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Volume-Controlled, Two-Carrier Currents in Solids: The Injected Plasma Case

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Double injection into semiconductors and insulators is studied under conditions where the injected electrons and holes are free (injected plasma), the current is volume-controlled, i.e., determined by distributed space charge, and the current is field-driven (diffusion negligible). The major results are, assuming a one-dimensional geometry and carrier lifetime independent of injection level, for extrinsic semiconductors, (i) an extended voltage region over which $J\propto V^2$ (J current density and V voltage), and (ii) *depression* of the current, at fixed voltage, in the square-law region through *increase* in the number of thermal *minority* carriers, $J\propto |n_T-p_T|$, with n_T, p_T the thermal-equilibrium densities of electrons and holes, respectively. This unusual behavior is shown to be a direct consequence of recombination kinetic requirements. For insulators, assuming trapping is negligible, $J\propto V^3$. A rigorous solution is obtained for the constant-lifetime problem, valid for both semiconductors and trap-free insulators. This solution furnishes a good approximation also for variable-lifetime cases, e.g., bimolecular recombination kinetics.

I. INTRODUCTION

DOUBLE injection into solids, that is, the simultaneous injection of electrons from a negative contact and holes from a positive contact, is a subject of both considerable complexity and diversity. In the theoretical studies of this subject over the past decade by far the greatest attention has been given to double injection into semiconductors, usually "short" *n-i-p* structures, in which the current is controlled by one or both contacts and the carrier density is determined by the solution of a diffusion equation.[1–4] More recently, double injection into solids has been theoretically studied under conditions where the currents are field-driven and volume controlled, i.e., limited by space charge which is distributed throughout the volume.[5,6] These latter studies have been confined to insulators in which there is a negligible density of free carriers in thermal equilibrium and, further, in which there is negligible trapping of injected carriers. Parmenter and Ruppel[5] have rigorously derived the current-voltage characteristic for two-carrier SCL (space-charge-limited) currents in a trap-free insulator assuming bimolecular recombination. Lampert[6] has extended their results, by an approximate analysis, to general recombination kinetics and arbitrary field-dependence of the mobilities, but still holding to insulators and to the assumption of a single lifetime for electrons and holes (the injected plasma case). The present paper extends this latter work to semiconductors, and also puts the theory on a more rigorous foundation.

Two outstanding features are revealed by the present analysis: (i) an extended voltage range, following the Ohm's law region, over which the current is proportional to the square of the voltage; (ii) *depression* of the current, at fixed voltage, in this square-law region through *increase* in the number of minority carriers in thermal equilibrium. The latter behavior is remarkable in that the addition of current carriers leads to a reduction in current at large injection levels. It will be seen in Secs. IV and V that this effect is a direct consequence of recombination kinetics.

Recent experiments[7] with germanium at this laboratory give the predicted square-law dependence of current on voltage following an Ohm's law region, and also give strong evidence for the predicted depression of the current by the minority carriers. A double-injection "breakdown" of iron-doped germanium at liquid nitrogen temperature was observed by Tyler[8] and interpreted by him in terms of volume-controlled currents, although a detailed theory has not, to date, been presented. Also the recently observed "oscillistor" phenomena[9] (current oscillations produced in a rod of semiconductor placed in a magnetic field) are known experimentally to involve field-driven, volume-controlled, double-injection currents.

The authors are confident that the systematic study, both experimental and theoretical, of two-carrier injection into solids will yield at least as great a wealth of useful information about localized imperfection states as has the study of one-carrier injection.[10]

II. ASSUMPTIONS

The assumptions on which our theory is based are as follows:

(i) The injected (excess) carriers are free, i.e., not bound in traps or recombination centers.

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[3] D. A. Kleinman, Bell System Tech. J. **35**, 685 (1956).

[4] V. I. Stafeev, J. Tech. Phys. (U.S.S.R.) **28**, 1631 (1958) [translation: Soviet Phys.—Tech. Phys. **3**, 1502 (1958)]; J. Solid State Phys. Acad. Sci. U.S.S.R. **1**, 841 and 848 (1959), (Translation: Soviet Phys.—Solid State **1**, 763 and 769 (1959)).

[5] R. H. Parmenter and W. Ruppel, J. Appl. Phys. **30**, 1548 (1959).

[6] M. A. Lampert, RCA Rev. **20**, 682 (1959).

[7] R. D. Larrabee, following paper [Phys. Rev. **121**, 37 (1961)].

[8] W. W. Tyler, Phys. Rev. **96**, 226 (1954).

[9] R. D. Larrabee and M. C. Steele, Bull. Am. Phys. Soc. **4**, 421 (1959) and J. Appl. Phys. (to be published).

[10] M. A. Lampert, A. Rose, and R. W. Smith, J. Phys. Chem. Solids **8**, 465 (1959).

For well-purified semiconductors, in the temperature region of extrinsic conductivity with the dominant impurity completely ionized, and at higher temperatures, the injected carrier densities, at injection levels of interest, will be large compared to recombination and trap state densities, and so will automatically be free. Further, changes in occupancy of the recombination centers under injection conditions are neglected. This is justified (see Appendix A) so long as the recombination state density is less than the *difference* between the thermal-equilibrium densities of majority and minority carriers. Again this will generally be the case for well-purified semiconductors in the current state of the art.

For insulators, even if highly purified, except at very high injection levels, quite generally at least one of the injected carriers, electrons or holes, will be largely trapped. Theory covering this case will be presented in a later publication.

(ii) The current is volume-controlled, i.e., the contacts impose no significant constraints on either the entering or exiting currents.

(iii) Diffusion currents are negligible.

These two assumptions are closely related to each other and therefore are discussed together. They mark a complete departure from the existing theory on double injection into semiconductors.[1–4] Assumption (iii), by definition, characterizes a "simplified" theory. In such a theory, assumption (ii) is precisely formulated in a pair of boundary conditions, namely that the electric field intensity vanishes at the cathode and anode. In practice such idealized contacts simply cannot be realized since, in the absence of diffusion, they imply infinite free-carrier densities at the contacts. Actually, where the electric field vanishes the current is necessarily a pure diffusion current. Nevertheless, so long as the diffusion currents are large only over narrow regions confined to the vicinity of the contacts, the "simplified" theory, with assumption (ii), will give an adequate description of the overall current flow, particularly of the current-voltage characteristic. Precisely the same considerations arise in the problem of a one-carrier SCL current injected via an "Ohmic" contact.[11]

In the case of double injection into a semiconductor with an *n-i-p* structure a condition for the applicability of the "simplified" theory is that the middle *i*-section be at least several diffusion lengths long. (See the discussion at the end of Sec. IV.) Such is the case in Larrabee's experiments.[7] Where the middle *i* section is less than a couple of diffusion lengths long, the assumptions (ii) and (iii) are no longer valid at practical voltages, the appropriate theory then being that already available in the literature.[1–4]

Although the *n-i-p* type of structure is indeed the most suitable, in the current state of the art, for studying double injection under highly controlled conditions, it is certainly not the only one available.[4,12] The problem of injecting contacts is one requiring considerably more experimental and theoretical study if greater understanding and control are to be realized for contacts other than those of the *p-n* junction type.

(iv) Low-field (field-independent) mobility conditions obtain.

This assumption is made to make the mathematical problem analytically tractable. Fortunately it will be realized in most, if not all, cases of practical interest. Where approximate arguments are employed it is, at least sometimes,[6] not necessary to restrict the discussion with this assumption. In the case of one-carrier SCL currents it has been shown that the analytical problem is tractable even with field-dependent mobilities, under fairly general conditions.[13] It remains to be seen whether such techniques can be applied to two-carrier problems.

III. PHYSICAL ARGUMENTS

In this section we derive some of the major results of the theory in a very simple manner, examining the average behavior of the injected carriers from the viewpoint of the underlying physical processes. The reasoning employed is a straightforward extension of that employed by Rose in his study of one-carrier SCL currents.[14]

Except where otherwise noted, the discussion throughout this paper refers to a one-dimensional, current-flow geometry and all formulas are expressed in mks units. In the following definitions of symbols, subscript n or p on a quantity indicates that the quantity refers to electrons or holes, respectively. J is the total current density; e is the electronic charge, P and N are the total number of injected, excess holes and electrons, respectively, per unit area, between cathode and anode; t_p is the "average" hole transit-time, $t_p=L^2/\mu_p V$, with L the cathode-anode spacing, μ_p the hole mobility, and V the applied voltage between anode and cathode, and similarly for t_n and μ_n; Q is the magnitude of charge, per unit area, of one sign between cathode and anode (see Appendix B); C is the geometric capacitance, per unit area, $C=\epsilon/L$ with ϵ the static dielectric constant; τ is the common, "average" lifetime for injected carriers; $t_{\Omega,n}$ is the so-called Ohmic (or dielectric) relaxation time for electrons, $t_{\Omega,n}=\epsilon/en_T\mu_n$ with n_T the thermal equilibrium density of electrons, and similarly for $t_{\Omega,p}$ and p_T.

