Spin in Classical and Quantum Theory

H. C. CORBEN
Space Technology Laboratories, Los Angeles, California
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The classical equations of motion of a charged point-particle with intrinsic spin under the influence of an external electromagnetic field are restated and compared with the Heisenberg equations of motion derived from the Dirac theory. The partition of angular momentum between particle and field in the classical theory is contrasted to the Dirac theory of electron spin. The analogy between the Dirac equation and the theory of parametric amplification is pointed out.

A free spinning point particle moving according to the laws of classical relativistic point-particle mechanics may move along a helix. The sum of the intrinsic spin $\boldsymbol{\sigma}$ and the angular momentum of the helical motion in this classical analog of zitterbewegung is an effective spin vector **S** which is a constant of the motion. Because of this internal motion, the effective mass M of the particle differs from the mass m which is ascribed to it in the equations of motion. Solutions are found in which **S** is parallel or antiparallel to the momentum, and the sign of M is determined by the helicity. When placed in a uniform electromagnetic field, the particle behaves as if it had a rest mass M and a magnetic moment $e\boldsymbol{\sigma}/Mc$, in addition to any explicit magnetic moment that may be ascribed to it.

I. INTRODUCTION

BECAUSE spin and quantum mechanics were discovered within a year of each other, these discoveries being quickly followed by that of the Dirac equation, the classical relativistic theory of spinning particles interacting with an electromagnetic field has not been widely used although the nonrelativistic theory is applied extensively. Indeed it was not until 1941 that the relativistic classical theory of charged spinning particles, including self-interaction, was developed.[1] In this theory, equations of motion were derived both for the spin and for the motion as a whole including not only the usual radiation damping arising from the acceleration of the charge but also the very complicated set of terms describing the radiation from this precessing accelerated magnet. The particle was described by a set of six parameters which referred to its mass, charge, rest magnetic moment, rest electric moment and moments of inertia along the directions of these moments, but the equations were also specialized to the case in which the particle had zero electric moment in its rest system.

Measurements of the magnetic moment of a particle are often made under conditions in which we have every reason to believe that relativistic classical theory is applicable. It is therefore of interest to examine the classical equations of motion which we might expect would be satisfied by a meson or baryon moving in a magnetic field and ask how measurements made on such a particle could be interpreted in terms of the intrinsic properties which could be described by a future quantum-mechanical theory of these particles. It is also of interest to note how the classical theory reflects the properties of the Dirac equation when applied to the electron or μ-meson. Although this question was discussed in reference 1, an unusual Heisenberg representation of the Dirac equation, developed in this paper, makes the relation more clear. In particular, we note that in classical theory the magnitude of the spin of a particle has its rest value only when the velocity and spin are parallel, whereas in the Dirac theory the magnitude of the spin is a constant of the motion under all conditions. Classically also the magnetic moment of a particle increases with velocity if the spin and velocity are not parallel, but on the other hand the rate of precession of the spin in a given uniform magnetic field decreases as the velocity increases. For a charged particle with spin moving in a Coulomb field, the sum of the orbital and spin angular momenta is not a constant of the motion, it being necessary to add also the angular momentum resident in the field. It is shown how this is consistent with the Dirac theory provided that the "spin of the electron" according to the latter is also taken to include the extra angular momentum that is due to the presence of the particle in the external field.

II. CLASSICAL THEORY

If we neglect all radiation damping effects, we may write the classical relativistic equation for the spin motion of a particle which has zero electric moment in its rest system in the form[2] [reference 1, Eq. (103), with $K=0$, IcS_{ij} replaced by σ_{ij}, g_2/I by g/c]

$$\dot{\sigma}_{ij}+\dot{S}_{ij}=\frac{g}{c}(f_{ik}\sigma_{kj}-\sigma_{ik}f_{kj}),$$

where

$$\dot{S}_{ij}\equiv(\sigma_{jm}v_i-\sigma_{im}v_j)\left(\dot{v}_m-\frac{g}{c}f_{mk}v_k\right), \tag{1}$$

$$\sigma_{ij}v_j=0. \tag{2}$$

(Sign differences with reference 1 are due to the fact that here $x_4=ict$). The dot denotes differentiation along

[1] H. J. Bhabha and H. C. Corben, Proc. Roy. Soc. (London) **A178**, 273 (1941).

[2] See V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Letters **2**, 435 (1959).

the space-time path of the particle, $v_i=\dot{x}_i$, so that

$$v_i\cdot v_i\equiv\frac{dx_i}{ds}\cdot\frac{dx_i}{ds}=-1, \tag{3}$$

and the magnetic moment tensor μ_{ij} has been assumed to be proportional to the spin tensor σ_{ij}:

$$\mu_{ij}=g\sigma_{ij}. \tag{4}$$

In vector notation, Eq. (2) is then $\boldsymbol{\varepsilon}=\mathbf{v}\times\boldsymbol{\mu}/c$, where $\boldsymbol{\varepsilon}\equiv i(\mu_{41},\mu_{42},\mu_{43})$ is the electric moment which then vanishes for $\mathbf{v}\equiv d\mathbf{x}/dt=0$.

