of the proton group leading to the 1.22-Mev Zr^{91} state. This is shown in Fig. 4. Cohen finds the same distribution for this state[3] and makes a $3s_{1/2}$ assignment for the state.

Figure 5 shows the total angular distribution for two proton groups, one leading to the 2.07-Mev state and the other to the 2.19-Mev state. The data indicated that the angular distributions for the two groups were different but they could not be measured separately. The curve in Fig. 5 represents a $2d_{3/2}$ shell-model state and is the experimental distribution for the ground-state group multiplied by 0.67. This curve is subtracted from the data and the resultant angular distribution is shown in Fig. 6. This distribution is fitted by a g-wave Butler curve. The expectation of shell-model states leads to a $2d_{3/2}$ assignment for the 2.07-Mev Zr^{91} state and a $1g_{7/2}$ assignment for the 2.19-Mev state.

Complete angular distributions were not measured for the other proton groups. A partial angular distribution for the proton group to the 2.56-Mev state was peaked at large angles. It is consistent with the expected $1h_{11/2}$ state but a definite assignment is not possible.

TABLE I. A comparison of the experimental and calculated reduced widths relative to the ground state.

Q	l	$(\gamma^2/\gamma^2$ gnd)[a] (Butler+expt)	$(\gamma^2/\gamma^2$ gnd)[b] (HO)
5.02	2	1.0	1.0
3.80	0	1.7	2.5
2.83	4	0.5	0.5

[a] This column shows the experimental reduced widths relative to the ground state when the Butler theory is assumed.
[b] This column shows calculated reduced widths relative to the ground state using harmonic oscillator wave functions for the neutron.

CONCLUSIONS

Angular distributions were measured for proton groups leading to four energy levels in Zr^{91}. All of the distributions were consistent with Butler theory calculations and shell-model expectations. Distorted-wave calculations by Tobocman[17] also support the assignments given here. Tobocman has calculated the distributions for the ground-state group and for the first excited state group and his assignments agree with the ones given here.

Finally, an extraction of the relative reduced widths of the states can be made by using the Butler theory. A comparison of the experimental widths with calculated widths is shown in Table I.

[17] W. Tobocman (private communication).

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Quantum Mechanical Three-Body Problem. II

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We present a method for treating the following quantum mechanical three-body problem: to find the ground-state eigenvalue and eigenfunction for a system of three identical particles between any pair of which there is an attractive central force. An essential point of the method is to assume the wave function Ψ has a special analytic form, $\Psi=\psi(\mathbf{r}_{12},\boldsymbol{\varrho}_3)+\psi(\mathbf{r}_{13},\boldsymbol{\varrho}_2)+\psi(\mathbf{r}_{23},\boldsymbol{\varrho}_1)$, where $\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2$, $\boldsymbol{\varrho}_3=\mathbf{r}_3-\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$ and $\mathbf{r}_{13}$, $\boldsymbol{\varrho}_2$ and $\mathbf{r}_{23}$, $\boldsymbol{\varrho}_1$ are defined analogously. The Schrödinger equation for the system can then be written as an integral equation for $\phi(\mathbf{k},\boldsymbol{\kappa})$, the Fourier transform of ψ. We expand this in Legendre polynomials,

$$\phi(\mathbf{k},\boldsymbol{\kappa})=\sum_{l=0}^{\infty}\phi_l(k,\kappa)P_l(\cos\gamma),$$

and this yields a set of coupled integral equations for the $\phi_l(k,\kappa)$. These can be truncated and to a good approximation one can neglect all ϕ_l except ϕ_0, thereby reducing the problem to a single integral equation for a function of two variables.

We propose an iterative scheme for solving this equation for the ground-state eigenfunction, and suggest a simple but accurate nonvariational method for deriving the energy eigenvalue therefrom. We test this proposed solution by working it out in detail for the case of exponential interparticle potentials. The results for the eigenvalue compare favorably with variational calculations by other authors. Finally, we discuss the accuracy of the approximations and the possible sources of error in the wave function.

I. INTRODUCTION

IN this paper we consider the quantum-mechanical problem of finding the ground-state energy eigenvalue and eigenfunction for a system of three identical particles in which identical attractive forces act between each pair. The general features of the method we discuss are applicable to other three-body and many-body problems, but for reasons that are more or less obvious the symmetrical three-body problem we mention is the simplest of these. The results we get in this problem are encouraging enough to make it hopeful that progress can be made along similar lines in more complicated problems. The symmetrical problem we dis-

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cuss is related to, but is one step closer to reality than one we have previously discussed,[1] in which we did not consider that potentials acted between pairs of particles, but instead required that the wave function satisfy a boundary condition at a prescribed interparticle distance. The relation between the two problems lies in the special functional form we choose for the wave function [Eq. (10) of this paper]; this form is essentially the same in both cases, account being taken of the differences between boundary conditions and potentials. One point of this paper is to emphasize that this form has special advantages; these are pointed out at appropriate places in the paper.

The method presented here is an approximate one, but it is not a variational method, and it has the advantage over the variational method that one can estimate the errors of the approximations that it is necessary to make.

The plan of the paper is as follows: In Sec. II we write some well known but basic equations. In Sec. III we discuss the motivation and the *rationale* for the particular form of the wave function mentioned above, and we write the Schrödinger equation for this form of the wave function as an integral equation in momentum space. In Sec. IV we expand this integral equation in partial waves to give an infinite set of coupled integral equations, and suggest how this set may be truncated, reducing it to a single integral equation for a function of two variables. In Sec. V we discuss an approximate solution for the ground-state eigenvalue and eigenfunction of this equation. In Sec. VI we work out this solution in detail for interparticle potentials of exponential shape. In Sec. VII we estimate the errors in the various approximations that must be made in deriving these results.

II. SOME BASIC EQUATIONS

Let $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ be the position vectors of the three particles in some reference frame. Then the Schrödinger equation we wish to solve is, in the usual notation,

$$\left(-\frac{\hbar^2}{2m}(\nabla_1^2+\nabla_2^2+\nabla_3^2)+V(r_{12})+V(r_{13})+V(r_{23})\right)\Psi = E\Psi. \quad (1)$$

As the notation indicates, the potential energy is taken to be the sum of central potentials acting between each of the three pairs. With a potential of this kind we can separate off the center-of-mass motion and so we introduce the center-of-mass coordinate $\mathbf{R}=\frac{1}{3}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3)$. In addition to $\mathbf{R}$ there are three equivalent pairs of coordinates that one might use to define the positions of the particles, and it is part of our method to treat each of these pairs on an equal footing. These pairs are

$$\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2 \quad \text{and} \quad \boldsymbol{\varrho}_3=\mathbf{r}_3-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2), \quad (2a)$$

or

$$\mathbf{r}_{13}=\mathbf{r}_1-\mathbf{r}_3 \quad \text{and} \quad \boldsymbol{\varrho}_2=\mathbf{r}_2-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_3), \quad (2b)$$

or

$$\mathbf{r}_{23}=\mathbf{r}_2-\mathbf{r}_3 \quad \text{and} \quad \boldsymbol{\varrho}_1=\mathbf{r}_1-\tfrac{1}{2}(\mathbf{r}_2+\mathbf{r}_3). \quad (2c)$$