In order to highlight the difference between one- and two-carrier injection, under field-driven conditions, we first review one-carrier injection (of holes) into a perfect insulator ($p_T=0$ and no trapping)[14]:

$$J=eP/t_p. \tag{1}$$

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[11] M. A. Lampert, Phys. Rev. **103**, 1648 (1956).

[12] R. W. Smith, Phys. Rev. **105**, 900 (1957).

[13] M. A. Lampert, J. Appl. Phys. **29**, 1082 (1958).

[14] A. Rose, Phys. Rev. **97**, 1538 (1955).

Charge and voltage are related in the usual manner,

$$Q = CV = (\epsilon/L)V. \tag{2}$$

It has been shown,[11] under very general conditions, that the distortion, by injected, distributed space charge, of the capacitance away from its geometric value is less than a factor of two. Correspondingly, an error of less than a factor of two is made by using the geometric value of the capacitance in Eq. (2). A similar situation is expected to obtain generally with two-carrier injection (see Appendix B).

Since all free carriers in the insulator are excess carriers,

$$Q = eP. \tag{3}$$

Combining Eqs. (1), (2) and (3),

$$J = CV/t_p = \epsilon\mu_p V^2/L^3. \tag{4}$$

The exact, analytical solution[15] for this problem differs from Eq. (4) only in that it is larger by the multiplicative factor 9/8.

The essential difference between one- and two-carrier injection is that in the latter case the injected carriers of one sign of charge can be largely neutralized through the accompanying injection of carriers of the opposite sign of charge. Thus, with double injection we are dealing basically with an injected plasma, so long as the injected carriers are mostly free [assumption (*i*)] of Sec. II). Nevertheless the small, residual space charge is of great importance in that it is precisely this space charge that limits the buildup of plasma density at any fixed voltage.

We consider specifically the case of double injection into an n-type semiconductor in the extrinsic region, $p_T \ll n_T$. Because of finite lifetime against recombination with electrons, the injected hole density necessarily decreases going from anode toward cathode. As pointed out above, the space charge associated with the injected holes can be largely "relaxed," i.e., neutralized, through the concomitant injection of electrons. However, since the motion of the electrons which "relax" the holes is in the opposite sense from the hole motion—ultimately, the "relaxing" electrons must come from the cathode, whereas the holes are injected at the anode—there will be a finite time delay, or relaxation time t_{rel}, associated with the neutralization process, as a result of which *complete* neutralization will clearly be impossible. So long as $t_{\mathrm{rel}} \ll \tau$ we would expect the ratio t_{rel}/τ to provide a measure of the incompleteness of the relaxation; i.e., of the residual space charge. Hence we write, for this case of double injection into an n-type semiconductor:

$$t_{\mathrm{rel}} \ll \tau: \quad Q = ePt_{\mathrm{rel}}/\tau. \tag{5}$$

Since $Q/e \ll P$, it follows automatically that $N \approx P$, i.e., that the double injection corresponds to an injected plasma (the free thermal electrons, of total number $n_T L$, per unit area, are neutralized by the positive donor ions).

We still use Eq. (1) for the *total* current, since we regard the electron motion as simply neutralizing the holes over their current path. This is, of course, valid only at high injection levels, $P > n_T L$. (For $P < n_T L$ we get simply Ohm's law.) Combining Eqs. (1), (2), and (5), we get

$$J = \frac{\tau CV}{t_{\mathrm{rel}} t_p} = \frac{\epsilon\tau\mu_p}{t_{\mathrm{rel}}} \frac{V^2}{L^3}. \tag{6}$$

In order to complete the picture we must identify t_{rel}. Two relaxation modes are available: (i) the "Ohmic" relaxation by the electrons distributed throughout the solid in thermal equilibrium; this has the characteristic relaxation time $t_{\Omega,n} = 5.5 \times 10^5 K/n_T\mu_n$ in practical units (K the dimensionless, relative dielectric constant, n_T in cm^{-3}, μ_n in cm^2/volt sec, and $t_{\Omega,n}$ in seconds), and (ii) the transit time t_n for electrons individually to traverse the solid after injection at the cathode. Whichever of $t_{\Omega,n}$, t_n is the shorter time will determine the actual mode of relaxation. Thus we have two different cases, for each of which the appropriate substitution for t_{rel} in Eq. (6) gives the desired current-voltage relationship:

$$t_{\Omega,n} < t_n, \tau; \quad t_{\mathrm{rel}} = t_{\Omega,n}: \quad J = e\tau\mu_n\mu_p n_T V^2/L^3, \tag{7'}$$

$$t_n < t_{\Omega,n}, \tau; \quad t_{\mathrm{rel}} = t_n: \quad J = \epsilon\tau\mu_n\mu_p V^3/L^5. \tag{8}$$

In both cases it is also necessary that $t_p < \tau$. If $t_p > \tau$, Ohm's law holds.

From the derivation, it is appropriate to refer to the square-law characteristic, Eq. (7′), as the "Ohmic relaxation regime." If $\tau < t_{\Omega,n}$ there will be no square-law portion in the current-voltage characteristic. At sufficiently high voltages where t_n, which varies as $1/V$, is less than $t_{\Omega,n}$, the carriers n_T, initially present thermally, no longer play the dominant role in determining the current. Therefore the solution, Eq. (8), is essentially the same as that previously obtained[6] for double injection into a perfect insulator ($n_T = p_T = 0$), and it is appropriate to refer to the cube-law characteristic, Eq. (8), as the "insulator regime." Actually there is missing from the right side of Eq. (8) a multiplicative, numerical factor ≈ 8. The missing factor is easily recovered by noting that in the insulator regime, as shown in Appendix A, there is almost complete symmetry between electrons and holes (unequal mobilities can *slightly* disturb this symmetry). Therefore, in concentrating on the injected holes in our derivation we have looked only at the right half of the solid, the anode half, and counted only half the current. Upon replacing J, V, and L in Eq. (8) by $J/2$, $V/2$, and $L/2$, respectively, the "correct" numerical factor appears.

Finally Eq. (7′) should be corrected for the presence of minority carriers p_T thermally generated in the

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[15] N. F. Mott and R. W. Gurney, *Electronic Processes in Ionic Crystals* (Oxford University Press, New York, 1940), p. 172.

n-type material. It is shown in the following Sec. IV, and more rigorously in Appendix A, that the corrected equation is,

$$J = e\tau\mu_n\mu_p(n_T - p_T)V^2/L^3. \tag{7}$$

Precise conditions for the validity of Eq. (7) are given in Appendix A. As shown there, the inequalities characterizing the different current-voltage regimes are slightly different than those given in Eqs. (7′) and (8).

It is seen in Sec. IV that the factor $n_T - p_T$ appears naturally in the analysis upon simple manipulation of particle conservation (recombination) equations—a matter explored further in Sec. V C. In the context of the physical arguments of the present section, Eq. (7) can also be interpreted as indicating that, in Ohmic relaxation, say by electrons, ***under double injection conditions***, the effective density of electrons available to "relax" the injected holes is $n_T - p_T$, and that the corresponding, effective Ohmic relaxation time is $t_{\Omega,n}' = \epsilon/e(n_T - p_T)\mu_n$. Replacing t_{rel} by $t_{\Omega,n}'$ in Eq. (6) then gives Eq. (7). However, we wish to stress that this behavior is very unusual and derives from recombination requirements. In the more usual charge-relaxation situation, e.g., if an excess charge were suddenly introduced into the injected plasma, the electrons and holes in the plasma would aid in the charge relaxation, i.e., their (mobility-weighted) sum, rather than difference, would be involved in the relaxation process.

Equation (7) would predict $J \to 0$ as $p_T \to n_T$, which is, of course, a spurious effect. Actually when p_T is very close to n_T the approximations characterizing the ohmic relaxation regime are no longer valid. In this case, with diffusion neglected, there is simply no voltage range over which $J \propto V^2$. The Ohm's law regime, valid at "low" voltages, is then followed directly by the insulator regime, $J \propto V^3$.