From (1) and (2) it follows that

$$\dot{\sigma}_{ij}v_j=-\sigma_{im}\left(\dot{v}_m-\frac{g}{c}f_{mk}v_k\right)+\frac{g}{c}(-\sigma_{ik}f_{kj}v_j),$$

or

$$\frac{d}{ds}(\sigma_{ij}v_j)=0,$$

so that (2) is consistent with the equation of spin motion, and the electric moment in the rest system remains zero throughout the motion.

For the special case in which $g=e/mc$ and the equation of motion is given sufficiently accurately by the Lorentz force equation

$$\dot{v}_m=\frac{e}{mc^2}f_{mk}v_k, \tag{5}$$

it follows that $\dot{S}_{ij}=0$ and

$$\dot{\sigma}_{ij}=\frac{e}{mc^2}(f_{ik}\sigma_{kj}-\sigma_{ik}f_{kj}),$$

or

$$\begin{aligned}\frac{d\boldsymbol{\sigma}}{dt}&=\frac{e}{mc^2}(\boldsymbol{\sigma}\times\mathbf{B}+\boldsymbol{\tau}\times\mathbf{E}),\\ \frac{d\boldsymbol{\tau}}{dt}&=\frac{e}{mc^2}(-\boldsymbol{\sigma}\times\mathbf{E}+\boldsymbol{\tau}\times\mathbf{B}),\end{aligned} \tag{6}$$

where

$$\boldsymbol{\tau}=(\mathbf{v}/c)\times\boldsymbol{\sigma}.$$

The rate of spin precession in a uniform magnetic field is then given by

$$\frac{d\boldsymbol{\sigma}}{dt}=\frac{e}{mc}(1-\beta^2)^{\frac{1}{2}}\boldsymbol{\sigma}\times\mathbf{B},$$

i.e., although the magnetic moment may increase with velocity, the resonant frequency decreases as the velocity of the particle increases. If, however, $g\neq e/mc$ or the motion of the particle departs for any reason from that given by the Lorentz force equation (5), $\dot{S}_{ij}$ will not vanish and extra terms must be added to (6).

From (1) and (2) it also follows that

$$\sigma_{ij}\frac{d}{ds}(\sigma_{ij})=0,$$

so that

$$\sigma^2-\tau^2=\sigma^2-\frac{1}{c^2}(\mathbf{v}\times\boldsymbol{\sigma})^2=\sigma_0{}^2=\text{const.} \tag{7}$$

Hence, if $\mathbf{v}\|\boldsymbol{\sigma}$, $\sigma^2=\sigma_0{}^2$ or the magnitude of the spin is independent of the velocity, whereas if $\mathbf{v}\perp\boldsymbol{\sigma}$ we have $\sigma^2=\gamma^2\sigma_0{}^2$ $[\gamma=(1-\beta^2)^{-\frac{1}{2}}]$. This result is in contrast to that of quantum theory according to which the magnitude of the spin is constant under all conditions. However, a simple example shows where this disparity arises. If we consider a transversely polarized particle of charge e, magnetogyric ratio e/mc, and rest mass m, projected from infinity in a straight line directly at a fixed charge Ze, the initial spin of the particle would be $\gamma_0\sigma_0$, where $\gamma_0=(1-\beta_0{}^2)^{-\frac{1}{2}}$, $\beta_0=v_0/c$, v_0 being the initial velocity. Such a particle would come to rest instantaneously at a distance r from the fixed charge, where

$$\gamma_0=1+\frac{Zr_0}{r}\quad(r_0=e^2/mc^2).$$

At this point its spin is σ_0, but the extra angular momentum in the electromagnetic field due to the presence of the transverse magnet of moment $\mu=(e/mc)\sigma_0$ at a distance r from Ze is $Ze\mu/cr=(Zr_0/r)\sigma_0$. Hence by the above energy integral, the angular momentum $\gamma_0\sigma_0$ stays constant throughout the motion, but as the particle approaches the fixed charge this angular momentum becomes divided between the spin of the particle and the angular momentum in the field. For the equivalent motion of a longitudinally polarized particle, however, the spin of the particle stays constant and no angular momentum appears in the field.

Thus, classically, the change in the magnitude of the spin of a particle is intimately connected to the angular momentum put into the electromagnetic field by the motion of the particle in the external field which is acting upon it. In quantum theory, the constant magnitude of the spin must therefore be interpreted to mean the constant magnitude of the intrinsic spin plus the angular momentum in the field. In the Dirac theory, the magnitude of one component of this quantity is constrained to be always $\hbar/2$ and this automatically means that a fast particle is polarized longitudinally. In order to have a theory in which free fast particles can be polarized transversely, it is necessary that the magnitude of the spin should be free to vary with momentum.