These sets of coordinates are of course not independent. The linear relations among them are of the form, for example,

$$\begin{aligned}\mathbf{r}_{13}&=\boldsymbol{\varrho}_1+\tfrac{1}{2}\mathbf{r}_{23}, \quad & \mathbf{r}_{12}&=\boldsymbol{\varrho}_1-\tfrac{1}{2}\mathbf{r}_{23},\\ \boldsymbol{\varrho}_2&=\tfrac{3}{4}\mathbf{r}_{23}-\tfrac{1}{2}\boldsymbol{\varrho}_1, \quad & \boldsymbol{\varrho}_3&=-\tfrac{3}{4}\mathbf{r}_{23}-\tfrac{1}{2}\boldsymbol{\varrho}_1.\end{aligned} \quad (3)$$

Of course, Eq. (1) has the same form in either of the coordinate sets (2a), (2b), (2c). For example, in $\mathbf{r}_{12}$, $\boldsymbol{\varrho}_3$ coordinates it is

$$(\nabla_{r_{12}}^2+\tfrac{3}{4}\nabla_{\rho_3}^2)\Psi+[v(r_{12})+v(r_{13})+v(r_{23})]\Psi=K^2\Psi, \quad (4)$$

where for a bound state (E negative), we use the definitions

$$K^2=m|E|/\hbar^2, \quad v(r)=mV(r)/\hbar^2.$$

Now we wish to write the integral equation equivalent of Eq. (4) (or of the other two similar equations). It is convenient to have a notation that deals with these three equations in a symmetrical way. Consider the coordinates $\mathbf{r}_{12}$, $\boldsymbol{\varrho}_3$, which in the center of mass system define the configuration of the particles. The same configuration is defined by $\mathbf{r}_{13}$, $\boldsymbol{\varrho}_2$ or $\mathbf{r}_{23}$, $\boldsymbol{\varrho}_1$, if these pairs are connected by equations like (3). Thus we can think of any one of the three pairs of coordinates (2a), (2b), (2c) as a kind of six-vector which defines the configuration, just as in three dimensions one can have different vectors (with different origins) which describe the same point in space. We shall let $\mathbf{P}$ stand for this general "point" in the six-dimensional configuration space, and Ψ will be a function of it. Stated differently, this simply means that $\mathbf{P}$ can be considered to be either $\mathbf{r}_{12}$, $\boldsymbol{\varrho}_3$ or $\mathbf{r}_{23}$, $\boldsymbol{\varrho}_1$ or $\mathbf{r}_{13}$, $\boldsymbol{\varrho}_2$, as we wish. Then the integral equation equivalent of (4) (or of its other two variants) is

$$\Psi(\mathbf{P})=\int\Psi(\mathbf{P}')v_t(\mathbf{P}')G_K(\mathbf{P}-\mathbf{P}')d\mathbf{P}', \quad (5)$$

where

$$v_t(\mathbf{P})=v(r_{12})+v(r_{13})+v(r_{23}). \quad (6)$$

The subscript t stands for "total." The function $G_K(\mathbf{P}-\mathbf{P}')$ which appears in Eq. (5) is a Green's function for Eq. (4). It has three equivalent forms corresponding to the three sets of coordinates (2). For example, in $\mathbf{r}_{12}$ $\boldsymbol{\varrho}_3$ coordinates it is

$$G_K=\frac{1}{(2\pi)^6}\int\int\frac{\exp[i\mathbf{k}\cdot(\mathbf{r}_{12}-\mathbf{r}_{12}')+i\boldsymbol{\kappa}(\boldsymbol{\varrho}_3-\boldsymbol{\varrho}_3')]}{k^2+\frac{3}{4}\kappa^2+K^2}\times d\mathbf{k}d\boldsymbol{\kappa}. \quad (7)$$

[1] L. Eyges, Phys. Rev. **115**, 1643 (1959).

It is obvious from this that G_K satisfies

$$(\nabla_{r_{12}}^2+\tfrac{3}{4}\nabla^2\rho_3-K^2)G_K=-\delta(\mathbf{r}_{12}-\mathbf{r}_{12}')\delta(\boldsymbol{\varrho}_3-\boldsymbol{\varrho}_3'), \quad (8)$$

and this last equation can be used directly to show that the integral equation (5) is indeed equivalent to the Schrödinger equation (4).

III. SPECIAL FORM FOR THE WAVE FUNCTION, AND THE EQUATION DETERMINING IT

We now wish to write the wave function in a special analytic form, which, as we shall try to show, has special advantages. To introduce this form, let us assume for the moment that we knew the wave function in *some* form, not necessarily analytic; for example we might suppose we were given the wave function as a multidimensional table, one entry for each small volume element in configuration space. Suppose now we wished to use this wave function to integrate over the primed variables on the right-hand side of Eq. (5) and so obtain an analytic form for ψ. This integration is the sum of three integrals, remembering that $v_t(\mathbf{P})$ is given by Eq. (6). Consider the integration over the first potential $v(r_{12})$ that appears in v_t. In integrating over it we can express the Green's function in $\mathbf{r}_{12}$, $\boldsymbol{\varrho}_3$ coordinates and the integration then gives rise to a function of $\mathbf{r}_{12}$, $\boldsymbol{\varrho}_3$ which we call ψ_{12} and which is defined by

$$\psi_{12}(\mathbf{r}_{12},\boldsymbol{\varrho}_3)=\int\int\Psi(\mathbf{P}')v(r_{12}')\times G_K(\mathbf{r}_{12}-\mathbf{r}_{12}',\ \boldsymbol{\varrho}_3-\boldsymbol{\varrho}_3')d\mathbf{r}_{12}'d\boldsymbol{\varrho}_3'. \quad (9a)$$

By doing the analogous thing for the other two potentials entering into v_t we are led to two other functions ψ_{13} and ψ_{23}

$$\psi_{13}(\mathbf{r}_{13},\boldsymbol{\varrho}_2)=\int\int\Psi(\mathbf{P}')v(r_{13}')\times G_K(\mathbf{r}_{13}-\mathbf{r}_{13}',\ \boldsymbol{\varrho}_2-\boldsymbol{\varrho}_2')d\mathbf{r}_{13}'d\boldsymbol{\varrho}_2', \quad (9b)$$

$$\psi_{23}(\mathbf{r}_{23},\boldsymbol{\varrho}_1)=\int\int\Psi(\mathbf{P}')v(r_{23}')\times G_K(\mathbf{r}_{23}-\mathbf{r}_{23}',\ \boldsymbol{\varrho}_1-\boldsymbol{\varrho}_1')d\mathbf{r}_{23}'d\boldsymbol{\varrho}_1'. \quad (9c)$$

From the way in which we have derived these equations it is clear that

$$\Psi(\mathbf{P})=\psi_{12}(\mathbf{r}_{12},\boldsymbol{\varrho})+\psi_{13}(\mathbf{r}_{13},\boldsymbol{\varrho}_2)+\psi_{23}(\mathbf{r}_{23},\boldsymbol{\varrho}_1). \quad (10)$$

This is the form of the wave function to which we refer in the introduction.