IV. APPROXIMATE ANALYTICAL ARGUMENTS

Additional insight is gained into the double-injection problem by somewhat more analytical, yet still simple, derivations. The complete equations, including diffusion, which define the problem are: the electron and hole current-flow equations,

$$\left.\begin{aligned} \frac{1}{e}J_n &= \mu_n(n+n_T)\mathcal{E} - D_n\frac{dn}{dx}, & \text{(9a)}\\ \frac{1}{e}J_p &= \mu_p(p+p_T)\mathcal{E} + D_p\frac{dp}{dx}, & \text{(9b)} \end{aligned}\right\} J = J_n + J_p;$$

the Poisson equation (neglecting changes in occupancy of localized, e.g., recombination, centers),

$$(\epsilon/e)(d\mathcal{E}/dx) = n - p, \tag{10}$$

and the particle conservation equations,

$$\left.\begin{aligned} -\frac{1}{e}\frac{dJ_n}{dx} &= r, & \text{(11a)}\\ \frac{1}{e}\frac{dJ_p}{dx} &= r, & \text{(11b)} \end{aligned}\right\} r = n/\tau_n = p/\tau_p.$$

Multiplying Eq. (11b) by $b = \mu_n/\mu_p$ and adding to the result Eq. (11a) gives, using Eqs. (9a) and (9b),

$$\frac{d}{dx}[(p-n)\mathcal{E}] - (n_T - p_T)\frac{d\mathcal{E}}{dx} + \frac{kT}{e}\frac{d^2}{dx^2}(n+p) = \frac{(b+1)r}{\mu_n} = \frac{(b+1)n}{\mu_n\tau_n} = \frac{(b+1)p}{\mu_n\tau_p}. \tag{12}$$

In the above equations, J_n is the electron current density; similarly for J_p; n and p are the injected, *excess* electrons and holes, respectively; $\mathcal{E}$ is the electric field intensity; D_n is the electron diffusion constant, $D_n = kT\mu_n/e$ with k = Boltzmann's constant and T the temperature in degrees Kelvin; similarly for D_p; x is the position coordinate; r is the recombination-rate density; τ_n is the electron lifetime, and similarly for τ_p. Other symbols have been previously defined in Sec. III.

Equation (12) is particularly useful for comparison of the different current-flow regimes. Of the three terms on the left-hand side of Eq. (12), previous theories[1–4] ignore the first two, take $n = p$ in the third term (hence $\tau_n = \tau_p = \tau$), and thereby obtain what we may call the "diffusion solution," with an effective diffusion length $L_{\mathrm{eff}} = (2D_n\tau/(b+1))^{\frac{1}{2}}$. Note that in obtaining this "diffusion solution" the field terms are neglected only in Eq. (12). The field terms are retained in Eqs. (9a) and (9b) and indeed, in references 1–4, these equations are used, in conjunction with the "diffusion solution" to Eq. (12), to obtain the electric field intensity. Interestingly enough, the "diffusion solution" yields a square-law regime at higher voltages, following the exponential current-voltage regime valid at low voltages.[3] This square law is unrelated to that characterizing the Ohmic-relaxation regime, Eq. (7).

The field-driven current regimes are determined neglecting the third (diffusion) term on the left-hand side of Eq. (12) as well as the diffusion terms in Eqs. (9a) and (9b). Now, it is obvious, and is verified in the exact solution of Appendix A, that until relatively high injection levels are reached, i.e., until $n \approx p > n_T, p_T$, the current-voltage characteristic is simply Ohm's law. In the non-Ohmic portions of the characteristic, i.e., at the high injection levels, it is permissible to neglect the thermal carrier densities n_T, p_T in the current expressions (9a) and (9b). The total current density can then be written, replacing $\mathcal{E}$ by the average field V/L,

$$J \simeq e(b+1)\mu_p\bar{n}V/L, \tag{13}$$

where $\bar{n}$ denotes the average value of n over the volume.

The Ohmic relaxation regime corresponds to keeping just the middle term on the left-hand side of Eq. (12):

$$-\mu_n(n_T-p_T)(d\mathcal{E}/dx)=(b+1)n/\tau_n. \tag{14}$$

A simple dimensional analysis of Eq. (14) gives the result immediately; namely replace $-d\mathcal{E}/dx$ by V/L^2 and n/τ_n by $\bar{n}/\tau$. This gives the useful relation,

$$(b+1)\bar{n}=\tau\mu_n(n_T-p_T)V/L^2, \tag{15}$$

which, substituted into Eq. (13), gives Eq. (7). (Note that, for the sake of definiteness, we are assuming n-type material, $n_T>p_T$.)

A somewhat more fastidious procedure for deriving Eq. (15) consists of replacing $d\mathcal{E}/dx$ in Eq. (14) by its value from Eq. (10), giving $e|n-p|=\epsilon(b+1)n/\mu_n\tau_n(n_T-p_T)$, substituting this into the relation, $|Q|=e\int_{\text{vol}}|n-p|dx\simeq\epsilon V/L$ (see Appendix B), and replacing $\int_{\text{vol}}ndx/\tau_n$ by $\bar{n}L/\tau$.

The insulator regime corresponds to keeping just the first term on the left-hand side of Eq. (12), and replacing (p-n) by its value in Eq. (10),

$$-\frac{\epsilon}{e}\mu_n\frac{d}{dx}\left(\mathcal{E}\frac{d\mathcal{E}}{dx}\right)=(b+1)\frac{n}{\tau_n}. \tag{16}$$

Again, a simple dimensional analysis, replacing $-(d/dx)[\mathcal{E}(d\mathcal{E}/dx)]$ by V^2/L^4 and n/τ_n by $\bar{n}/\tau$, gives the useful result,

$$(b+1)\bar{n}=\frac{\epsilon\tau\mu_n}{e}\frac{V^2}{L^4}, \tag{17}$$

which, substituted into Eq. (13), gives the final result, Eq. (8). More detailed examinations of the insulator regime are given elsewhere.[5,6]

The present treatment of double injection, in which we neglect the diffusion term and retain the field terms in the particle conservation equation (12), is applicable to semiconductor structures of the n-i-p type if the middle i section is sufficiently long compared to a diffusion length. The reason for this requirement is that the n-i (p-i) junction blocks the exit of holes (electrons), whereas our theory assumes that there are no constraints on currents at the contacts. Consequently at each junction there is needed an accommodation region in the i section over which a diffusion-dominated solution adjusts to our field-dominated solution. At the onset of the Ohmic relaxation regime the length of each of the accommodation regions is approximately one diffusion length.[16] At higher currents their length grows logarithmically with current.[16] In Larrabee's experiments,[7] at the highest currents the total length of the two accommodation regions is small enough compared to the specimen length that we expect at least a qualitative check of his results with our predictions, and indeed this check is provided.

V. RECOMBINATION KINETICS

It is clear from the entire discussion of Sec. III, and from the appearance of the lifetime τ in the final results, Eqs. (7) and (8), that recombination kinetics play a crucial role in double injection currents. We discuss in this section three aspects of recombination: the variation of lifetime with position, the variation of lifetime with injection level, and the connection of recombination with the "n_T-p_T effect," Eq. (7).

A. Variation of Lifetime with Position

We recall that the "physical" arguments of Sec. III and, to some extent, also the approximate mathematical arguments of Sec. IV, were based on an examination of the *average* properties of the injected plasma. The reason why the study of average quantities yields accurate results for the current-voltage characteristic is that, in truth, the various quantities of interest, such as carrier densities and electric field intensity, vary only weakly with position over most of the solid, as demonstrated by the "exact" solutions in Appendix A. The "exact" solutions, in turn, are derived making the approximation that either the electron or hole lifetime is constant with position. Now even where this approximation is poorest, namely with bimolecular recombination, the carrier lifetimes have only the same degree of positional variation as the free carrier densities. Since the calculated free-carrier densities have, indeed, only weak positional dependence, the approximation of constant lifetime is mathematically a self-consistent one, even in this "worst" case. This explains why the rigorous solution for the case of double injection into an insulator with bimolecular recombination[5] is reproduced with considerable accuracy by the constant-lifetime calculation of Appendix A.

Actually, in the more usual circumstance of recombination through localized centers the approximation of lifetime constant with position will be an extremely good one for either electrons or holes, so long as the recombination cross sections, $\bar{\sigma}_n$ and $\bar{\sigma}_p$ for electrons and holes, respectively, are very unequal in magnitude. For we may write $\tau_n^{-1}=\bar{v}\bar{\sigma}_np_R$, $\tau_p^{-1}=\bar{v}\bar{\sigma}_pn_R$, $p_R\simeq\bar{\sigma}_pN_R/(\bar{\sigma}_n+\bar{\sigma}_p)$, and $n_R\simeq\bar{\sigma}_nN_R/(\bar{\sigma}_n+\bar{\sigma}_p)$. Here $\bar{v}$ is the thermal velocity of the free carriers (taken equal for electrons and holes, for the sake of simplicity), n_R and p_R are the density of filled and empty recombination centers, respectively, and $n_R+p_R=N_R$, the total density of recombination centers (of a single class dominating recombination). If, for example, $\bar{\sigma}_n\gg\bar{\sigma}_p$, then $n_R\approx N_R$ irrespective of modest variations of carrier densities with position, and τ_p will be constant with position to a high degree of approximation. In this case, τ_n will be "constant" with position to a lower degree of approximation, namely it will vary with position to the same degree as the "almost-constant" ratio n/p. If $\bar{\sigma}_p\gg\bar{\sigma}_n$, the situation is reversed and τ_n is the lifetime most constant with position.