The classical relativistic equation for the motion of the particle as a whole, again neglecting radiation damping and setting $K=0$, $IcS_{ij}=\sigma_{ij}$, $g_2/I=g/c$,

$g_1 = e/c^2$ in Eq. (101) of reference 1, is

$$\frac{d}{ds}\left[mv_i - \frac{1}{c}\dot{\sigma}_{ij}v_j - \frac{g}{2c^2}v_i\sigma_{kl}f_{kl} - \frac{g}{c^2}\sigma_{ij}f_{jk}v_k\right] = \frac{e}{c^2}f_{ik}v_k + \frac{g}{2c^2}\sigma_{kl}\partial_i f_{kl}. \quad (8)$$

Thus on multiplying Eq. (8) by v_i we have

$$\frac{1}{2}\frac{g}{c^2}\frac{d}{ds}(\sigma_{kl}f_{kl}) - \frac{g}{c^2}v_i\frac{d}{ds}(\sigma_{ij}f_{jk}v_k) = \frac{g}{2c^2}\sigma_{kl}f_{kl},$$

or

$$\sigma_{kl}f_{kl} = 2v_i\dot{\sigma}_{ij}f_{jk}v_k.$$

This last equation follows from Eqs. (1) and (2) for the spin motion, so that Eq. (8) is consistent with the condition (3) that $v_iv_i = -1$.

However, although the above classical equations of motion are self-consistent, and appear to lead to the correct nonrelativistic limit if we set $g = e/mc$, it is important to realize that, as pointed out in reference 1, these equations do not give the correct cross section for the scattering of light by an electron, and that in order to describe the interaction of the electron with an electromagnetic field it is necessary to put $g = 0$. A value of g different from zero gives the classical limit of the Dirac theory to which Pauli terms have been added.

According to Eq. (8), a free spinning point-particle is not restricted to motion in a straight line, but may move along a circle or a helix. It is this orbital motion which gives the particle a magnetic moment which is the classical analog of the magnetic moment of the Dirac theory. Before discussing this motion in Sec. IV we rewrite the Dirac theory in a form suitable for comparison with the classical theory.

III. DIRAC THEORY

The relations between the classical equations of motion of the last section and those obtained from the Heisenberg representation of the Dirac equation may be seen most easily by defining the operator

$$\frac{d}{ds} \equiv \frac{\gamma_4}{c}\frac{d}{dt}, \quad (9)$$

where

$$\frac{dX}{dt} = \frac{\partial X}{\partial t} - \frac{i}{\hbar}(XH - HX),$$

and H is the Dirac Hamiltonian:

$$H = e\phi + c\boldsymbol{\alpha}\cdot\mathbf{P} + \rho_3 mc^2,$$

$$(\mathbf{P} = \mathbf{p} - (e/c)\mathbf{A}, \quad (\gamma_1,\gamma_2,\gamma_3) = \rho_2\boldsymbol{\sigma}, \quad \gamma_4 = \rho_3).$$

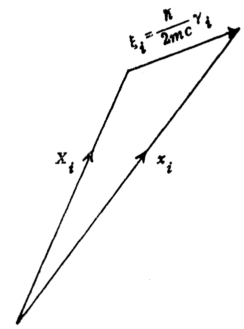

FIG. 1. Mnemonic for the operators X_i, ξ_i.

It then follows that

$$\frac{dx_i}{ds} = i\gamma_i,$$
$$\frac{d\gamma_i}{ds} = 2\kappa\frac{dx_i}{ds} - \frac{2}{\hbar}P_i, \quad (10)$$
$$\frac{dP_i}{ds} = \frac{e}{c}f_{ij}\frac{dx_j}{ds},$$

$(i, j = 1\cdots 4, \kappa = mc/\hbar)$.

If we define[3] (see Fig. 1)

$$X_i = x_i - \xi_i, \quad \left(\xi_i \equiv \frac{\hbar}{2mc}\gamma_i\right),$$

it follows from (10) that

$$mc\frac{dX_i}{ds} = P_i \equiv p_i - \frac{e}{c}A_i,$$

and

$$\frac{d^2x_i}{ds^2} = \frac{e}{mc^2}f_{ik}\frac{dx_k}{ds} + \frac{\hbar}{2imc}\frac{d^3x_i}{ds^3}. \quad (11)$$

The above equations are relations between operators acting on a solution of the Dirac equation, which may be written

$$\frac{dx_i}{ds}\frac{dX_i}{ds} = -1, \quad (12)$$

or, using (10),

$$\frac{dx_i}{ds}\frac{dx_i}{ds} + \frac{i\hbar}{2mc}\frac{dx_i}{ds}\frac{d^2x_i}{ds^2} = -1. \quad (12')$$