We now discuss the reasons for choosing to write the wave function in the form (10). To do this we digress for a moment to consider a well-known problem which is mathematically similar to, but simpler than the three-body problem. This is the problem of a single particle bound to more than one fixed potential (for example a one-electron multicenter problem, like the hydrogen molecule-ion). To be concrete, suppose we simply have two spherically symmetric potentials, one centered at $\mathbf{d}_1$ and the other at $\mathbf{d}_2$. If $\mathbf{r}_1$ and $\mathbf{r}_2$ refer to coordinate systems centered at each potential, then the potential energy of the system is $V(r_1)+V(r_2)$. Now an integral equation identical in form to Eq. (5) holds for this system, except that the symbols must be reinterpreted. For this case $\mathbf{P}$ means a three-dimensional position vector, call it $\mathbf{r}$; the Green's function is $G_K(\mathbf{r}-\mathbf{r}')$ the ordinary three-dimensional Green's function; and v_t is just $v(r_1)+v(r_2)$.[2]

This problem is suggestive for the three-body one in the following way. A standard and advantageous way of writing the wave function in this problem is the so-called LCAO (linear combination of atomic orbitals) representation. In this one tries to express the solution as a sum of two functions (orbitals) the first orbital in coordinates appropriate to the first potential, and the second in coordinates appropriate to the second. The advantage of this procedure is that each of the "orbitals" at least qualitatively resembles the wave function of a particle bound to a single potential and a corollary of this is that an expansion of the orbitals in partial waves converges well. This is a very important advantage. We can see this if we consider the two limiting cases of the potentials very close together and very far apart. When the potentials are very close together there is not much difference between a single-center representation (the wave function expressed in terms of a single coordinate system, say halfway between the potentials) and the LCAO multicenter representation. But when the potentials are far apart the two-center representation is much more compact. For in it the wave function essentially becomes two S waves, one about each potential. If, however, we were to try to expand these two S waves in terms of the single center representation the convergence would be very poor indeed. The simple LCAO description involving essentially a single-particle wave becomes a slowly convergent one involving many partial waves.

To return now to our three-body problem, it is the same convergence advantage we seek in writing the wave function in the form (10). For we have done here, in the higher dimensional space corresponding to $\mathbf{r}_{12}$, $\boldsymbol{\varrho}_3$, etc., exactly the same thing as in the LCAO case. We have three potentials, and there is an obvious set of coordinates appropriate to each. We have then tried to write the solution as a sum of "two-body orbitals," i.e., as a sum of three functions, each one in the coordinates appropriate to a given potential. We hope then, and we shall see later that this hope is fulfilled, that these "orbital" functions have a rapidly convergent partial wave expansion. This is an important and even decisive advantage of writing the wave function in the form (10).

Now we return to the discussion of Eqs. (9a,b,c) for

[2] For this case $v(r)=2mV(r)/\hbar^2$ and $K^2=2m|E|/\hbar^2$.

the three-body problem. We have defined three different functions $\psi_{12}, \psi_{13}, \psi_{23}$. But since the problem is symmetric (identical masses for the particles and identical potentials acting between them) it is clear that at least for the ground state, a state of maximum symmetry, that all these functions have the same functional form, e.g., that ψ_{12} is the same function of $\mathbf{r}_{12}$, ϱ_3 that ψ_{23} is of $\mathbf{r}_{23}$, ϱ_1. Hence the subscripts on these functions are superfluous and we can call them all ψ.

$$\psi_{12}=\psi_{13}=\psi_{23}\equiv\psi. \tag{11}$$

Thus, the three equations (9) are in fact identical and we need consider only one of them, say the first. This becomes, substituting on the right-hand side for $\Psi(\mathbf{P}')$ from Eq. (10) and dropping the subscripts according to Eq. (11),

$$\psi(\mathbf{r}_{12},\varrho_3)=\int\int v(\mathbf{r}_{12}')\times[\psi(\mathbf{r}_{12}',\varrho_3')+\psi(\mathbf{r}_{13}',\varrho_2')+\psi(\mathbf{r}_{23}',\varrho_1')]\times G_K(\mathbf{r}_{12}-\mathbf{r}_{12}',\ \varrho_3-\varrho_3')d\mathbf{r}_{12}'d\varrho_3'. \tag{12}$$

This is our basic equation. It is a homogeneous integral equation for the unknown function ψ, and being homogeneous presumably has solutions only for certain values of K, the eigenvalues we seek.

The integration in Eq. (12) is over $\mathbf{r}_{12}'$ and ϱ_3'. We must then have the integrand completely expressed in these variables, and that means that we must express $\psi(\mathbf{r}_{13}',\varrho_2')$ and $\psi(\mathbf{r}_{23}',\varrho_1')$ in terms of them. The obvious way to do this is to express these functions as Fourier integrals, in which the dependence on $\mathbf{r}_{13}'$, ϱ_2' and $\mathbf{r}_{23}'$, ϱ_1' is in an exponent, and then use Eqs. (3) to transform from one coordinate system to another. Accordingly we write, e.g.,

$$\psi(\mathbf{r}_{13},\varrho_2)=\frac{1}{(2\pi)^3}\int\int\phi(\mathbf{k},\boldsymbol{\kappa})\exp[i(\mathbf{k}\cdot\mathbf{r}_{13}+\boldsymbol{\kappa}\cdot\varrho_2)]d\mathbf{k}d\boldsymbol{\kappa}$$

$$=\frac{1}{(2\pi)^3}\int\int\phi(\mathbf{k},\boldsymbol{\kappa})\exp[i\mathbf{k}\cdot(\tfrac{1}{2}\mathbf{r}_{12}-\varrho_3)-i\boldsymbol{\kappa}\cdot(\tfrac{3}{4}\mathbf{r}_{12}+\tfrac{1}{2}\varrho_3)]d\mathbf{k}d\boldsymbol{\kappa}, \tag{13}$$

where we have used $\mathbf{r}_{13}=\mathbf{r}_{12}/2-\varrho_3$, $\varrho_2=-3\mathbf{r}_{12}/4-\frac{1}{2}\varrho_3$. Having defined ϕ for the reasons given above it turns out that it is in fact easier to work with ϕ than with ψ itself. Hence we Fourier transform Eq. (12) using Eq. (13) and the integral representation for the Green's function. This leads to an eight-fold integration on the right-hand side of the transformed Eq. (12), but some of the integrations lead to δ functions and the right-hand side simplifies considerably, to give the following equation for ϕ:

$$\phi(\mathbf{k},\boldsymbol{\kappa})=-\frac{1}{(2\pi^3)(k^2+\frac{3}{4}\kappa^2+K^2)}\int\int v(r)\times\{\phi(\mathbf{k}',\boldsymbol{\kappa})\exp[i\mathbf{r}\cdot(\mathbf{k}'-\mathbf{k})]+\phi(\tfrac{1}{2}\mathbf{k}',\mathbf{k}'-2\boldsymbol{\kappa})\times\exp[i\mathbf{r}\cdot(\mathbf{k}'-\mathbf{k}-\tfrac{3}{2}\boldsymbol{\kappa}]+\phi(\tfrac{1}{2}\mathbf{k}',\mathbf{k}'-2\boldsymbol{\kappa})\times\exp[i\mathbf{r}\cdot(-\mathbf{k}'-\mathbf{k}+\tfrac{3}{2}\boldsymbol{\kappa})]\}d\mathbf{k}'d\mathbf{r}. \tag{14}$$

IV. PARTIAL WAVE EXPANSION AND THE FUNDAMENTAL EQUATION

In Eq. (14), $\phi(\mathbf{k},\boldsymbol{\kappa})$ is ostensibly a function of the two variables $\mathbf{k}$ and $\boldsymbol{\kappa}$, i.e., a function of six scalar variables. But for the symmetrical ground state, which we assume to be an S state ($\mathbf{L}=0$) symmetry considerations limit the functional dependence much more severely. Returning to $\psi(\mathbf{r}_{12},\varrho_3)$ for a moment, we observe that if this corresponds to an S state it cannot single out a direction in $\mathbf{r}_{12}$, ϱ_3 space, but must depend only on $\mathbf{r}_{12}$, ϱ_3 and the angle between them. A similar remark then applies to the dependence of ϕ on $\mathbf{k}$ and $\boldsymbol{\kappa}$; ϕ must be a function only of k, κ and $\mathbf{k}\cdot\boldsymbol{\kappa}$. In this case we can expand it in the following form

$$\phi(\mathbf{k},\boldsymbol{\kappa})=\sum_{l=0}^{\infty}\phi_l(k,\kappa)P_l(\cos\gamma), \tag{15}$$

where γ is the angle between $\mathbf{k}$ and $\boldsymbol{\kappa}$. If we put this into Eq. (14) we can then equate coefficients of $P_l(\cos\gamma)$ on the left- and right-hand sides and so get an infinite set of coupled equations for the functions $\phi_l(k,\kappa)$. We derive these equations now.

