Zimmermann,[2] and Haag[3] that there is no difference between a composite particle and an elementary particle as far as the theory of scattering is concerned. This is true to the extent that it is possible to have an initial or a final state in which the composite particle moves as a single entity at a distance from all other particles. However, it is not quite arbitrary to regard a particle as elementary or as composite, since there is some experimental indication even in scattering when two "elementary" particles form a stable "composite" particle. Thus, a positive scattering length, when an attractive force acts between two colliding "elementary" particles, suggests that there is a bound state, i.e., a "composite" particle formed of the two "elementary" particles, of not too large a binding energy.[4]

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[1] K. Nishijima, Progr. Theoret. Phys. (Kyoto) **17**, 765 (1957); Phys. Rev. **111**, 995 (1958).
[2] W. Zimmermann, Nuovo cimento **10**, 597 (1958).
[3] R. Haag, Phys. Rev. **112**, 669 (1958).
[4] See, for instance, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 68.

It was shown by the author[5] that the well-known theorem on the zero energy limit of the scattering phase shift is connected with the sign of the scattering length, and that this theorem is a direct consequence of the fact that the scattering states are orthogonal to a bound state. It was suggested in I that the formation of a bound state is a sudden change in the character of the system of the two colliding particles, and that the Born series for the S matrix, which is based on the idea of the adiabatic change of the system, does not apply at low energies if a bound state is actually present. Incidentally, the converse of the last statement was established by Davies.[6] The Born series does converge when there is no bound state when one deals with potential scattering by a nonsingular potential of finite range.

In the case of potential scattering studied in I, thorough consideration was given to the "orthogonality phase shift." This paper is concerned with the orthogonality phase shift in a channel of more complicated nature. The points in which we are particularly interested are summarized as follows.

(1) The phase factor concerned with the orthogonality phase shift is factored out of the S matrix. Actually we have

$$S=S_{\mathrm{orth}}{}^{\frac{1}{2}}S_{\mathrm{res}}S_{\mathrm{orth}}{}^{\frac{1}{2}}, \tag{1}$$

where S_{orth} is the part of the S matrix concerned exclusively with the orthogonality phase shift, and S_{res} is what we shall call the residual part of the S matrix. The Born series, which fails to apply in the straightforward calculation of S, is supposed to apply to the calculation of S_{res}.

(2) The final-state interaction introduces the factor $(\sin\delta/\delta)$ into any matrix element for inelastic scattering. Here δ denotes a suitably defined part of the scattering phase shift for the elestic scattering in the exit channel. Since the inelastic scattering is properly accounted for by S_{res}, it turns out that the δ which should be substituted into $(\sin\delta/\delta)$, is the residual phase shift δ_{res}, defined as the total phase shift minus the orthogonality phase shift:

$$\delta_{\mathrm{res}}=\delta_{\mathrm{tot}}-\delta_{\mathrm{orth}}. \tag{2}$$

The total phase shift δ_{tot} must be used in the discussion of the elastic scattering.

(3) δ_{res} vanishes at the threshold, while δ_{orth}, and so δ_{tot}, assumes there the value $n\pi$, where n is the number of bound states. There is therefore a crucial difference in the threshold behavior of the scattering matrix element, depending on whether Eq. (1) holds or not, since the factor $(\sin\delta_{\mathrm{tot}}/\delta_{\mathrm{tot}})$ would vanish at the threshold [Eq. (1) not valid], while $(\sin\delta_{\mathrm{res}}/\delta_{\mathrm{res}})$ does not vanish there [Eq. (1) valid]. It is found that our result for the effect of the final-state interaction agrees with that of Watson[7] in spite of the differences in approach.

(4) It is unlikely that electrodisintegration of a deuteron into the triplet S state is forbidden at threshold just because of the existence of a stable deuteron state. If however this process is forbidden, Eq. (1) is not valid so that an experimental investigation of the appropriate electrodisintegration cross section at threshold will test our theory.

Finally, let us comment on the form of Eq. (1). Once the S matrix is fixed as a whole, there are, of course, many other ways of decomposing or factoring it. The particular factorization shown in Eq. (1) keeps track of the order in which the mathematical operations are made and eliminates the effect of the formation of a bound state *before* the Born series is summed.[8]

2. THE STRUCTURE OF THE S MATRIX

Let us specialize the discussion, for the sake of definiteness, to the study of an inelastic scattering of the type

$$a+\alpha\rightarrow b+\beta, \tag{3}$$

where there is a strong attraction between the two particles b and β, so that there is a bound state $c\equiv[b\beta]$ composed of these two particles. We denote the entrance channel by (A) and the exit channel by (B). As discussed in I, it is convenient to introduce a set of independent creation-annihilation operators for the particle c. Accordingly we have to perform a unitary transformation to deal with the nonlocal potential which represents the effect of the bound state c. Other kinds of physical effects in the channel (B) will be computed by summing up the Born series in the transformed representation.