In the limit $\hbar \to 0$, (11) becomes the Lorentz force

[3] Similar coordinate operators for a Dirac field have been introduced in discussing the center of mass and localized states of the field and the passage to the nonrelativistic, extreme relativistic and classical limits, see M. H. L. Pryce, Proc. Roy. Soc. (London) **A150**, 166 (1935); **195**, 62 (1948). L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950). T. A. Newton and E. P. Wigner, Revs. Modern Phys. **21**, 400 (1949). F. Bopp, Z. angew. Phys. **1**, 387 (1949). H. Hönl, Ergeb. exakt. Naturw. **26**, 291 (1952). H. Hönl and A. Papapetrou, Z. Physik **116**, 153 (1940).

equation (5), and (12) reduces formally to (3). Because of (10), however,

$$\frac{dx_i}{ds}\frac{dx_i}{ds}=-\gamma_i\gamma_i=-4,$$

so that

$$\frac{dx_i}{ds}\frac{d^2x_i}{ds^2}=-6i\kappa, \tag{13}$$

which tends to infinity rather than zero in the classical limit. On the other hand, from (12)

$$-1=\frac{dX_i}{ds}\left[\frac{dX_i}{ds}-\frac{i}{2\kappa}\frac{d^2X_i}{ds^2}-\frac{1}{4\kappa^2}\frac{d^3X_i}{ds^3}\cdots\right], \tag{12''}$$

which reduces to

$$\frac{dX_i}{ds}\frac{dX_i}{ds}=-1,$$

in the classical limit.

In terms of the internal coordinates ξ_i, Eq. (11) becomes

$$\frac{d^2\xi_i}{ds^2}-2i\kappa\frac{d\xi_i}{ds}+\frac{2ie}{\hbar c}f_{ij}\xi_j=0, \tag{11'}$$

with

$$\xi_i\frac{d\xi_i}{ds}=\frac{3i}{2\kappa}.$$

The natural oscillations of the ξ_i thus become forced by an external field which in general couples their normal modes. Equation (11′) may also be written in either of the forms

$$\begin{aligned}&\frac{d^2\xi_i}{dt^2}-\frac{2i}{\hbar}(H-e\phi)\frac{d\xi_i}{dt}+\frac{2iec}{\hbar}f_{ij}\xi_j=0,\\ &\frac{d^2\xi_i}{dt^2}+\frac{2i}{\hbar}\frac{d\xi_i}{dt}(H-e\phi)-\frac{2iec}{\hbar}(f_{ij}\xi_j-2f_{i4}\xi_4)=0.\end{aligned} \tag{11''}$$

For an external magnetic field B_z, Eq. (11′) becomes

$$\frac{d^2Z}{ds^2}-2i\kappa\frac{dZ}{ds}+\frac{2e}{\hbar c}B_zZ=0, \tag{14}$$

where $Z=\xi_i+i\xi_2$, and ξ_3, ξ_4 are unaffected by the presence of the field. Thus in the classical limit of an electron at rest, cd/ds is to be interpreted as a time derivative, and a periodic magnetic field

$$B_z=B_0\cos\Omega t,$$

leads to a Mathieu equation to describe the motion of the internal coordinate

$$\frac{d^2y}{d\omega^2}+(a+q\cos 2\omega)y=0, \tag{15}$$

with

$$y=(\xi_1+i\xi_2)\exp(-imc^2t/\hbar),$$
$$\omega=\tfrac{1}{2}\Omega t,\quad a=(2mc^2/\hbar\Omega)^2,\quad q=8ecB_0/\hbar\Omega^2.$$

Standard plots of the a-q diagram for Eq. (15) show the values of these parameters for which the motion is stable and in particular show that for small q, i.e., small B_0, the motion is unstable for $a=n^2$ (n integer) or

$$n\hbar\Omega=2mc^2. \tag{16}$$

These instabilities in the internal motion therefore occur at just those frequencies of the driving field for which n quanta provide enough energy for pair production. From this point of view pair production bears a close analogy with parametric amplification, the natural frequency of a resonant circuit being replaced by the natural frequency of the electron's zitterbewegung.

There is no indication in (11) of the interaction of the magnetic and electric moments of the electron with the field f_{ik}, but this appears if we use the equation itself to eliminate the third derivative in terms of the fourth. Introducing the spin operator

$$\sigma_{ij}\equiv\frac{\hbar}{2i}(\gamma_i\gamma_j-\delta_{ij})=\frac{\hbar}{4i}(\gamma_i\gamma_j-\gamma_j\gamma_i), \tag{17}$$

so that

$$(\sigma_{23},\sigma_{31},\sigma_{12})=\tfrac{1}{2}\hbar\boldsymbol{\sigma},$$
$$(\sigma_{14},\sigma_{24},\sigma_{34})=\tfrac{1}{2}\hbar\rho_1\boldsymbol{\sigma},$$

we have

$$\frac{d^2x_i}{ds^2}=\frac{e}{mc^2}f_{ij}\frac{dx_j}{ds}+\frac{e}{2m^2c^3}\sigma_{kj}\partial_if_{kj}+\frac{ie\hbar}{2m^2c^3}\left[j_i-f_{ik}\frac{d^2x_k}{ds^2}+\frac{i\hbar c}{2e}\frac{d^4x_i}{ds^4}\right]. \tag{18}$$

In this form the equation of motion shows in the second term on the right-hand side the force due to the interaction of its magnetic moment

$$\mu_{ij}\equiv\frac{e}{mc}\sigma_{ij} \tag{19}$$

with the inhomogeneous electromagnetic field. The four vector $j_i=\partial_kf_{ik}$ is the source of this field.