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[16] D. O. North (private communication).

B. Variation of Lifetime with Injection Level

The major results of the theory, Eqs. (7) and (8), express the current in terms of the average, common, carrier lifetime τ as well as the voltage V and other, fixed parameters. At injection levels where the common lifetime is independent of injected carrier density this is a suitable manner in which to express the current-voltage characteristic. However, where the lifetime varies with injection level it follows, e.g., from Eqs. (15) or (17), that the lifetime is also a function of voltage, $\tau=\tau(V)$, and the complete voltage dependence of the current is known only when $\tau(V)$ is specified. Double injection into insulators with bimolecular recombination,[5] described by Eq. (8), is a case in point. Here $\tau \propto 1/V$, so that finally $J \propto V^2$, not V^3.

The numerous possibilities for dependence of τ on injection level, hence on V, and the consequent effects on the current-voltage characteristic are a subject for further, detailed investigations. We confine our remarks here to two observations.

First it will be true quite generally that the dependence of lifetime on injection level in double injection experiments on a given material will be exactly the same as the dependence of lifetime on excitation level in bulk photoconductivity experiments on the same material using bandgap excitation. The lifetimes are, of course, to be compared at the same carrier densities. The only difference in the two types of experiments relates to the method of introducing excess carriers. The bulk photogeneration of excess carriers will, or can, be quite uniform, whereas in the electrical experiment excess carriers are introduced at contacts or electrodes. However, as we have already discussed, the injection of excess carriers from contacts nonetheless leads to a fairly uniform distribution of the carriers throughout the volume—hence the equivalence of the two types of experiments for lifetime studies. Photoconductive lifetime variations with excitation level are discussed in two recent review articles.[17,18]

The second remark pertains to the use of the functional form n/τ_n or p/τ_p to express recombination rates. The use of this form implies that the thermal re-emission rate from the recombination center to the corresponding carrier band is negligible, since net recombination is equal to the difference between the capture rate, as expressed above, and the re-emission rate. It is easily seen[17] that for a single, dominant recombination center, at the high injection levels of interest, $n \approx p \gtrsim n_T, p_T$, although the thermal re-emission rate can indeed be substantial to one band, it will then be negligible to the second band. For the recombination centers would have to be close, energetically, to the former band if re-emission is to be substantial, and hence they would be too far away, energetically, from the latter band for re-emission to this band to be significant. Since recombination for one sign of carrier, say holes, can be written properly in the form p/τ_p, we can use the "equation," $n/\tau_n=p/\tau_p$, as a formal means for defining the lifetime τ_n of the other sign of carrier. This mode of definition automatically sifts out the "capture-followed-by-thermal-re-emission" events from the recombination lifetime. These observations justify the use of the simple, "capture"-type functional form above in the theoretical analysis.

There is an additional question as to whether the net recombination rate should include the thermal-equilibrium density of carriers, i.e., whether the electron recombination rates, for example, should be written as n/τ_n or $(n+n_T)/\tau_n$. Actually this depends on the details of the recombination kinetics. For example, if the recombination centers are essentially full in thermodynamic equilibrium, and essentially empty under double injection (because $\bar{\sigma}_p \gg \bar{\sigma}_n$), then clearly the net electron recombination rate is given by $(n+n_T)/\tau_n$. This situation obviously involves a change in electron lifetime as a function of injection level. In the analytical formulation of the problem in Appendix A both possibilities for the functional form of the recombination are considered. At high injection levels, $n > n_T$, the results obviously will not depend strongly on the particular choice between the two functional forms.

C. Recombination and the Role of Minority Carriers in Ohmic Relaxation

The appearance of (n_T-p_T) in Eq. (7) is so unusual that further understanding of its origin is desirable. Some insight can be gained by concentrating on the requirements imposed by particle conservation, i.e., by the recombination Eqs. (11a) and (11b). For the sake of simplicity we consider the case of equal mobilities, $\mu_n=\mu_p=\mu$. From (9a) and (9b), neglecting diffusion, we can write: $\mathcal{E}=K/(n+n_T+p+p_T)$ with $K=J/e\mu = \text{constant}$. Putting this into Eq. (11a), we obtain

$$-\frac{d}{dx}\left[\frac{n+n_T}{2(n+n_T)-(n-p)-(n_T-p_T)}\right]=\frac{r}{K\mu}. \quad (18)$$

For the insulator case, $n_T=p_T=0$, we see that double-injection ($J\neq 0, r\neq 0$) requires a finite, if small, deviation from neutrality, $n\neq p$. This leads to the insulator regime characteristic, Eq. (8). Obviously for semiconductors with $n_T \approx p_T$ a similar situation obtains; particle conservation, i.e., a nonvanishing current divergence in Eq. (18), requires that $n\neq p$ for significant injection to occur, with a concomitant deviation from Ohm's law. Further, the insulator regime, Eq. (8), will describe the high-injection-level situations in this case.

However, for extrinsic semiconductors, $n_T \neq p_T$, it is clear that the term (n_T-p_T) in the denominator of the current term in Eq. (18) can also lead to nonvanishing current divergence, even with $n \simeq p$ in Eq. (18). Indeed

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[17] A. Rose, *Progress in Semiconductors* (Heywood and Company, Ltd., London, England, 1957), Vol. 2, p. 111.

[18] A. Many and R. Bray, *Progress in Semiconductors* (Heywood and Company, Ltd., London, England, 1958), Vol. 3.

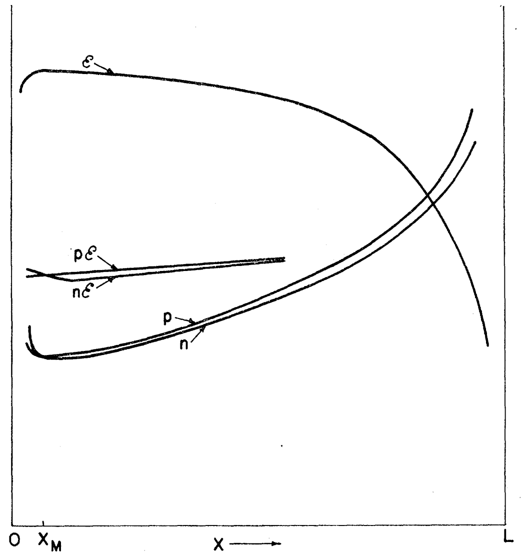

FIG. 1. Qualitative plots of several variables of interest for double injection into an extrinsic semiconductor in the Ohmic-relaxation regime. x_M is the position of the electric field maximum.

this is the case in the Ohmic relaxation regime, Eq. (7). Further, in an expansion of the left-hand side of Eq. (18) in powers of $(n_T-p_T)/2(n+n_T)$ (taking $p-n=0$), the lowest order, nonvanishing term is proportional to (n_T-p_T). Thus the recombination rate, and hence the current, are proportional to (n_T-p_T), as in Eq. (7). The same conclusions are reached if $\mu_n \neq \mu_p$, only the algebra is more involved.

It is also instructive to consider the contributions to recombination of the individual terms in the Eqs. (11a) and (11b), rewritten here, neglecting diffusion, as:

$$\left[-\frac{d}{dx}(n\mathcal{E})\right]_1+\left[-n_T\frac{d\mathcal{E}}{dx}\right]_2=\frac{r}{\mu_n}, \tag{19a}$$

$$\left[\frac{d}{dx}(p\mathcal{E})\right]_3+\left[p_T\frac{d\mathcal{E}}{dx}\right]_4=\frac{r}{\mu_p}. \tag{19b}$$

For insulators, $n_T=p_T=[\quad]_2=[\quad]_4=0$ and $[\quad]_1$ and $[\quad]_3$ are the sole contributors to recombination. In lowest order, $n\mathcal{E}=p\mathcal{E}=$ constant. In the next order, which gives the lowest order and dominant, contribution to recombination, $n\mathcal{E}$ and $p\mathcal{E}$ vary in opposite senses with position, namely decreasing and increasing monotonically, respectively, from cathode to anode. For extrinsic semiconductors, say $n_T \gg p_T$, in the Ohmic relaxation regime, the situation is different. Qualitative plots of $\mathcal{E}$, n, p, $n\mathcal{E}$ and $p\mathcal{E}$, determined from the solution in Appendix A, are given in Fig. 1. We are interested in the region between cathode and anode which begins at a distance on the order of x_M to the right of the neutrality plane at x_M, i.e., slightly to the right of the plane where $n\mathcal{E}$ has its minimum. This region comprises most of the volume of the solid, absorbs most of the applied voltage and hence determines the current-voltage characteristic. Our remarks are confined to this region. Again in lowest order $n\mathcal{E}=p\mathcal{E}=$ constant; in the next order $n\mathcal{E}\simeq p\mathcal{E}$ still, both increasing monotonically towards the anode in the region of interest, as shown in Appendix A. From Fig. 1 it is clear that the term $[\quad]_1$ in Eq. (19a) is now, in a mathematical sense, *opposing* recombination and the sole contribution to electron recombination is provided by the thermal-carrier term $[\quad]_2$. Since in lowest (nonvanishing) order, $[\quad]_1=-[\quad]_3$, it follows, adding Eqs. (19a) and (19b), and neglecting $[\quad]_4$, that $[\quad]_2\simeq(b+1)r/b\mu_p\simeq(b+1)[\quad]_3/b$. Including the small term $[\quad]_4$, it opposes recombination, and on adding Eqs. (19a) and (19b) $[\quad]_2$ is replaced by

$$[\quad]_2+[\quad]_4=-(n_T-p_T)d\mathcal{E}/dx.$$

The above considerations bring out the crucial role played by the thermally present carriers in the Ohmic relaxation regime. Although, to be sure, the separation of currents into those carried by thermal and injected carriers respectively is only a mathematical artifact, it is, as the above analysis shows, a useful one for understanding the final results.