When all terms which are irrelevant for the study of the reaction (3) are ignored, the S matrix is given by (1), where

$$S_{\mathrm{orth}}{}^{\frac{1}{2}}=\exp[i\sum_{\mathbf{kk}'}{}' B_{\mathbf{k}}{}^*(\mathbf{k}|\delta_{\mathrm{orth}}|\mathbf{k}')B_{\mathbf{k}'}], \tag{4}$$

$$S_{\mathrm{res}}=\exp[2i\sum_{\mathbf{kk}'}{}' B_{\mathbf{k}}{}^*(\mathbf{k}|\delta_{\mathrm{res}}|\mathbf{k}')B_{\mathbf{k}'} +i\sum_{\mathbf{kk}'}{}' B_{\mathbf{k}}{}^*(\mathbf{k}|M|\mathbf{k}')A_{\mathbf{k}'} +i\sum_{\mathbf{kk}'}{}' A_{\mathbf{k}}{}^*(\mathbf{k}|M^\dagger|\mathbf{k}')B_{\mathbf{k}'}]. \tag{5}$$

Several conventions on notations are introduced in Eqs. (4)–(5). First, the product of the creation operators for the a and the α (for the b and the β) is abbreviated as A^* (B^*). The suffixes $\mathbf{k}$, $\mathbf{k}'$ attached to the creation-annihilation operators denote the center-of-mass-frame linear momenta and angular momenta of the pairs of

[5] S. Tani, Phys. Rev. **117**, 252 (1960); this paper is referred to as I in the following.

[6] H. Davies, Nuclear Phys. **14**, 465 (1960).

[7] K. M. Watson, Phys. Rev. **88**, 1163 (1952).

[8] Whenever one divides the interparticle interaction into two parts, one of which is taken into account exactly at the beginning, one observes that the S matrix assumes a form similar to Eq. (1). See, for instance, p. 217 in W. Brenig and R. Haag, Fortschr. Physik **7**, 183 (1959).

particles in their respective channels.[9] The eigenfunctions of angular momentum are absorbed in the definition of the matrix element. (See Appendix for details.) The prime attached to the summation symbol implies that the sum over k and k' runs only over values which are consistent with the conservation of energy.

The δ_{orth} is the orthogonality phase shift which is caused by the bound state c. The δ_{res} is the residual phase shift, which together with δ_{orth} is used in the discussion of the elastic scattering in the channel (B):

$$b+\beta \rightarrow b+\beta. \quad (6)$$

The quantity $(\mathbf{k}|M|\mathbf{k}')$ in S_{res} is the matrix element of the inelastic scattering (3) with the effect of the final-state interaction on the energy shell excluded. It is important to note that $(\mathbf{k}|M|\mathbf{k}')$ is porportional to k^L near threshold, when the orbital angular momentum concerned is L.[10] This follows simply from kinematical considerations in first Born approximation and the same threshold behavior obtains from higher order Born approximations. Consequently, this threshold behavior must be true in general if our conjecture about the existence of the well-defined Born series for S_{res} is valid. A similar argument shows that near threshold δ_{res} is proportional to k^{2L+1}. A straightforward Born series is, however, ruled out for δ_{tot} because the individual terms of this series vanish at threshold, so that their sum can never give a multiple of π. In fact, the threshold behavior of $\delta_{\text{tot}}=\delta_{\text{res}}+\delta_{\text{orth}}$ is given by:

$$(\text{const})k^{2L+1}+n\pi \cong n\pi.$$

3. THE FINAL-STATE INTERACTION

Let us study the matrix element of the inelastic scattering (3), starting with the S matrix as given by (1), (4), and (5). We first introduce a further assumption which simplifies the analysis without sacrificing our general goal. Thus we consider cases where the inelastic process (3), produced in lowest order by a "weak" interaction, is followed by a "strong" final state interaction (6).

Accordingly we retain only the terms linear in $(\mathbf{k}|M|\mathbf{k}')$ in the following calculation. The higher order terms in $(\mathbf{k}|M|\mathbf{k}')$ take into account the "shadow" in channel (B) caused by the inelastic transition into channel (A); in other words, the effect of these higher order terms is equivalent to the replacement of δ_{res} by a complex phase shift.