It is easily verified that the spin operator σ_{ij} satisfies the conditions

$$\begin{aligned}\sigma_{ij}\frac{dX_j}{ds}&=\frac{\hbar}{2i}\frac{d\xi_i}{ds},\\ \sigma_{ij}\frac{dx_j}{ds}&=\frac{3\hbar}{2i}\frac{dx_i}{ds}=3mc\xi_i.\end{aligned} \tag{20}$$

We now compare the classical equations (4), (2), (8) with the Dirac equations (19), (20), and (18), respectively. Equation (19) gives the correct value of the magnetic moment, apart from radiative corrections, while in (4) g appears as an arbitrary parameter. Clearly in taking the classical limit by letting $\hbar\to 0$ all spin properties disappear and the equation of motion (11) reduces to the Lorentz force equation without spin. However, if we take the classical limit but keep the spin finite by setting $\hbar=0$ in (18), the extra term that remains on the right-hand side of (18) is identical with the extra term on the right-hand side of (8), with $g=e/mc$. This is a source of confusion, for the classical limit of the Dirac equation is obtained by setting $g=0$ in the classical equation (8). It is shown below that the extra term on the left-hand side of (8) gives rise automatically to the correct gyromagnetic ratio (apart from radiative corrections).

In addition to the spin operator (17) we may consider the following angular momentum tensors:

$$L_{ij}\equiv M_{ij}+F_{ij},$$
$$L_{ij}=x_ip_j-x_jp_i,$$
$$M_{ij}=mc\left[x_i\frac{dX_j}{ds}-x_j\frac{dX_i}{ds}\right], \tag{21}$$
$$F_{ij}=\frac{e}{c}[x_iA_j-x_jA_i].$$

It then follows from the equations of motion that

$$\begin{aligned}\frac{d}{ds}[M_{ij}+\sigma_{ij}]&=mc\left[x_i\frac{d^2X_j}{ds^2}-x_j\frac{d^2X_i}{ds^2}\right]\\&=\frac{e}{c}[x_if_{jk}-x_jf_{ik}]\frac{dx_k}{ds},\\ \frac{d}{ds}[L_{ij}+\sigma_{ij}]&=\frac{e}{c}\left[\frac{dx_i}{ds}A_j-\frac{dx_j}{ds}A_i\right.\\&\qquad\left.+\frac{dx_k}{ds}(x_i\partial_j-x_j\partial_i)A_k\right].\end{aligned} \tag{22}$$

For the space components these equations may be written

$$\begin{aligned}\frac{d}{dt}(\mathbf{M}+\boldsymbol{\sigma})&=e\mathbf{r}\times\left(\mathbf{E}+\frac{1}{c}\frac{d\mathbf{r}}{dt}\times\mathbf{B}\right),\\ \frac{d}{dt}(\mathbf{L}+\boldsymbol{\sigma})&=e\left[\frac{1}{c}\frac{d\mathbf{r}}{dt}\times\mathbf{A}+\mathbf{r}\times\nabla\left(\frac{\mathbf{A}}{c}\cdot\frac{d\mathbf{r}}{dt}-\phi\right)\right].\end{aligned} \tag{22'}$$

The first of these equates the rate of change of the spin plus *mechanical* angular momentum to the torque about the origin. The second equation reduces to the first for $\mathbf{A}=0$.

IV. MOTION OF A CLASSICAL FREE SPINNING PARTICLE

The equations of motion (1), (8) may be written in the form

$$\dot\sigma_{ij}=\frac{g}{c}(f_{ik}\sigma_{kj}-\sigma_{ik}f_{kj})-(v_ip_j-v_jp_i), \tag{1'}$$

$$\frac{d}{ds}\left[m'v_i+\frac{p_i}{c}\right]=\frac{e}{c^2}f_{ik}v_k+\frac{g}{2c^2}\sigma_{kl}\partial_if_{kl}, \tag{8'}$$

where

$$m'=m-\frac{g}{2c^2}\sigma_{kl}f_{kl}=m-\frac{g}{c^2}(\boldsymbol{\sigma}\cdot\mathbf{B}+\boldsymbol{\tau}\cdot\mathbf{E}),$$
$$p_i=\sigma_{ij}Y_j,\quad (v_ip_i=0)$$
$$Y_j=\dot v_j-\frac{g}{c}f_{jk}v_k.$$