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APPENDIX A. THE RIGOROUS SOLUTION

The current-flow, Poisson, and particle-conservation equations defining the problem are:

$$\frac{J}{e}=(n+n_T)\mu_n\mathcal{E}+(p+p_T)\mu_p\mathcal{E}=\text{constant}, \tag{A1}$$

$$\frac{\epsilon}{e}\frac{d\mathcal{E}}{dx}=n-p+z_R, \tag{A2}$$

$$-\mu_n\frac{d}{dx}[(n+n_T)\mathcal{E}]=r, \tag{A3a}$$

$$\mu_p\frac{d}{dx}[(p+p_T)\mathcal{E}]=r. \tag{A3b}$$

Equation (A1) is the sum of Eqs. (9a) and (9b) with diffusion neglected. Equation (A2) is the Poisson Eq. (10) with an additional term z_R allowing for a change in occupancy of the recombination centers. Note that z_R can be either positive or negative. Equations (A3a) and (A3b) are the same as Eqs. (11a) and (11b), respectively. The appropriate functional form for r depends on the detailed properties of the localized

TABLE I. Parameters dependent on the details of recombination kinetics.[a]

	r	r_T/n_T	α	β/Θ
a	n/τ_n	ν	$1-\Phi-\zeta+\Theta$	$b+\Phi+\zeta$
b	p/τ_p	π	$1-\Phi-\zeta-b\Theta$	$b+\Phi-b\zeta$
c	$(n+n_T)/\tau_n$	$\nu+1$	$1-\Phi-\zeta+\Theta$	$\Phi+\zeta-1$
d	$(p+p_T)/\tau_p$	$\pi+1$	$1-\Phi-\zeta-b\Theta$	$\Phi-b\zeta-1$

[a] The letters designating rows are used as identifying subscripts on the symbols designating columns; thus $r_a=n/\tau_n$, $r_{cT}/n_T=\nu+1$, etc.

defect states in the semiconductor. The four possibilities, discussed in Secs. VA and VB, are listed in the first column of Table I.

The particle-conservation equation most useful for the analytical discussion is that obtained by adding Eq. (A3b), multiplied by $b=\mu_n/\mu_p$, to Eq. (A3a):

$$\frac{d}{dx}[(p-n)\mathcal{E}]-(n_T-p_T)\frac{d\mathcal{E}}{dx}=\frac{(b+1)}{\mu_n}r. \tag{A4}$$

It is convenient to obtain the solution in terms of the dimensionless position, field, and potential variables of reference 11, Appendix A:

$$w=e^2n_T{}^2\mu_nx/\epsilon J,\quad u=en_T\mu_n\mathcal{E}/J,\quad v=e^3n_T{}^3\mu_n{}^2V/\epsilon J^2. \tag{A5}$$

Here V is the potential at position x, $V=V(x)$, and the applied potential is $V_L=V(L)$. (In the main body of this paper, V_L is written simply as V, since the potential distribution is not discussed there.) It is convenient, in the definitions (A5), to use the majority-carrier thermal density; thus the choice n_T would refer to an n-type or intrinsic semiconductor.

Further dimensionless variables and constants are:

$$\begin{gathered}\nu=n/n_T,\quad \pi=p/n_T,\quad \zeta=z_R/n_T,\\ \Theta=t_{\Omega,n}/\tau,\quad \Phi=p_T/n_T,\end{gathered} \tag{A6}$$

with $t_{\Omega,n}=\epsilon/en_T\mu_n$; τ is the appropriate, constant lifetime, τ_n or τ_p, appearing in r in the first column of Table I, and $\Phi\leq 1$.

Equations (A1), (A2), and (A4) become, respectively,

$$1/u=\nu+1+(1/b)(\pi+\Phi), \tag{A7}$$

$$du/dw=\nu-\pi+\zeta, \tag{A8}$$

$$\frac{d}{dw}[(\pi-\nu)u]-(1-\Phi)\frac{du}{dw}=\Theta(b+1)\frac{r_T}{n_T}. \tag{A9}$$

Equations (A7), (A8), and (A9), with r_T/n_T taken from Table I, define our problem, subject to the boundary conditions $\mathcal{E}=0$ at $x=0$ (the cathode) and at $x=L$ (the anode), i.e.:

$$u=0\ \text{at}\ w=0\ \text{and at}\ w=w_L=e^2n_T{}^2\mu_nL/\epsilon J. \tag{A10}$$

From Eqs. (A7) and (A8), ν and π are given by

$$\nu=\frac{1}{b+1}\left\{\frac{b}{u}+\frac{du}{dw}-(b+\Phi+\zeta)\right\}, \tag{A11a}$$

$$\pi=\frac{1}{b+1}\left\{\frac{b}{u}-b\frac{du}{dw}-(b+\Phi-b\zeta)\right\}. \tag{A11b}$$

Substitution of Eqs. (A11a) and (A11b) into Eq. (A9) gives finally:

$$u\frac{d}{dw}\left(u\frac{du}{dw}\right)+\alpha u\frac{du}{dw}-\beta u+\gamma=0, \tag{A12}$$

where $\gamma=\Theta b$ and the appropriate expressions for α and β are given in Table I. Note that we have taken ζ to be a constant (positive *or* negative), independent of position; this will be true generally to a high degree of approximation for constant-lifetime situations such as discussed here.

Equation (A12) is reduced to a differential equation which is integrable by inspection by the substitution:

$$u\frac{d}{dw}=\frac{d}{dy}\quad\text{or}\quad u=\frac{dw}{dy}, \tag{A13}$$

namely:

$$\frac{d^2u}{dy^2}+\alpha\frac{du}{dy}-\beta u+\gamma=0. \tag{A14}$$

The solution to this equation is written as

$$u=B'\exp A_1y+C'\exp A_2y+\gamma/\beta, \tag{A15}$$

where B' and C' are arbitrary constants, to be determined, and

$$A_1=-\tfrac{1}{2}\alpha+\tfrac{1}{2}(\alpha^2+4\beta)^{\frac{1}{2}},\quad A_2=-\tfrac{1}{2}\alpha-\tfrac{1}{2}(\alpha^2+4\beta)^{\frac{1}{2}}. \tag{A16}$$

In writing the solution in this form we are assuming that $\alpha^2+4\beta>0$. From Table I it appears that there may be some unusual situations where this inequality does not hold. However it is not our intention to present here an exhaustive treatment of all possible cases.

From this point on we confine the discussion to recombination possibilities a and b in Table I. At high injection levels, $n, p>n_T, p_T$ cases c and d are substantially the same as a and b, respectively, as already pointed out in Sec. VB. Further we assume that both ζ and Θ are sufficiently small to be neglected in α and β/Θ. Thus,

$$\begin{gathered}\alpha=\alpha_a=\alpha_b=1-\Phi>0;\\ \beta=\beta_a=\beta_b=\Theta(b+\Phi);\quad \gamma=\Theta b.\end{gathered} \tag{A17}$$

From (A16) it follows that

$$A_1>0,\quad A_2<-\alpha<0,\quad -A_2>A_1. \tag{A18}$$

From the manner in which the variable y is introduced, (A13), it is clear that only differences in y, and not the absolute values of y, are significant in the solution.

Therefore, to simplify the algebra we set

$$y=0 \text{ at } w=0;\text{ also let } y=y_L \text{ at } x=L,\ w=w_L. \qquad \text{(A19)}$$

Further simplification results from the following changes of variables and parameters:

$$\bar{y}=y/y_L,\quad \bar{u}=(\beta/\gamma)u,\quad \bar{w}=(\beta/\gamma y_L)w,$$
$$\bar{v}=(\beta^2/\gamma^2 y_L)v,\quad \bar{A}_1=A_1y_L,\quad \bar{A}_2=A_2y_L. \qquad \text{(A20)}$$

Equation (A15) can now be rewritten as

$$\bar{u}=B\exp\bar{A}_1\bar{y}+C\exp\bar{A}_2\bar{y}+1, \qquad \text{(A21)}$$

with B and C constants to be determined.