We have to expand S into power series in the A's and B's. Insofar as we consider only one pair of a and α, or b and β, it turns out that the following expression for the commutation relations is sufficient; even any difference of statistics does not matter. We can take

$$[A_{\mathbf{k}},A_{\mathbf{k}'}^*]=[B_{\mathbf{k}},B_{\mathbf{k}'}^*]=\delta(\mathbf{k}-\mathbf{k}'), \quad (7)$$

with all other commutators vanishing. With use of Eqs. (7), (4), (5) we expand Eq. (1) as

$$S=1+\sum_{\mathbf{k}\mathbf{k}'}{}' B_{\mathbf{k}}^*(\mathbf{k}|\exp[2i\delta_{\text{tot}}]-1|\mathbf{k}')B_{\mathbf{k}'}$$
$$+i\sum_{\mathbf{k}\mathbf{k}'}{}' B_{\mathbf{k}}^*(\mathbf{k}|\exp[i\delta_{\text{tot}}]\times(\sin\delta_{\text{res}}/\delta_{\text{res}})$$
$$\times M|\mathbf{k}')A_{\mathbf{k}'}+\cdots. \quad (8)$$

Equation (8) shows that the expected result is obtained for the elastic scattering; the total phase shift δ_{tot} appears here. Watson showed[7] that the inelastic scattering matrix element including the effect of the final state interaction is proportional to

$$\exp[i\delta](\sin\delta/k^{L+1}). \quad (9)$$

The other factors being independent of k; in Eq. (9) δ should be identified with our δ_{tot}. To compare Eq. (9) with our result in Eq. (8) we note first of all that the phase factor $\exp(i\delta)$ is identical in both equations. Secondly, $\sin\delta_{\text{tot}}=\sin(\delta_{\text{res}}+\delta_{\text{orth}})$ in Eq. (9) is almost equivalent to $\pm\sin\delta_{\text{res}}$; this is because δ_{orth} is nearly $n\pi$ around the threshold. Thirdly, as mentioned in the last section, δ_{res} and the $(\mathbf{k}|M|\mathbf{k}')$ vary as k^{2L+1} and k^L near threshold so that $(\mathbf{k}|(1/\delta_{\text{res}})\cdot M|\mathbf{k}')$ varies as $1/k^{L+1}$. Thus our result given by Eq. (8) is essentially equivalent to Watson's result in Eq. (9), although the derivations of the two results are quite different. (More details of the necessary kinematical considerations are given in the Appendix.)

4. DISCUSSION

Summarizing, we can say that, if our conjecture about the structure of the S matrix involving a bound state is valid, we get the same threshold behavior of inelastic scattering cross sections as Watson.[7] Watson based his investigation on the asymptotic form of the scattering state wave function in configuration space. In our approach, we worked in the momentum space representation exclusively, since it is useful at high energies and since we are interested in its consistent development in all connections. To the extent that we do not measure the orthogonality phase shift directly, its distinction from the other part of the total scattering phase shift might appear ambiguous. What is to be emphasized, however, is that one can restore the applicability of the Born series, if one follows our prescription.

It is interesting to test the predicted threshold behavior by an experiment. The most promising experiment seems to be the electrodisintegration of a deuteron

$$e+d \rightarrow e+p+n.$$

This reaction does not fall exactly into the category as schematized by Eq. (3), but since the electron does nothing but break up the deuteron by means of the impulse it provides, the final-state interaction can be

[9] Such a device is discussed by B. Lippmann and J. Schwinger; see p. 476 in B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[10] It is important to note that the contribution of the normalization of the angular momentum eigenfunction is included here. See the last paragraph of Appendix.

treated in the same way as in the last section. Experiments on the electrodisintegration of the deuteron were done by Friedman[11] at an electron energy of 175 Mev. But the energy resolution at such high energies, being comparable to the deuteron binding energy, unfortunately does not allow any definite conclusions to be drawn on the threshold behavior. It would, however, be interesting to extend these experiments to lower energies, since, as discussed by Jankus,[12] one can expect that at lower energies the contribution of the electron-proton Coulomb interaction, which produces $p+n$ in the triplet S state, is relatively enhanced.

If the predicted threshold behavior is confirmed, one will establish the inapplicability of the Born series for the inelastic scattering matrix element in a formulation in which the orthogonality phase shift is not factored out as in Eq. (1). The inapplicability of the Born series for the calculation of the elastic scattering phase shift was noted already at the end of Sec. 2.