In vector notation, these equations are

$$\begin{aligned}\frac{d\boldsymbol{\sigma}}{dt}&=\frac{g}{\gamma}[\boldsymbol{\sigma}\times\mathbf{B}+\boldsymbol{\tau}\times\mathbf{E}]-(\mathbf{v}\times\mathbf{p}),\\ \frac{d\boldsymbol{\tau}}{dt}&=\frac{g}{\gamma}[-\boldsymbol{\sigma}\times\mathbf{E}+\boldsymbol{\tau}\times\mathbf{B}]\\&\qquad+c\left(\mathbf{p}-\mathbf{v}\frac{(\mathbf{v}\cdot\mathbf{p})}{c^2}\right),\end{aligned} \tag{1''}$$

$$\begin{aligned}\frac{d}{dt}(m'\mathbf{v}\gamma+\mathbf{p})&=e\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{B}}{c}\right)+\frac{g}{\gamma}[\nabla(\boldsymbol{\sigma}\cdot\mathbf{B}+\boldsymbol{\tau}\cdot\mathbf{E})],\\ \frac{d}{dt}(m'\gamma c^2+\mathbf{v}\cdot\mathbf{p})&=e\mathbf{v}\cdot\mathbf{E}-\frac{g}{\gamma}\left(\boldsymbol{\sigma}\cdot\frac{\partial\mathbf{B}}{dt}+\boldsymbol{\tau}\cdot\frac{\partial\mathbf{E}}{\partial t}\right),\end{aligned} \tag{8''}$$

with

$$p_i=\mathbf{p}\quad(i=1,2,3),\quad p_4=(i/c)\mathbf{v}\cdot\mathbf{p},$$
$$\mathbf{p}=-\frac{\boldsymbol{\sigma}}{c^2}\times\left\{\gamma^2\frac{d\mathbf{v}}{dt}-\frac{cg}{\gamma}\left[\mathbf{E}+\frac{\gamma^2}{c}\mathbf{v}\times\left(\mathbf{B}-\frac{\mathbf{v}\times\mathbf{E}}{c}\right)\right]\right\},$$

The ten equations (1″) (8″) are six independent equations for the spin motion and the motion as a whole. In a uniform magnetic field they become

$$\frac{d}{dt}(m'\mathbf{v}\gamma+\mathbf{p})=\frac{e}{c}(\mathbf{v}\times\mathbf{B}),$$
$$\frac{d\boldsymbol{\sigma}}{dt}=\frac{g}{\gamma}\boldsymbol{\sigma}\times\mathbf{B}-\mathbf{v}\times\mathbf{p},$$

with

$$m'=m-(g/c^2)\boldsymbol{\sigma}\cdot\mathbf{B},$$
$$m'\gamma c^2+\mathbf{v}\cdot\mathbf{p}=\text{const},$$
$$\mathbf{p}=-\frac{\gamma^2}{c^2}\boldsymbol{\sigma}\times\left(\frac{d\mathbf{v}}{dt}-\frac{g}{\gamma}\mathbf{v}\times\mathbf{B}\right).$$

The extra terms, which include Thomas precession effects, have the important consequence that in a uniform magnetic field $\boldsymbol{\sigma}\cdot\mathbf{B}$ is not necessarily a constant of the motion, even in this approximation in which radiation damping is neglected;

$$\frac{d}{dt}(\boldsymbol{\sigma}\cdot\mathbf{B})=\frac{\gamma^2}{c^2}\left[\frac{dv^2}{2dt}(\boldsymbol{\sigma}\cdot\mathbf{B})-(\mathbf{v}\cdot\boldsymbol{\sigma})\left(\mathbf{B}\cdot\frac{d\mathbf{v}}{dt}\right)\right].$$

The extra terms also lead to the interesting result that a free spinning particle does not necessarily move in a straight line, but that classically such a particle can move on a helical path which is the classical analog of the Dirac zitterbewegung.[4] In such a motion the spin and orbital angular momenta are in general not separately constants of the motion. To see this we note that in the absence of any external field at all, Eqs. (1) and (8) give

$$mcv_i+\sigma_{ij}\dot{v}_j=P_i,\qquad \dot{\sigma}_{ij}=(P_iv_j-P_jv_i), \tag{23}$$

where P_i is a constant four vector such that $v_iP_i=-mc$. Thus, $v_i=P_i/mc$, $\dot{\sigma}_{ij}=0$ is a solution, but it is not the only solution. Writing $P_i=(\mathbf{P},iW/c)$ we have

$$m\gamma\mathbf{v}-\frac{\gamma^2}{c^2}\boldsymbol{\sigma}\times\frac{d\mathbf{v}}{dt}=\mathbf{P},\qquad mc^2\gamma-\frac{\gamma^2}{c^2}\mathbf{v}\cdot\boldsymbol{\sigma}\times\frac{d\mathbf{v}}{dt}=W,\qquad d\boldsymbol{\sigma}/dt=(\mathbf{P}\times\mathbf{v}), \tag{24}$$

with

$$\gamma(W-\mathbf{v}\cdot\mathbf{P})=mc^2.$$

Thus for $\mathbf{P}=0$, we have $\boldsymbol{\sigma}=\text{const}$, and

$$\mathbf{v}=\frac{\gamma}{mc^2}\boldsymbol{\sigma}\times\frac{d\mathbf{v}}{dt},\quad W=\frac{mc^2}{\gamma}.$$