Integration of Eq. (A13) gives

$$\bar{w}=\frac{B}{\bar{A}_1}\exp\bar{A}_1\bar{y}+\frac{C}{\bar{A}_2}\exp\bar{A}_2\bar{y}+\bar{y}+D, \qquad \text{(A22)}$$

with D a constant of integration.

The boundary conditions (A10), with subsidiary condition (A19), are re-written as follows:

$$\text{At } \bar{y}=0,\ \bar{u}=0 \text{ and } \bar{w}=0;$$

$$\text{at } \bar{y}=1,\ \bar{u}=0 \text{ and } \bar{w}=\bar{w}_L=\frac{\beta}{\gamma y_L}w_L=\frac{\beta e^2 n_T^2\mu_n L}{\gamma y_L\epsilon J}. \qquad \text{(A23)}$$

Applying these boundary conditions to Eqs. (A21) and (A22) yields the following four equations for the five unknowns B, C, D, w_L, y_L:

$$0=B+C+1, \qquad \text{(A24a)}$$

$$0=(B/\bar{A}_1)+(C/\bar{A}_2)+D, \qquad \text{(A24b)}$$

$$0=B\exp\bar{A}_1+C\exp\bar{A}_2+1, \qquad \text{(A24c)}$$

$$\bar{w}_L=(B/\bar{A}_1)\exp\bar{A}_1+(C/\bar{A}_2)\exp\bar{A}_2+1+D. \qquad \text{(A24d)}$$

Elimination of B, C, and D yields the following *characteristic equation:*

$$\bar{w}_L-1=\left(\frac{1}{\bar{A}_1}-\frac{1}{\bar{A}_2}\right)\frac{(\exp\bar{A}_1-1)(\exp\bar{A}_2-1)}{\exp\bar{A}_1-\exp\bar{A}_2}. \qquad \text{(A25a)}$$

This is more clearly seen as an equation relating the unknowns w_L and y_L by re-writing it as

$$\frac{\beta}{\gamma}w_L-y_L=\left(\frac{1}{A_1}-\frac{1}{A_2}\right)\frac{(\exp A_1y_L-1)(\exp A_2y_L-1)}{\exp A_1y_L-\exp A_2y_L}, \qquad \text{(A25b)}$$

with A_1, A_2 constants given by (A16).

The final equation determining the solution to the problem is:

$$V_L=\int_0^L \mathcal{E}dx,\ \text{or}\ v_L=\int_0^{y_L}u^2dy,\ \text{or}\ \bar{v}_L=\int_0^1\bar{u}^2d\bar{y}. \qquad \text{(A26)}$$

Equations (A24) yield the following useful relations:

$$B=\frac{\exp\bar{A}_2-1}{\exp\bar{A}_1-\exp\bar{A}_2},\quad C=-\frac{\exp\bar{A}_1-1}{\exp\bar{A}_1-\exp\bar{A}_2}, \qquad \text{(A27)}$$

with D then given by Eq. (A24b).

Carrying out the integration in Eq. (A26), and using Eq. (A27), we obtain

$$\bar{v}_L=\tfrac{1}{2}(\exp\bar{A}_1-\exp\bar{A}_2)^{-2}\left[\frac{1}{\bar{A}_1}(\exp\bar{A}_2-1)^2(\exp2\bar{A}_1-1)\right.$$
$$+\frac{1}{\bar{A}_2}(\exp\bar{A}_1-1)^2(\exp2\bar{A}_2-1)$$
$$-\frac{4}{\bar{A}_1+\bar{A}_2}(\exp\bar{A}_1-1)(\exp\bar{A}_2-1)$$
$$\left.\times(\exp(\bar{A}_1+\bar{A}_2)-1)\right]+2\bar{w}_L-1. \qquad \text{(A28)}$$

The various regions of the solution are determined by the relative magnitudes of $\bar{A}_1$ and $-\bar{A}_2$ with respect to unity. These regions are:

(i) $\bar{A}_1\gg1$, hence $-\bar{A}_2\gg1$:
Ohm's-law regime, Eq. (A30b),

(ii) $\bar{A}_1\ll1$, $-\bar{A}_2\gg1$:
Ohmic-relaxation regime, Eq. (A33b),

(iii) $-\bar{A}_2\ll1$, hence $\bar{A}_1\ll1$:
Insulator regime, Eq. (A38b).

Where $\bar{A}_1$ or $-\bar{A}_2$ passes through unity as a function of applied voltage, we obtain a transition from one regime to another. In such transition regions the current-voltage characteristic cannot be expressed in a simple, analytical form.

We consider the different regions separately.

Ohms's-law regime: $\bar{A}_1\gg1$, hence $-\bar{A}_2\gg1$. $\text{Exp}\bar{A}_2\ll1$ is neglected everywhere. Equations (A25a) and (A28) become respectively, using (A20):

$$\bar{w}_L\simeq1-\left(\frac{1}{\bar{A}_1}-\frac{1}{\bar{A}_2}\right)\simeq1\rightarrow w_L\simeq\frac{\gamma}{\beta}y_L, \qquad \text{(A29a)}$$

$$\bar{v}_L\simeq2\bar{w}_L-1+\frac{1}{2}\left(\frac{1}{\bar{A}_1}-\frac{1}{\bar{A}_2}\right)\simeq1\rightarrow$$
$$v_L\simeq\left(\frac{\gamma}{\beta}\right)^2y_L. \qquad \text{(A29b)}$$

Dividing Eq. (A29a) by Eq. (A29b), we obtain, using (A5) and (A17):

$$\frac{w_L}{v_L}=\frac{JL}{en_T\mu_nV_L}\simeq\frac{\beta}{\gamma}=\left(1+\frac{\Phi}{b}\right), \qquad \text{(A30a)}$$

or

$$J\simeq e(n_T\mu_n+p_T\mu_p)V_L/L, \qquad \text{(A30b)}$$

which is just Ohm's law.

Ohmic-relaxation regime: $\bar{A}_1 \ll 1$, $-\bar{A}_2 \gg 1$. These conditions on $\bar{A}_1$ and $\bar{A}_2$ require that $\beta \ll \alpha$, hence $\gamma \ll \alpha$, in (A17), or from (A17) that $\Theta = t_{\Omega,n}/\tau \ll (1-\Phi)/(b+\Phi)$. Equation (A16) can then be written:

$$\beta \ll \alpha: \quad A_1 \simeq \beta/\alpha, \quad A_2 \simeq -\alpha, \quad A_1 \ll -A_2. \qquad \text{(A31)}$$

Again $\exp\bar{A}_2$ is neglected everywhere. Equations (A25a), (A27), (A21), and (A26) become, respectively, using (A20) and (A31):

$$\bar{w}_L \simeq \bar{A}_1\left(\frac{1}{2} + \frac{1}{\bar{A}_2}\right) \simeq \tfrac{1}{2}\bar{A}_1 \rightarrow$$

$$w_L \simeq \frac{1}{2}\frac{\gamma}{\beta} A_1 y_L^2 \simeq \frac{1}{2}\frac{\gamma}{\alpha} y_L^2, \qquad \text{(A32a)}$$

$$B \simeq -\exp -\bar{A}_1 \simeq -1; \quad C = -1 - B \simeq 0, \qquad \text{(A32b)}$$

$$\bar{u} \simeq 1 - \exp -\bar{A}_1(1-\bar{y}) \simeq \bar{A}_1(1-\bar{y}), \qquad \text{(A32c)}$$

$$\bar{v}_L \simeq \int_0^1 \bar{A}_1^2 (1-\bar{y})^2 d\bar{y} = \tfrac{1}{3}\bar{A}_1^2 \rightarrow$$

$$v_L \simeq \frac{1}{3}\left(\frac{\gamma}{\beta}A_1\right)^2 y_L^3 \simeq \frac{1}{3}\left(\frac{\gamma}{\alpha}\right)^2 y_L^3. \qquad \text{(A32d)}$$

Dividing the cube of Eq. (A32a) by the square of Eq. (A32d), we obtain, using (A5) and (A17):

$$\frac{w_L^3}{v_L^2} = \frac{L^3 J}{\epsilon \mu_n V_L^2} = \frac{9}{8}\frac{\alpha}{\gamma} = \frac{9}{8}\frac{1-\Phi}{\Theta b} = \frac{9}{8}\frac{e n_T \mu_p \tau (1-\Phi)}{\epsilon}, \qquad \text{(A33a)}$$

or

$$J \simeq (9/8) e \tau \mu_n \mu_p (n_T - p_T) V_L^2 / L^3. \qquad \text{(A33b)}$$

The transition from the Ohm's-law regime to the Ohmic-relaxation regime occurs approximately at the voltage $V_{1\rightarrow 2}$ where the curves (A30b) and (A33b) intersect, namely where

$$t_{p,1\rightarrow 2} = \frac{L^2}{\mu_p V_{1\rightarrow 2}} \simeq \tau \frac{1-\Phi}{1+(\Phi/b)},$$

or $\qquad \text{(A34)}$

$$V_{1\rightarrow 2} \simeq V_{\tau,p} \frac{1+(\Phi/b)}{1-\Phi}, \quad \text{with} \quad V_{\tau,p} = \frac{L^2}{\mu_p \tau}.$$

The inherent asymmetry of the Ohmic-relaxation regime is brought out by calculation of the position x_M where the electric field intensity is a maximum: $d\mathcal{E}/dx = 0$ at x_M and $\mathcal{E}(x_M)$ is a maximum, i.e., $d\bar{u}/d\bar{w} = 0$ at $\bar{w}_M$ corresponding to x_M, and $\bar{u}(\bar{w}_M)$ is a maximum.