Since ϕ depends only the angle between $\mathbf{k}$ and $\boldsymbol{\kappa}$, it is permissible (and convenient) to take $\boldsymbol{\kappa}$ along the z axis. For the other two vectors $\mathbf{r}$, $\mathbf{k}'$, that appear in Eq. (14) we specify that they, resepctively, make angles θ and γ' with the z axis. Then in the integrand of Eq. (14) we can write

$$\phi(\mathbf{k}',\boldsymbol{\kappa})=\sum_{l=0}^{\infty}\phi_l(k',\kappa)P_l(\cos\gamma'), \tag{16}$$

and

$$\phi(\tfrac{1}{2}\mathbf{k}',\mathbf{k}'-2\boldsymbol{\kappa})=\sum_{l=0}^{\infty}\phi_l(\tfrac{1}{2}k',(k'^2+4\kappa^2-4k'\kappa\cos\gamma')^{\frac{1}{2}})\times P_l\left(\frac{\mathbf{k}'\cdot(\mathbf{k}'-2\boldsymbol{\kappa})}{|\mathbf{k}'||(\mathbf{k}'-2\boldsymbol{\kappa})|}\right);$$

or with the definition

$$h(\cos\gamma')=(k'^2+4\kappa^2-4k'\kappa\cos\gamma')^{\frac{1}{2}},$$

$$\phi(\tfrac{1}{2}\mathbf{k}',\mathbf{k}'-2\boldsymbol{\kappa})=\sum_{l=0}^{\infty}\phi_l(\tfrac{1}{2}k',h(\cos\gamma'))\times P_l\left(\frac{k'-2\kappa\cos\gamma'}{h(\cos\gamma')}\right). \tag{17}$$

Now we expand the exponentials in the integrals in (14) using, as necessary, either the general formula

$$\exp(i\mathbf{a}\cdot\mathbf{b})=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l j_l(ab)Y_{lm}(\Omega_a)Y_{lm}^*(\Omega_b), \quad (18)$$

or the special case of it when $\mathbf{b}$ is along the z axis, and α is the angle between $\mathbf{a}$ and the z axis

$$\exp(i\mathbf{a}\cdot\mathbf{b})=\sum_{l=0}^{\infty} i^l(2l+1)j_l(ab)P_l(\cos\alpha). \quad (19)$$

We consider the first integral in Eq. (14)

$$\int\int v(r)\phi(\mathbf{k}',\kappa)\exp[i\mathbf{r}\cdot(\mathbf{k}'-\mathbf{k})]d\mathbf{k}'d\mathbf{r} = \int \phi(\mathbf{k}',\kappa)w(\mathbf{k}-\mathbf{k}')d\mathbf{k}', \quad (20)$$

where

$$w(\mathbf{k}-\mathbf{k}')=\int v(r)\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]d\mathbf{r}. \quad (21)$$

If v is a function only of the magnitude of $\mathbf{r}$, as the notation indicates, w is a function only of $|\mathbf{k}-\mathbf{k}'|$. We use the expansion (18) for $\exp(i\mathbf{k}\cdot\mathbf{r})$ and $\exp(-i\mathbf{k}'\cdot\mathbf{r})$ and do the integration over the solid angle $d\Omega_r$ to get

$$w(\mathbf{k}-\mathbf{k}')=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} w_l(k,k')Y_{lm}^*(\Omega_k)Y_{lm}(\Omega_{k'}), \quad (22)$$

where

$$w_l(k,k')=4\pi\int_0^{\infty} v(r)j_l(kr)j_l(k'r)r^2dr. \quad (23)$$

We put the expansion (23) into Eq. (20) and do the integration over the solid angle of $d\Omega_{k'}$ and (20) becomes

$$\begin{aligned}&\int\int v(r)\phi(\mathbf{k}',\kappa)\exp[i\mathbf{r}\cdot(\mathbf{k}'-\mathbf{k})]d\mathbf{k}'d\mathbf{r}\\ &=4\pi\int\int\Big(\sum_{l=0}^{\infty}\phi_l(k',\kappa)P_l(\cos\gamma')\Big)\\ &\quad\times\Big(\sum_{l_1=0}^{\infty}\sum_{m_1=-l_1}^{l_1} w_{l_1}(k,k')Y_{l_1m_1}^*(\Omega_k)Y_{l_1m_1}(\Omega_{k'})\Big)k'^2dk'd\Omega_{k'}\\ &=4\pi\sum_{l=0}^{\infty}\Big(P_l(\cos\gamma)\int_0^{\infty}\phi_l(k',\kappa)w_l(k,k')k'^2dk'\Big). \quad (24)\end{aligned}$$

This is the desired expansion in $P_l(\cos\gamma)$. Now we consider the second integral

$$\int\int v(r)\phi(\tfrac{1}{2}\mathbf{k}',\mathbf{k}'-2\kappa)\times\exp[i\mathbf{r}\cdot(\mathbf{k}'-\mathbf{k}-\tfrac{3}{2}\kappa)]d\mathbf{k}'d\mathbf{r} \quad (25)$$

If we use Eq. (17), expand $\exp(i\mathbf{r}\cdot\mathbf{k}')$ by Eq. (18) and do the integration over the solid angle of $\Omega_{k'}$ ($d\Omega_{k'}=2\pi k'^2\sin\gamma' dk'd\gamma'$) we find that (25) becomes

$$\begin{aligned}&2\pi\int\int\int v(r)\Big\{\sum_{l_1=0}^{\infty}\phi_{l_1}(\tfrac{1}{2}k',h(\cos\gamma'))P_{l_1}\Big(\frac{k'-2\kappa\cos\gamma'}{h(\cos\gamma')}\Big)\Big\}\\ &\quad\times\Big\{\sum_{l_2=0}^{\infty} i^{l_2}(2l_2+1)j_{l_2}(k'r)P_{l_2}(\cos\theta)P_{l_2}(\cos\gamma')\Big\}\\ &\quad\times\exp[-i\mathbf{r}\cdot(\mathbf{k}+\tfrac{3}{2}\kappa)]k'^2\sin\gamma' dk'd\gamma' d\mathbf{r}. \quad (26)\end{aligned}$$

Now since κ is along the z axis,

$$\exp[-i\mathbf{r}\cdot\tfrac{3}{2}\kappa]=\sum_{l_3=0}^{\infty}(-i)^{l_3}(2l_3+1)j_{l_3}(\tfrac{3}{2}\kappa r)P_{l_3}(\cos\theta).$$