This equation is satisfied by a particle moving in a circle normal to $\boldsymbol{\sigma}$, and of radius $(\sigma/mc)(\gamma^2-1)^{\frac{1}{2}}$.

$$\mathbf{v}=\boldsymbol{\Omega}\times\mathbf{r},\quad \boldsymbol{\Omega}=(-mc^2/\gamma\sigma^2)\boldsymbol{\sigma}. \tag{25}$$

For $\mathbf{P}\neq 0$, the equations are also satisfied for arbitrary constant $\mathbf{J}$ by

$$\boldsymbol{\sigma}=\mathbf{J}-\mathbf{r}\times\mathbf{P},\qquad \mathbf{v}=\frac{mc^2}{\gamma(\mathbf{J}\cdot\mathbf{P})^2}[\gamma^2\sigma_0{}^2\mathbf{P}-\mathbf{J}\cdot\mathbf{P}\boldsymbol{\sigma}], \tag{26}$$

where σ_0 is the spin of the particle in its rest system:

$$\sigma_0{}^2=\frac{\sigma^2}{\gamma^2}+\frac{(\mathbf{v}\cdot\boldsymbol{\sigma})^2}{c^2}=\frac{1}{\gamma^2}\left(\sigma^2+\frac{(\mathbf{J}\cdot\mathbf{P})^2}{m^2c^2}\right), \tag{27}$$

and $\gamma=(1-\beta^2)^{-\frac{1}{2}}=\text{const}$. Other constants of the motion are σ^2, $\boldsymbol{\sigma}\cdot\mathbf{P}=\mathbf{J}\cdot\mathbf{P}$, $\mathbf{v}\cdot\mathbf{p}$, $\mathbf{v}\cdot\boldsymbol{\sigma}$, but $\mathbf{v}$ and $\boldsymbol{\sigma}$ are not constant in direction:

$$d\boldsymbol{\sigma}/dt=\boldsymbol{\Omega}\times\boldsymbol{\sigma},\qquad \mathbf{v}=\text{const}+\boldsymbol{\Omega}\times\mathbf{r},$$

with

$$\boldsymbol{\Omega}=\frac{-mc^2}{\gamma\mathbf{J}\cdot\mathbf{P}}\mathbf{P}. \tag{28}$$

The position of the particle at any time may be written

$$\mathbf{r}(t)=\frac{c^2t}{W}\mathbf{P}+\boldsymbol{\varrho}(t)+\mathbf{R}_0, \tag{29}$$

where

$$d\boldsymbol{\varrho}/dt=\boldsymbol{\Omega}\times\boldsymbol{\varrho}, \tag{30}$$

$$\mathbf{R}_0=\mathbf{P}\times\mathbf{J}/P^2=\text{const}, \tag{31}$$

$$W^2=P^2c^2+M^2c^4, \tag{32}$$

$$M=m\sigma_0P/\mathbf{J}\cdot\mathbf{P},\quad \gamma=mW/M^2c^2. \tag{33}$$

From (29) the velocity may be written

$$\mathbf{v}=\mathbf{v}_0+\boldsymbol{\Omega}\times\boldsymbol{\varrho}, \tag{34}$$

where $v_0=c^2P/W$ is the forward velocity on which is superposed the motion in a circle. On squaring Eq. (34) and using (33), we find the radius ρ_0 of the circle:

$$\rho_0=\frac{\sigma_0}{Mc}\left(\frac{m^2}{M^2}-1\right)^{\frac{1}{2}}. \tag{35}$$

Hence $M\leq m$, so that since $W\geq Mc^2$, it follows that $\gamma\geq 1$ as required. We then have

$$P=M\gamma_0v_0,\quad W=Mc\gamma_0,$$

where $\gamma_0=(1-v_0{}^2/c^2)^{-\frac{1}{2}}$, so that M is the effective mass of the particle.

The effective spin $\mathbf{S}$ of the particle may be defined as the sum of its intrinsic spin $\boldsymbol{\sigma}$ and the angular momentum due to the helical motion:

$$\mathbf{S}=\boldsymbol{\sigma}+\boldsymbol{\varrho}\times\mathbf{P}. \tag{36}$$

From (26) and (31) it follows that

$$\mathbf{S}=\frac{\mathbf{P}\cdot\mathbf{J}}{P^2}\mathbf{P}. \tag{37}$$

[4] See D. Bohm, P. Hillion, T. Takabayasi, and J. P. Vigier, Progr. Theoret. Phys. (Kyoto) **23**, 496 (1960), C. Møller, Ann. inst. Henri Poincaré **11–12**, 251 (1949).