We first find $\bar{y}_M$ at which $\bar{u}$ is a maximum: $d\bar{u}/d\bar{y} = 0$ at $\bar{y} = \bar{y}_M$. From (A21), $B\bar{A}_1 \exp\bar{A}_1\bar{y}_M + C\bar{A}_2 \exp\bar{A}_2\bar{y}_M = 0$; using (A27) and (A31) this gives $\bar{y}_M \simeq \ln|\bar{A}_2|/|\bar{A}_2| \simeq \ln\alpha y_L/\alpha y_L$. For voltages V_L well below the transition voltage $V_{2\rightarrow 3}$ from the square-law to the cube-law current-voltage dependences, $\bar{y}_M \ll 1$ since $\alpha y_L \gg 1$, namely $\alpha y_L = 3\alpha^2 v_L / 2\gamma w_L = 4(1-\Phi)V_{\Omega,n}/3V_L \approx 8V_{2\rightarrow 3}/V_L$, from (A32a), (A32d), (A33b), and (A39). It remains to show that $\bar{y}_M \ll 1$ implies $\bar{w}_M/\bar{w}_L = x_M/L \ll 1$. From (A22), (A24b), and (A32b) it follows that

$$\bar{w} \simeq \bar{A}_1 \bar{y} - \tfrac{1}{2}\bar{A}_1 \bar{y}^2. \qquad \text{(A35)}$$

Note that (A35) checks (A32a), taking $\bar{y} = 1$. Thus $\bar{w}_M \simeq \bar{A}_1 \bar{y}_M$ and $\bar{w}_M/\bar{w}_L \simeq 2\bar{y}_M \ll 1$, as asserted above.

We next show that the density-field product $n\mathcal{E}$ or νu (i) is constant with position to a good approximation, and (ii) has a slight monotonic increase from somewhat beyond the position x_M of the field maximum to the anode, $x = L$. From (A11a) (taking $\zeta = 0$), (A13), and (A20), $(b+1)\nu u = b + \gamma\beta^{-1}[y_L^{-1} d\bar{u}/d\bar{y} - (b+\Phi)\bar{u}]$. From (A32c) and (A30a) this gives $\nu u = \text{constant} - \bar{A}_1 b(b+1)^{-1}(1-\bar{y})$ where the constant is approximately $b/(b+1)$. This approximation is inadequate (higher order terms must be included) in the immediate vicinity of position x_M, roughly within a distance of order x_M. Beyond this distance, up to the anode, it is valid. Since $\bar{A}_1 \ll 1$ and $\bar{y}$ increases monotonically from 0 to 1 between the cathode and anode, both assertions (i) and (ii) above are proved. These same assertions are, of course, likewise valid for the product $p\mathcal{E}$ or πu.

Finally we note that the free-carrier densities do not vary strongly with position over most of the solid. Denoting $n(x_M)$ by n_M, $n_M/n(x) = \nu_M/\nu \simeq \bar{u}/\bar{u}_M \simeq 1 - \bar{y}$ from (A32c). From (A35), $x/L = \bar{w}/\bar{w}_L \simeq 2\bar{y} - \bar{y}^2$. Taking $\bar{y} = \frac{1}{2}$, $n(x)/n_M \simeq 2$ where $x/L = \frac{3}{4}$; i.e., n varies by less than a factor of 2 over 75% of the solid, or, taking $\bar{y} = \frac{2}{3}$, by less than a factor of 3 over approximately 90% of the solid.

Insulator regime: $-\bar{A}_2 \ll 1$, hence $\bar{A}_1 \ll 1$. Expanding $\exp\bar{A}_1$ and $\exp\bar{A}_2$, (A25a) gives, in lowest order:

$$\bar{w}_L \simeq -\tfrac{1}{12}\bar{A}_1\bar{A}_2 \rightarrow w_L = -\tfrac{1}{12}(\gamma/\beta)A_1 A_2 y_L^3. \qquad \text{(A36)}$$

From (A26) and (A21), $\bar{v}_L = \Sigma_1^2/3 + \Sigma_1\Sigma_2/4 + (\Sigma_2^2/4 + \Sigma_1\Sigma_3/3)/5 + \cdots$ with $\Sigma_n = B\bar{A}_1^n + C\bar{A}_2^n$, $n = 1, 2, \cdots$. From (A27), to the lowest order, $\Sigma_1 \simeq -\bar{A}_1\bar{A}_2/2 = -\Sigma_2/2$ and Σ_3, Σ_4, etc. are negligible. Thus,

$$\bar{v}_L \simeq \frac{1}{120}\bar{A}_1^2\bar{A}_2^2 \rightarrow v_L = \frac{1}{120}\left(\frac{\gamma}{\beta}A_1 A_2\right)^2 y_L^5. \qquad \text{(A37)}$$

Dividing the fifth power of Eq. (A36) by the cube of Eq. (A37) we obtain, using (A5), (A17), and (A16):

$$\frac{w_L^5}{v_L^3} = \frac{e n_T L^5 J}{\epsilon^2 \mu_n V_L^3} = -\frac{125}{18}\frac{\beta}{\gamma A_1 A_2} = \frac{125}{18\gamma}, \qquad \text{(A38a)}$$

or

$$J = (125/18)\epsilon\tau\mu_n\mu_p V_L^3/L^5. \qquad \text{(A38b)}$$

The transition from the Ohmic-relaxation regime to the insulator regime occurs approximately at the voltage $V_{2\rightarrow 3}$ where the curves (A33b) and (A38b)

TABLE II. Domains of validity for the several current-voltage regimes (double injection into an n-type semiconductor: $n_T > p_T$ or $\Phi < 1$).[a]

Regime	Current-voltage characteristic	Domain of hole transit-time t_p	Domain of applied voltage V_L
Ohm's law	Eq. (A30b)	$t_p > \tau \dfrac{1-\Phi}{1+(\Phi/b)}$	$V_L < V_{\tau,p} \dfrac{1+(\Phi/b)}{1-\Phi}$
Ohmic relaxation	Eq. (A33b)	$\tau \dfrac{1-\Phi}{1+(\Phi/b)} > t_p > \dfrac{500}{81} \dfrac{b}{1-\Phi} t_{\Omega,n}$	$V_{\tau,p} \dfrac{1+(\Phi/b)}{1-\Phi} < V_L < \dfrac{81}{500}(1-\Phi) V_{\Omega,n}$
Insulator	Eq. (A38b)	$\dfrac{500}{81} \dfrac{b}{1-\Phi} t_{\Omega,n} > t_p$	$\dfrac{81}{500}(1-\Phi) V_{\Omega,n} < V_L$

[a] $V_{\tau,p} = L^2/\mu_p\tau$; $V_{\Omega,n} = L^2/\mu_n t_{\Omega,n}$. For $1-\Phi \lesssim \{(500/81)(b+1)t_{\Omega,n}/\tau\}^{\frac{1}{2}}$, the Ohmic-relaxation regime no longer exists.

intersect, namely where

$$t_{n,2\to3} = \frac{L^2}{\mu_n V_{2\to3}} = \frac{500}{81} \frac{1}{1-\Phi} t_{\Omega,n}$$

or

$$V_{2\to3} \simeq \frac{81}{500}(1-\Phi) V_{\Omega,n}, \quad \text{with} \quad V_{\Omega,n} = \frac{L^2}{\mu_n t_{\Omega,n}}. \tag{A39}$$

Using (A34) and (A39), the conditions characterizing the various current-voltage regimes can be specified as in Table II. Setting $V_{1\to2} = V_{2\to3}$ it follows that for Φ such that $1-\Phi < \{500(b+1)t_{\Omega,n}/81\tau\}^{\frac{1}{2}}$ there will be no Ohmic-relaxation regime.