Also

$$\exp[-i\mathbf{r}\cdot\mathbf{k}]=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^l j_l(kr)Y_{lm}(\Omega_r)Y_{lm}^*(\Omega_k).$$

Putting these into Eq. (25) and doing the integration over the solid angle $d\Omega_r$, we find that by orthogonality all terms with $m\neq 0$ integrate to zero and we are left with

$$\begin{aligned}&(2\pi)^2\int_0^{\infty}\sin\gamma' d\gamma'\int_0^{\pi}\sin\theta d\theta\int_0^{\infty}r^2dr\int_0^{\infty}k'^2dk'\\ &\times\Big\{v(r)\Big[\sum_{l_1=0}^{\infty}\phi_{l_1}(\tfrac{1}{2}k',h(\cos\gamma'))P_{l_1}\Big(\frac{k'-2\kappa\cos\gamma'}{h(\cos\gamma')}\Big)\Big]\\ &\times\Big[\sum_{l_2=0}^{\infty}(2l_2+1)i^{l_2}j_{l_2}(k'r)P_{l_2}(\cos\theta)P_{l_2}(\cos\gamma')\Big]\\ &\times\Big[\sum_{l_3=0}^{\infty}(2l_3+1)(-i)^{l_3}j_{l_3}(\tfrac{3}{2}\kappa r)P_{l_3}(\cos\theta)\Big]\\ &\times\Big[\sum_{l=0}^{\infty}(2l+1)(-i)^l j_l(kr)P_l(\cos\theta)P_l(\cos\gamma)\Big]\Big\}. \quad (27)\end{aligned}$$

This is the expansion in terms of $P_l(\cos\gamma)$ that we seek. We simplify its aspect by defining a function W_{rst},

$$W_{rst}(k,k',\kappa)=4\pi\int_0^{\infty}v(r)j_r(kr)j_s(k'r)j_t(\tfrac{3}{2}\kappa r)r^2dr, \quad (28)$$

and use the well-known result for the integral of three Legandre polynomials

$$\int_0^{\pi}P_lP_{l_2}P_{l_3}\sin\theta d\theta\equiv C(l,l_2,l_3)=2\begin{pmatrix} l & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix}^2, \quad (29)$$

where in the notation of Edmonds[3] $\begin{pmatrix} l & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix}$ is a

[3] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1951).

3-j coefficient. With these definitions, (26) becomes

$$\pi\sum_{l=0}^{\infty}(2l+1)(-i)^l P_l(\cos\gamma)\Big\{\int_0^{\pi}\sin\gamma' d\gamma'\int_0^{\infty}k'^2dk'\Big[\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{l_3=0}^{\infty}(2l_2+1)(2l_3+1)i^{l_2}(-i)^{l_3}\phi_{l_1}(\tfrac{1}{2}k',h(\cos\gamma'))$$
$$\times P_{l_1}\Big(\frac{k'-2\kappa\cos\gamma'}{h(\cos\gamma')}\Big)W_{ll_2l_3}(k,k',\kappa)C(l,l_2,l_3)P_{l_2}(\cos\gamma')\Big]\Big\}. \quad (30)$$

We do a similar expansion for the last integral in Eq. (14), which simply involves replacing i by $-i$ in two places and put it, Eq. (30), and Eq. (24) back into (14) and equate coefficients of $P_l(\cos\gamma)$ to get the final set of coupled equations for the functions $\phi_l(k,\kappa)$

$$\phi_l(k,\kappa)=-\frac{1}{2\pi^2(k^2+\frac{3}{4}\kappa^2+K^2)}\Big\{\int_0^{\infty}\phi_l(k',\kappa)w_l(k,k')k'^2dk'+(-i)^l\frac{(2l+1)}{4}\int_0^{\pi}\sin\gamma'd\gamma'\int_0^{\infty}k'^2dk'$$
$$\times\Big[\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{l_3=0}^{\infty}i^{l_2+l_3}(2l_2+1)(2l_3+1)((-)^{l_2}+(-)^{l_3})\phi_{l_1}(\tfrac{1}{2}k',h(\cos\gamma'))P_{l_1}\Big(\frac{k'-2\kappa\cos\gamma'}{h(\cos\gamma')}\Big)$$
$$\times W_{ll_2l_3}(k,k',\kappa)C(l,l_2,l_3)P_{l_2}(\cos\gamma')\Big]\Big\}. \quad (31)$$

One point of writing the wave function as we did was the expectation that this would lead to a good convergence for the expansion (15). In the light of this we shall, in this section, discuss the equation obtained by truncating Eqs. (31) and keeping only the first term, i.e., assuming only ϕ_0 is different from zero. We estimate the error involved in this truncation in Sec. VII; for the moment we simply take it as a working hypothesis.

In Eq. (31) then, we set $l=0$ and on the right-hand side retain only the first term in the sum over l_1. With $l=0$, $C(0,l_2,l_3)$ is zero unless $l_2=l_3$ so the double sum over l_2 and l_3 reduces to a single one. Finally we set $y=\cos\gamma'$ and get the basic homogeneous integral equation for $\phi_0(k,\kappa)$,

$$\phi_0(k,\kappa)=-\frac{1}{2\pi^2(k^2+\frac{3}{4}\kappa^2+K^2)}\Big\{\int_0^{\infty}\phi_0(k',\kappa)w_0(k,k')k'^2dk'$$
$$+\int_0^{\infty}k'^2dk'\int_{-1}^{1}\phi_0(\tfrac{1}{2}k',(k'^2+4\kappa^2-4k'\kappa y)^{\frac{1}{2}})\sum_{l=0}^{\infty}(2l+1)W_{0ll}(k,k',\kappa)P_l(y)dy\Big\}. \quad (32)$$

V. APPROXIMATE SOLUTION: ALGORITHM FOR THE EIGENVALUE

In this section we study Eq. (32) and propose a method for finding its lowest eigenvalue and corresponding eigenfunction.

We first note that the interparticle potential enters Eq. (32) through $w_0(k,k')$ and $W_{0ll}(k,k',\kappa)$. For a given potential *shape* each of these quantities is proportional to the potential *strength*, and the solution of Eq. (32) gives K as a function of this strength. In what follows it is convenient to turn the problem around and to imagine that K is given, and the potential strength is the eigenvalue we seek. This is equally acceptable, since it enables us to find what we want, the relation between potential strength and the eigenvalue K.

Supposing K given, one obvious way to try to solve Eq. (32) is by iteration, i.e., by putting some reasonable approximation for ϕ_0 into the integrals on the right of Eq. (32) and getting an improved approximation to ϕ_0 on the left-hand side. This is the method we adopt. Before we expand on it, however, it is worth commenting briefly on the solution of an equation which resembles Eq. (32) somewhat, although it is much simpler. This is the equation which is the analog of Eq. (32) for a two-body system, i.e., of two particles bound by a potential $v(r)$.