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APPENDIX

In accordance with our convention for the notation for $\mathbf{k}$, the operators which appear in Eqs. (4) and (5) are given explicitly by

$$\sum_{\mathbf{k}\mathbf{k}'}{}' B_k^*(\mathbf{k}|\delta|\mathbf{k}')B_{\mathbf{k}'} = \int\int d^3k d^3k' \sum_{L,m} B_{kLm}^* B_{k'Lm} Y_{Lm}(\theta,\varphi) \times \frac{1}{kk'}\delta(k-k')\delta^{(L)}(k) Y_{Lm}(\theta',\varphi') \quad (\text{A.1})$$

$$\sum_{\mathbf{k}\mathbf{k}'}{}' B_{\mathbf{k}}^*(\mathbf{k}|M|\mathbf{k}')A_{\mathbf{k}'} = \int\int d^3k d^3k' \sum_{L,m} B_{kLm}^* Y_{Lm}(\theta,\varphi) \times \frac{1}{kk'}\delta(k'-C(k))M^{(L)}(k,k') Y_{Lm}(\theta',\varphi') A_{k'Lm}, \quad (\text{A.2})$$

θ and φ (θ' and φ') are the angular variables for $\mathbf{k}(\mathbf{k}')$. We assume that the particles are spinless and the interaction is spherical symmetric, for simplicity. The conservation of energy in the inelastic scattering is represented by

$$\frac{k^2}{2M_B}+Q=\frac{k'^2}{2M_A}, \quad (\text{A.3})$$

where M_A (M_B) is the reduced mass in the channel (A) ((B)). $C(k)$ in (A.2) is defined by

$$C(k)=[2M_AQ+(M_A/M_B)k^2]^{\frac{1}{2}}. \quad (\text{A.4})$$

In our notation the radial wave equation is treated exactly as a one-dimensional problem. The commutation relation in Eq. (7) reads as

$$[B_{kLm},B_{k'L'm'}^*]=\delta_{LL'}\delta_{mm'}\delta(k-k'). \quad (\text{A.5})$$

Obviously one gets

$$\exp[i\sum{}' B_{\mathbf{k}}^*(\mathbf{k}|\delta|\mathbf{k}')B_{\mathbf{k}'}] = 1+\int\int d^3kd^3k' \sum_{Lm} B_{kLm}^*B_{k'Lm}Y_{Lm}(\theta,\varphi)Y_{Lm}(\theta',\varphi') \times\delta(k-k')(1/kk')\{\exp[i\delta^{(L)}(k)]-1\}+\cdots \quad (\text{A.6})$$

and other relations as used to derive Eq. (8). Note that the extra factor $(1/kk')$ in (A.1) is canceled by the volume element in the integrations over the intermediate states.

Due to the fact that we are dealing with the one-dimensional problem (the free radial wave function for $L=0$ is $\sin kr$), a matrix element of the interaction in any channel is proportional to $(k)^{L+1}(k')^{L+1}$ when both k and k' are small. By multiplying the delta function to take into account the energy conservation, we get the k dependence of $\delta^{(L)}(k)$ and $M^{(L)}(k,k')$ in (A.1–2). For the elastic scattering, energy conservation is taken into account by multiplication with

$$2\pi\delta\left(\frac{k^2}{2M_B}-\frac{k'^2}{2M_B}\right)=2\pi\left(\frac{M_B}{k}\right)\delta(k-k'), \quad (\text{A.7})$$

and for the inelastic scattering, by

$$2\pi\delta\left(\frac{k^2}{2M_B}+Q-\frac{k'^2}{2M_A}\right)=2\pi\left(\frac{M_A}{k'}\right)\delta(C(k)-k'). \quad (\text{A.8})$$

Thus, we find near threshold $\delta^{(L)}(k)$ is proportional to K^{2L+1}, and $M^{(L)}(k,k')$ to K^{L+1}. ($k'\cong$const near threshold.) This situation for δ has been described in the text by the statement that δ_{res} is proportional to k^{2L+1}.

In order to get the threshold behavior of the scattering amplitude and compare it with the result of Watson, we have to take out the energy conservation factor (A.8) from the S-matrix element given by (8) in the text. We find that $(1/k)$ due to the normalization of the radial wave function is left. This reduces the k dependence of $M^{(L)}(k,k')$ to k^L; this situation has been described in the text by the statement that $(\mathbf{k}|M|\mathbf{k}')$ is proportional to k^L. [The extra factor $(1/k)$ for the elastic scattering can be shown to give rise to the well-known formula for the cross section: $\sigma=(4\pi/k^2)\times\sum_L(2L+1)\sin^2\delta^{(L)}(k)$.]

[11] J. I. Friedman, Phys. Rev. **116**, 1257 (1959).
[12] V. Z. Jankus, Phys. Rev. **102**, 1586 (1956).