The basic symmetry of the insulator regime is brought out by calculation of the position x_M where the electric field intensity is a maximum, i.e., where $d\mathcal{E}/dx = 0$ or $d\bar{u}/d\bar{w} = 0$ or, from (A13), $d\bar{u}/d\bar{y} = 0$. From (A21), $d\bar{u}/d\bar{y} = B\bar{A}_1 \exp\bar{A}_1\bar{y} + C\bar{A}_2 \exp\bar{A}_2\bar{y}$. Expanding the exponentials, $d\bar{u}/d\bar{y} = 0$ at $\bar{y}_M$ given by $\Sigma_1 + \Sigma_2\bar{y}_M = 0$ or $\bar{y}_M = -\Sigma_1/\Sigma_2 = \frac{1}{2}$, independent of the mobility ratio to this lowest order. On the other hand, expanding the exponentials in (A22) and using (A24a) and (A24b), $\bar{w} \simeq \Sigma_1\bar{y}^2/2 + \Sigma_2\bar{y}^3/6 = -\Sigma_2(\bar{y}^2/4 - \bar{y}^3/6)$. Taking $\bar{y} = 1$ this gives $\bar{w}_L = -\Sigma_2/12$, checking (A36); taking $\bar{y} = \bar{y}_M = \frac{1}{2}$ the corresponding $\bar{w}_M$ is $-\Sigma_2/24 = \bar{w}_L/2$. Thus the field maximum is located, to lowest order, exactly at the center of the solid.

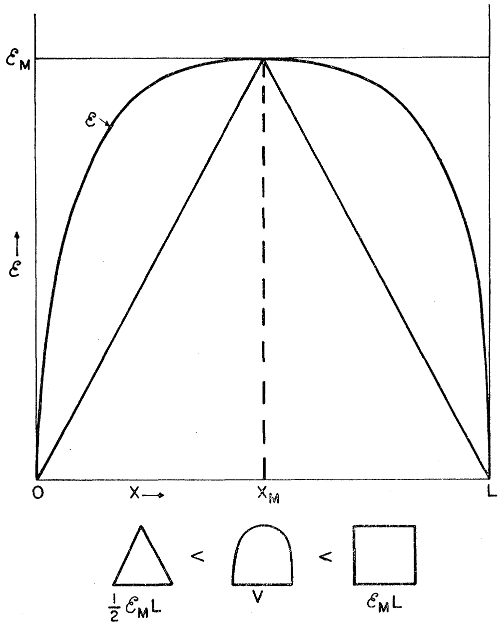

FIG. 2. An inequality derived from the convexity of the electric field distribution. The convexity is a sufficient, though not necessary, condition that the triangle lie inside the area bounded by the $\mathcal{E}$ curve and the x axis. The inequality of area yields directly the inequality: $V/L < \mathcal{E}_M < 2V/L$.

APPENDIX B. THE RELATION $Q = CV$

By integration of the Poisson equation (10), it follows that there can be no *net* charge between the cathode and anode with $\mathcal{E} = 0$ at both electrodes. Letting the electric field distribution be represented by a curve such as shown in Fig. 2, with a maximum $\mathcal{E}_M$ at position x_M (the particular curve might be appropriate for the insulator regime), the region between $x = 0$ and $x = x_M$ is one in which there is everywhere excess negative charge, of total amount $|Q_-|$, and the region between $x = x_M$ and $x = L$ correspondingly one of excess positive charge, of total amount $Q_+ = |Q_-|$. The quantity Q in the relation $Q = CV$ is defined by

$$Q = Q_+ = |Q_-| = e\int_0^{x_M} (n-p)dx = \epsilon\mathcal{E}_M, \tag{B1}$$

with the last equality following from Eq. (10).

In the approximate arguments of Secs. III and IV we have used the relation $Q = CV$ taking for C the *geometric* value for the capacitance: $C = \epsilon/L$. From (B1) it is evident that this is equivalent to using for $\mathcal{E}_M$ the *average* field V/L. Since $\mathcal{E}_M > V/L$, it is important to show that the relation $Q \simeq \epsilon V/L$ is not seriously in error. The principal observation we wish to make is the following:

So long as the electric field distribution is everywhere convex, the relation $Q \simeq \epsilon V/L$ is correct to within a factor of two.

The proof, which is geometric in character, is based simply on a comparison of areas, and is given in Fig. 2.

It remains to prove that the electric field distribution is indeed convex. For the set of Eqs. (A1), (A2), (A3a), and (A3b) in their full generality and with the boundary conditions: $\mathcal{E}=0$ at $x=0$ and at $x=L$, we can only conjecture at this point that the $\mathcal{E}$ curve has the postulated convexity. For the class of cases studied analytically in Appendix A it can be verified algebraically that the field is convex, and $\mathcal{E}_M/(V/L)$ can be directly computed and shown to be less than two. However, for investigation of the more general cases a combination of geometrical and analytical methods[11] will likely prove more fruitful.

We illustrate here the application of this technique to the case of the perfect insulator, $n_T=p_T=z_R=0$. Substituting for p-n from Eq. (A2) in Eq. (A4), $(d/dx)[\mathcal{E}(d\mathcal{E}/dx)]=-e(b+1)r/\epsilon\mu_n$ and is therefore everywhere negative. Assuming a single maximum in $\mathcal{E}$, say at x_M, then the plot of $\mathcal{E}$ vs x must be convex. If it were not, say it was as plotted in Fig. 3, then the plot of $d\mathcal{E}/dx$ would have to be as shown. Taking two positions, say x_1, and $x_2>x_1$, such that $(d\mathcal{E}/dx)_1=(d\mathcal{E}/dx)_2$, it follows from Fig. 3 that $[\mathcal{E}(d\mathcal{E}/dx)]_2>[\mathcal{E}(d\mathcal{E}/dx)]_1$, whence $d/dx[\mathcal{E}(d\mathcal{E}/dx)]$ must be positive somewhere between x_1 and x_2, and we have reached a contradiction. In like manner it is shown that $\mathcal{E}$ can have only a single maximum between $x=0$ and $x=L$.

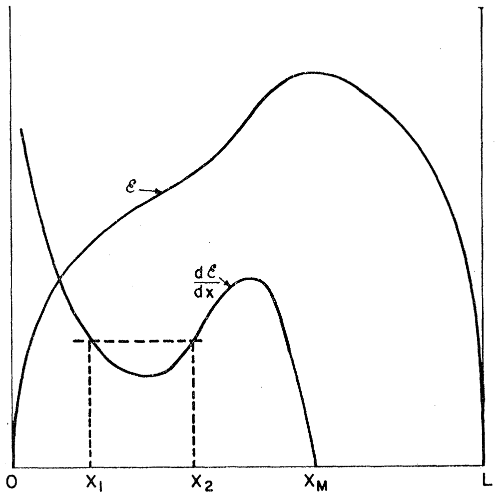

FIG. 3. Proof, by contradiction (see text), that the electric-field distribution must be convex for double injection into a perfect insulator.

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Current-Voltage Characteristics of Forward Biased Long *p-i-n* Structures

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The current-voltage characteristics have been observed in several germanium *p-i-n* structures in which the *n* side was biased negative and the *p* side positive so as to cause a double injection of electrons and holes into the structure. The middle *i* section was constructed of good quality germanium (approximately 2×10^{13} donors/cm^3) and was many minority carrier diffusion lengths long. The observed *I-V* characteristics display a low-field region in which the current is proportional to the voltage followed by a higher field region in which the current is proportional to the square of the voltage. In the square-law region, the current is a function of the difference, rather than the sum, of the thermal densities of the electrons and holes. These observations lend experimental support to the basic theories of Lampert and Rose regarding volume-controlled double injection into a semiconductor.

THE preceding paper[1] outlined a theory regarding the current-voltage characteristics of long semiconductor specimens provided with one contact which injected electrons and another which injected holes. This paper will describe some relatively simple experiments in which the measured current-voltage characteristics of suitable specimens lend support to these theoretical considerations.

Germanium was selected as the most suitable material for study since its technology is so far advanced compared to the other semiconductors. Since it was desired to measure the sample characteristics in the presence of double injection in which the density of injected carriers was large compared to the extrinsic density, it was necessary to utilize the highest resistivity material available. A germanium crystal with an extrinsic density of about 2×10^{13} electrons/cm^3 and a low level bulk lifetime of approximately 600 μsec was selected. It should be noted, however, owing to the perturbing influence of the surfaces of the specimens utilized and the high injection levels employed, it is not certain that this low-level bulk lifetime should be used in the quantitive comparison of theory and experiment.

Since the theoretical considerations were concerned with the case in which the diffusion length could be

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[1] M. Lampert and A. Rose, preceding paper [Phys. Rev. **121**, 26 (1961)].