If we call $\tilde{\phi}_{00}(k)$ the S-wave part of the momentum space wave function, then $\tilde{\phi}_{00}(k)$ satisfies the following equation, which is derived in the Appendix:

$$\tilde{\phi}_{00}(k)=-\frac{1}{2\pi^2(k^2+K^2)}\int_0^{\infty}\tilde{\phi}_{00}(k')w_0(k,k')k'^2dk'. \quad (33)$$

This equation can be solved exactly for the exponential potential, and the results are given in the Appendix. The points we wish to make about it are two. First,

for forces of finite range [i.e., for which $v(r)$ is essentially zero when r is greater than some radius r_0], the behavior of $\tilde{\phi}_{00}(k)$ for small k is given by the factor $1/(K^2+k^2)$ that stands in front of the integral. This is because this factor is the Fourier transform of e^{-Kr}/r, the space wave function for $r>r_0$. The smaller r_0 is the larger the range of k over which $1/(K^2+k^2)$ is a good approximation to the wave function. The second point is that if one tries an iteration procedure with this equation, using $1/(K^2+k^2)$ as the zeroth order iterating function, one gets quite good results, even the first iterate being a fair approximation to the wave function. Results of such an iteration are derived in the Appendix and plotted in Fig. 3.

These results make it at least suggestive that a similar iteration procedure can be applied to Eq. (32) for the three-body problem. Again a reasonable first approximation to $\phi_0(k,\kappa)$ is the function that stands in front of the integral, namely $1/(k^2+\frac{3}{4}\kappa^2+K^2)$. The reason is the same as in the two-body case: for short-range forces this factor is essentially the Fourier transform of that part of the coordinate space function $\psi(\mathbf{r},\boldsymbol{\varrho})$ which corresponds to free motion of the particles, i.e., when they are outside the range of each others force. As the range of the forces gets smaller, one would expect that the range of k and κ over which $1/(k^2+\frac{3}{4}\kappa^2+K^2)$ is a good approximation to the wave function becomes larger and larger. Actually it is not difficult to imagine better first iterates but we shall confine ourselves to this one, since our aim in this paper is as much to establish the general validity of the equations and estimate the errors of the various approximations as to get very precise numerical results. Moreover, as we shall see shortly, one can find quite a precise eigenvalue with even a relatively poor wave function, and in any case it is always interesting to see how far one can get with the simplest approximation.

Assuming we have found $\phi_0(k,\kappa)$, by iteration or otherwise, the question remains: how to find the relation between K and the potential strength? Of course, this could be done variationally, but this has the great disadvantage that the integrals that arise are tedious, if they can be done at all. Here we shall present a method which is very much simpler, which appears to be accurate, and which seems to share the virtue of the variational method that the accuracy of the eigenvalue is appreciably better than the accuracy of the eigenfunction.

We begin the discussion of this method by setting $\kappa=0$ in Eq. (32). On the left this gives $\phi_0(k,0)$. Now, this quantity has a simple interpretation in terms of $\psi(\mathbf{r},\boldsymbol{\varrho})$. For we have from the inverse of Eq. (13)

$$\phi(\mathbf{k},\boldsymbol{\kappa})=\frac{1}{(2\pi)^3}\int\int\psi(\mathbf{r},\boldsymbol{\varrho})\,\exp[-i(\mathbf{k}\cdot\mathbf{r}+\boldsymbol{\kappa}\cdot\boldsymbol{\varrho})]d\mathbf{r}d\boldsymbol{\varrho},$$

$$\phi(\mathbf{k},0)=\frac{1}{(2\pi)^3}\int\exp(-i\mathbf{k}\cdot\mathbf{r})\left[\int\psi(\mathbf{r},\boldsymbol{\varrho})d\boldsymbol{\varrho}\right]d\mathbf{r}.$$

Thus $\phi(\mathbf{k},0)$ is the Fourier transform of a kind of "effective radial wave function," by which we mean the wave function $\psi(\mathbf{r},\boldsymbol{\varrho})$ integrated over the variable $\boldsymbol{\varrho}$, and $\phi_0(k,0)$ is of course the S-wave part of $\phi(\mathbf{k},0)$. Then it is not too surprising that, as we shall see, $\phi_0(k,0)$ satisfies an equation which closely resembles a two-body S-wave equation.

From the defining equation (28) we see that

$$W_{0ll}(k,k',0)$$

is different from zero only for $l=0$. If then we set $\kappa=0$ in Eq. (32) the sum over l reduces to a single term. Moreover,

$$W_{000}(k,k',0)=w_0(k,k'),$$

so that Eq. (32) becomes, with a little rearrangement,

$$\phi_0(k,0)=-\frac{1}{2\pi^2(k^2+K^2)}\int_0^\infty\phi_0(k',0)\left\{1+\frac{2\phi_0(\frac{1}{2}k',k')}{\phi_0(k',0)}\right\}\times w_0(k,k')k'^2dk'. \quad (34)$$

Except for the factor in curly brackets this equation looks very much like the two-body equation (33). We can put it into an even more similar form in the following way. We define a function $g(k)$:

$$g(k)=\left(1+\frac{2\phi_0(\frac{1}{2}k,k)}{\phi_0(k,0)}\right)^{\frac{1}{2}}. \quad (35)$$

Then Eq. (34) can be written

$$g(k)\phi_0(k,0)=-\frac{1}{2\pi^2(k^2+K^2)}\int_0^\infty g(k')\phi_0(k',0)\times w_0(k,k')g(k)g(k')k'^2dk'.$$

If now we define new "effective" functions,

$$\phi_{\text{eff}}(k)=g(k)\phi_0(k,0) \quad (36)$$

and

$$w_{\text{eff}}(k,k')=w_0(k,k')g(k)g(k'), \quad (37)$$

we find that

$$\phi_{\text{eff}}(k)=-\frac{1}{2\pi^2(k^2+K^2)}\int_0^\infty\phi_{\text{eff}}(k')w_{\text{eff}}(k,k')k'^2dk'. \quad (38)$$

This has just the form of Eq. (33), the two-body equation in momentum space. The ground-state eigenvalue of the three-body problem is the lowest eigenvalue of this "effective" two-body equation. We can express this in terms of an effective space potential if we remember that in an equation like (38) w_{eff} defines an effective potential $v_{\text{eff}}(r)$, using Eq. (A8) of the Appendix.

$$v_{\text{eff}}(r)=\frac{\sqrt{3}}{2\pi^2}\int_0^\infty w_0(k,0)g(k)j_0(kr)k^2dk. \quad (39)$$

We now briefly discuss the physical meaning of this

effective potential. As we shall see later, $g(k)$ is a slowly varying function of k whose average magnitude is somewhat larger than unity (and is in fact between unity and the square root of three). Thus $v_{\text{eff}}(r)$ is somewhat larger than $v(r)$. To see why this must be so, consider the potential energy which is felt by one particle, call it particle 1. When all three particles are far apart, its potential energy is zero (for short-range forces). When particle 1 is close to particle 2 and the third is far away, then the first particle feels a potential $v(r)$. If however particle 2 and 3 coincide, particle 1 feels the central potential $3v(r)$. Finally, when particles 2 and 3 are close, but do not coincide, particle 1 sees a potential which is no longer spherically symmetric, and which has a magnitude between $v(r)$ and $3v(r)$. Equation (39) is then the mathematical statement of this situation in which one particle "screens" the other. This is very similar to the He atom, for example, where crudely one can say that one electron "screens" the nucleus so that the second electron feels a weakened value of the nuclear charge. Of course the "screening" we describe above works in the opposite direction: it effectively strengthens the interparticle potential, but physically it is the same mechanism.