Thus in the motion described by (29), **S** is directed parallel or antiparallel to **P** according to the sign of $\mathbf{J}\cdot\mathbf{P}$, which by (33) also determines the sign of the effective mass M. Hence, $\mathbf{S}\cdot\mathbf{P}$ has the sign of M, and the magnitude of **S** is $S=m\sigma_0/M$. Since $M\leq m$, it follows that $S\geq\sigma_0$. The magnitude of the angular velocity of rotation is M^2c^4/WS, and its sense is opposite to that of **S**.

From (26) and (31) we note that **J** may be broken into two parts which are separately constants of the motion for a free particle:

$$\mathbf{J}=\mathbf{L}+\mathbf{S},$$

where

$$\mathbf{L}=\mathbf{R}_0\times\mathbf{P},$$

and $\mathbf{R}_0$ is the vector drawn from the origin perpendicular to the axis of the helix (see Fig. 2). The radius ρ_0 (Eq. 35) may then be written

$$\rho_0=(S^2-\sigma_0^2)^{\frac{1}{2}}/Mc. \tag{38}$$

It was pointed out in reference 1 that the classical limit of the Dirac equation is obtained by setting $g=0$ in the above theory. In this limit, Eqs. (1′) and (8′) become

$$\begin{aligned}\dot{\sigma}_{ij}&=-(v_iP_j-v_jP_i),\\ dP_i/ds&=(e/c)f_{ik}v_k,\end{aligned} \tag{39}$$

where P_j is defined by (23) but is now no longer a constant four-vector. As in the Dirac theory, the classical theory in this form does not exhibit an explicit interaction between the spin and the external field. The first of Eqs. (39) may be written in the form

$$\frac{d}{ds}(M_{ij}+\sigma_{ij})=\frac{e}{c}(x_if_{jk}-x_jf_{ik})v_k,$$

where

$$M_{ij}=(x_iP_j-x_jP_i).$$

This is of the same form as (22), and the second of Eqs. (39) is of the same form as (10). Similarly it follows that even if an electromagnetic field is present,

$$v_iP_i=-mc,$$

which is to be compared with (12).

Finally we note that for $g=0$,

$$\begin{aligned}P_i\dot{P}_i&=(e/c)f_{ik}v_k\sigma_{ij}\dot{v}_j,\\ \dot{\sigma}_{ik}f_{ik}&=2f_{ik}v_k\sigma_{ij}\dot{v}_j,\end{aligned}$$

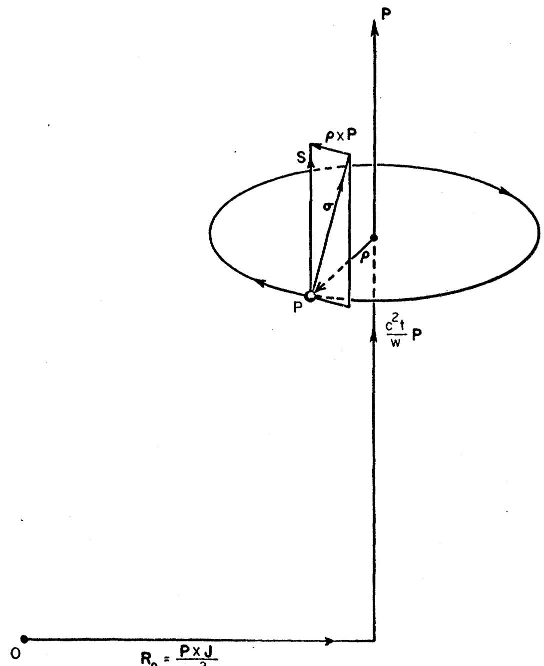

FIG. 2. Classical motion of a free spinning particle.

so that

$$\frac{d}{ds}\left(P_iP_i-\frac{e}{c}\sigma_{ik}f_{ik}\right)=-\frac{e}{c}\sigma_{ik}\dot{f}_{ik}.$$

Thus in a uniform and static field,

$$P_iP_i-(e/c)\sigma_{ik}f_{ik}=-M^2c^2$$

is a constant of the motion. We may write this in the form

$$W^2=c^2P^2+M^2c^4-2ec(\boldsymbol{\sigma}\cdot\mathbf{B}+\boldsymbol{\tau}\cdot\mathbf{E}),$$

or, nonrelativistically

$$W=Mc^2+\frac{P^2}{2M}-\frac{e}{Mc}(\boldsymbol{\sigma}\cdot\mathbf{B}+\boldsymbol{\tau}\cdot\mathbf{E}),$$

where $\mathbf{P}=\mathbf{p}-(e/c)\mathbf{A}$, $W=E-e\phi$, p, E being the canonical momentum and total energy, respectively. Thus in this limit the particle behaves as if it had a rest mass M, magnetic moment $e\sigma/Mc$, and electric moment $e\tau/Mc$. However M is not the bare mass m assumed for the particle, and in time- or space-varying fields M is not necessarily a constant of the motion.