We can then find the eigenvalue of the three-body problem if we can calculate $v_{\text{eff}}(r)$, which means that we must be able to calculate the ratio $\phi_0(\frac{1}{2}k,k)/\phi_0(k,0)$. To calculate this ratio we will of course use the approximate iterated solutions described above. The point we wish to emphasize is that this ratio turns out to be a very slowly varying function of k, and it seems very likely that in forming it from an approximate or inexact ϕ_0, that the accuracy of the ratio is much better than the accuracy of either the numerator or denominator, any errors in the function ϕ_0 itself tending to cancel out. The situation is perhaps similar to that in the variational principle, where even an inaccurate wave function can lead to a relatively accurate eigenvalue. The difference of course is that the variational principle is based on a mathematical theorem, and the "principle" we have stated above is simply based on hope. One test of these ideas is of course to try them on some special potential, and compare the results with other calculations, where they exist. This is done in the next section.

VI. EXAMPLE: EXPONENTIAL POTENTIAL

We shall work out the method outlined above for the special case of exponential interparticle potentials. The exponential potential has the advantage that it is one of the few continuous potentials for which the two-body problem (for $l=0$) can be solved and this simplifies somewhat the numerical calculations for the three-body problem. In this section we shall make whatever approximations we consider necessary without trying to justify them at the time. In the next section we then discuss the accuracy of these approximations.

The exponential interparticle potential $v(r)$ is written as

$$v(r)=-b^2e^{-r/d}. \tag{40}$$

The integral in Eq. (23) defining $w_0(k,k')$ is then elementary and yields

$$w_0(k,k')=-\frac{2\pi b^2 d}{kk'}\left(\frac{1}{1+d^2(k-k')^2}-\frac{1}{1+d^2(k+k')^2}\right). \tag{41}$$

We use the following notation for the iteration procedure. We call the zeroth order iterate $\phi_0^{(0)}$. As we have discussed, this is just the factor[4] that stands in front of the integral in Eq. (32):

$$\phi_0^{(0)}=\frac{1}{k^2+\frac{3}{4}\kappa^2+K^2}. \tag{42}$$

The first iterate $\phi_0^{(1)}$ is of course the function obtained by putting this into the right-hand side of Eq. (32) and evaluating the integral, and the nth iterate $\phi_0^{(n)}$ is defined analogously, although in practice of course one is limited to one or possibly two analytic iterations, since the complexity of the integrations grows rapidly with n. From Eq. (32) the first iterate is

$$\phi_0^{(1)}(k,\kappa)=\frac{1}{k^2+\frac{3}{4}\kappa^2+K^2}\left\{\int_0^\infty \frac{w_0(k,k')k'^2dk'}{k'^2+\frac{3}{4}\kappa^2+K^2}\right.$$
$$\left.+\int_0^\infty k'^2dk'\int_{-1}^1 \frac{\sum_{l=0}^\infty (2l+1)W_{0ll}(k,k',\kappa)P_l(y)dy}{k'^2+3\kappa^2-3k'\kappa y+K^2}\right\}. \tag{43}$$

To help in doing the integrations, we expand the denominator in the second integral in Eq. (43)

$$\frac{1}{K^2+k'^2+3\kappa^2-3k'\kappa y}=\sum_{m=0}^\infty \frac{(3k'\kappa y)^m}{(K^2+k'^2+3\kappa^2)^{m+1}}. \tag{44}$$

We then have

$$\phi_0^{(1)}(k,\kappa)=\frac{1}{k^2+\frac{3}{4}\kappa^2+K^2}$$
$$\times\left(f(k,\kappa)+\sum_{l=0}^\infty\sum_{m=0}^\infty (2l+1)F_{lm}(k,\kappa)\right), \tag{45}$$

where

$$f(k,\kappa)=\int_0^\infty \frac{w_0(k,k')k'^2dk'}{k'^2+\frac{3}{4}\kappa^2+K^2}, \tag{46}$$

and

$$F_{lm}(k,\kappa)=\int_0^\infty \frac{k'^2W_{0ll}(k,k',\kappa)(3k'\kappa)^m dk'}{(k'^2+3\kappa^2+K^2)^{m+1}}\int_{-1}^1 P_l(y)y^m dy. \tag{47}$$

[4] We drop irrelevant numerical factors.

We note that F_{lm} vanishes unless the y integrand is even, that is, unless l and m are both even or both odd. As we shall see, the contributions of the higher F_{lm} to the series for $\phi_0^{(1)}$ fall off rapidly with l and m and we shall make the approximation of truncating the series, keeping only F_{00} and F_{11}.

The integrals involved in f, F_{00}, and F_{11} can be done by using in them the definitions (23) and (28) for w_0 and W_{0ll} and interchanging the order of integration in the resultant double integrals. We first quote the results and then outline their derivation. The results are:

$$f(k,\kappa)=-\frac{2\pi^2(bd)^2}{k_0^2+[1+\beta(\kappa_0)]^2}, \tag{48}$$

$$F_{00}(k,\kappa)=-\frac{2\pi^2(bd)^2}{3k_0\kappa_0}\ln\frac{[1+\alpha(\kappa_0)]^2+(k_0+\frac{3}{2}\kappa_0)^2}{[1+\alpha(\kappa_0)]^2+(k_0-\frac{3}{2}\kappa_0)^2}, \tag{49}$$

$$F_{11}(k,\kappa)=-\frac{2\pi^2(bd)^2}{9k_0\kappa_0}\left\{\ln\frac{[1+\alpha(\kappa_0)]^2+(k_0+\frac{3}{2}\kappa_0)^2}{[1+\alpha(\kappa_0)]^2+(k_0-\frac{3}{2}\kappa_0)^2}-\frac{3(k_0+\frac{3}{2}\kappa_0)\kappa_0}{[1+\alpha(\kappa_0)]^2+(k_0+\frac{3}{2}\kappa_0)^2}-\frac{3(k_0-\frac{3}{2}\kappa_0)\kappa_0}{[1+\alpha(\kappa_0)]^2-(k_0+\frac{3}{2}\kappa_0)^2}\right\}, \tag{50}$$

where

$$K_0=Kd,\quad k_0=kd,\quad \kappa_0=\kappa d, \tag{51a}$$

$$\alpha(\kappa_0)=(K_0^2+3\kappa_0^2)^{\frac{1}{2}}, \tag{51b}$$

$$\beta(\kappa_0)=(K_0^2+\tfrac{3}{4}\kappa_0^2)^{\frac{1}{2}}. \tag{51c}$$

By rewriting Eq. (49) and expanding as follows, we get a useful series expression for F_{00}.

$$F_{00}(k,\kappa)=-\frac{2\pi^2(bd)^2}{3k_0\kappa_0}\times\ln\left\{\frac{1+\dfrac{3k_0\kappa_0}{[1+\alpha(\kappa_0)]^2+k_0^2+(9\kappa_0^2/4)}}{1-\dfrac{3k_0\kappa_0}{[1+\alpha(\kappa_0)]^2+k_0^2+(9\kappa_0^2/4)}}\right\} \tag{52}$$

$$=-4\pi^2(bd)^2\left\{\frac{1}{[1+\alpha(\kappa_0)]^2+k_0^2+(9\kappa_0^2/4)}+\frac{3(k_0\kappa_0)^2}{\{[1+\alpha(\kappa_0)]^2+k_0^2+(9\kappa_0^2/4)\}^3}+\cdots\right\}.$$

The above results were derived as follows. For $f(k,\kappa)$, for example, we have, on using (23) and (40),

$$f(k,\kappa)=-4\pi(bd)^2\int_0^\infty e^{-x}j_0(k_0x)x^2dx\int_0^\infty\frac{j_0(xy)y^2dy}{y^2+\beta^2(\kappa_0)}.$$

The y integration gives $(\pi/2x)\exp[-x\beta(\kappa_0)]$, and the x integration is then elementary. For F_{00} we get in the same way

$$F_{00}=-4\pi^2(bd)^2\int_0^\infty\exp\{-x[1+\alpha(\kappa_0)]\}\times j_0(k_0x)j_0(\tfrac{3}{2}\kappa_0x)xdx.$$

This can be done by using an integral representation[5] for j_0. Alternatively, we note that it is a special case of a standard integral in the theory of Bessel functions,[6] viz.

$$\int_0^\infty e^{-at}j_n(bt)j_n(ct)tdt=\frac{1}{2bc}Q_n\left(\frac{a^2+b^2+c^2}{2bc}\right), \tag{53}$$

where Q_n is a Legandre polynomial of the second kind. For F_{11} we are led to

$$F_{11}=-8\pi(bd)^2\kappa_0\int_0^\infty e^{-x}j_0(k_0x)j_0(\tfrac{3}{2}\kappa_0x)dx\times\int_0^\infty\frac{j_1(xy)y^2dy}{[y^2+\alpha^2(\kappa_0)]^2}.$$

The y integration in this last expression deserves some comment. It is

$$\int_0^\infty\frac{j_1(xy)y^2dy}{(y^2+\alpha^2)^2}=\tfrac{1}{4}\pi e^{-\alpha x},$$

for x different from zero. For $x=0$ the integral is zero; therefore the integral is a discontinuous function of x at $x=0$, but this causes no difficulty for our purposes.

Now we wish to calculate the function $g(k)$ defined by Eq. (35) for it is this that gives the effective potential according to Eq. (39). At this point it is useful to be a little more explicit about $g(k)$ and what it depends on. For the exponential potential we deal with we shall write $g_{\exp}(k_0,K_0)$ instead of $g(k)$, to emphasize that it depends on k through the product kd and is a function of Kd as well.

$$g(k)\longrightarrow g_{\exp}(k_0,K_0).$$

We note first that if we form $g_{\exp}(k_0,K_0)$ with the zeroth order iterate, it is independent of k_0 and K_0 and is just equal to the square root of three.

$$g_{\exp}(k_0,K_0)=\sqrt{3},\text{ calculated with }\phi_0^{(0)}.$$

In the next approximation we evaluate $g_{\exp}(k_0,K_0)$ with $\phi_0^{(1)}$, using the above results for f, F_{00}, and F_{11}

[5] P. More and H. Feshbach, *Methods of Theoretical Physics* (McGraw Hill Book Company, Inc., New York, 1953), p. 622.
[6] P. Morse and H. Feshbach, reference 5, p. 1575.

(still dropping higher F_{lm}). We get

$$g_{\exp}(k_0,K_0) = \left\{1+\tfrac{2}{3}[k_0^2+(1+K_0)^2]\left[\frac{1}{\frac{1}{4}k_0^2+[1+\beta(k_0)]^2} + \frac{4}{3k_0^2}\ln\left(\frac{[1+\alpha(k_0)]^2+4k_0^2}{[1+\alpha(k_0)]^2+k_0^2}\right)+\frac{2}{[1+\alpha(k_0)]^2+k_0^2} - \frac{4}{[1+\alpha(k_0)]^2+4k_0^2}\right]\right\}^{\frac{1}{2}}. \quad (54)$$

In Fig. 1 we plot this expression for $g_{\exp}(k_0K_0)$. We see that for given K_0 it is a function which drops sharply from its value at the origin and then varies very slowly. This has an interesting consequence for the effective potential which we calculate from Eq. (39). For if $g_{\exp}$ were strictly a constant, we see that the effective potential $v_{\text{eff}}(r)$ would be of exactly the same *shape* as the interparticle potential $v(r)$ but would be stronger by the factor $\sqrt{3}g_{\exp}$. Now for $g_{\exp}$ of the form shown in Fig. 1, it is still a good approximation to assume that $v_{\text{eff}}(r)$ is the same *shape* but of different *strength*, especially since in the calculation of $v_{\text{eff}}(r)$ from Eq. (39) the values of $g_{\exp}$ near the origin (where it varies most rapidly) are not weighed heavily due to the factor k^2 in the integrand.

We can put this more precisely as follows. For the exponential potential, Eq. (39) for v_{eff} becomes

$$v_{\text{eff}}(r) = -\frac{4\sqrt{3}b^2}{\pi}\int_0^\infty \frac{g_{\exp}(k_0,K_0)j_0(k_0r/d)k_0^2dk_0}{(1+k_0^2)^2}. \quad (55)$$

Now we define a constant $\bar{g}$, which is essentially the mean value of $g_{\exp}(k_0,K_0)$ [with weight function $w_0(k,0)$]:

$$\bar{g} = \int_0^\infty w_0(k_0,0)g_{\exp}(k_0,K_0)k_0^2dk_0 \Big/ \int_0^\infty w_0(k_0,0)k_0^2dk_0 = \frac{4}{\pi}\int_0^\infty \frac{g_{\exp}(k_0,K_0)k_0^2dk_0}{(1+k_0^2)^2}.$$

By adding and subtracting the same quantity in Eq. (55), it can be rewritten in terms of $\bar{g}$:

$$v_{\text{eff}}(r) = -\sqrt{3}\frac{b^24}{\pi}\left\{\bar{g}\int_0^\infty \frac{j_0(k_0r/d)k_0^2dk_0}{(1+k_0^2)^2} + \int_0^\infty \frac{[g_{\exp}(k_0,K_0)-\bar{g}]j_0(k_0r/d)k_0^2dk_0}{(1+k_0^2)^2}\right\},$$

or

$$v_{\text{eff}}(r) = -b^2\bar{g}\sqrt{3}e^{-r/d} - \frac{4b^2\sqrt{3}}{\pi}\int_0^\infty \frac{[g_{\exp}(k_0,K_0)-\bar{g}]j_0(k_0r/d)k_0^2dk_0}{(1+k_0^2)^2}. \quad (56)$$

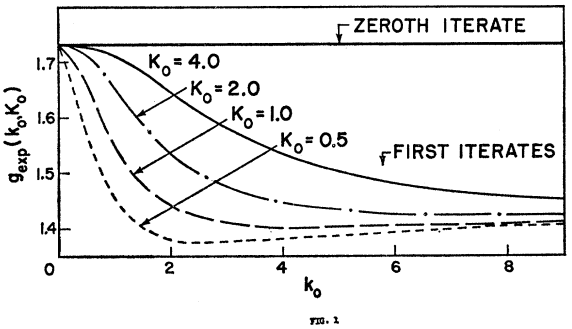

FIG. 1. The function $g_{\exp}(k_0,K_0)$ which gives the effective two-body potential according to Eq. (55). As indicated, $g_{\exp}(k_0,K_0)$ has the constant value $\sqrt{3}$ when evaluated with the zeroth order iterate, Eq. (42). The curves give $g_{\exp}$ as evaluated from the first iterate, Eq. (45).

Here we have written $v_{\text{eff}}(r)$ as an exponential potential with a strength $b^2\bar{g}\sqrt{3}$, plus a correction represented by the integral term in the last equation. The point of choosing $\bar{g}$ as we have is that, comparing (39) with (56), it makes the exponential potential have the correct value at the origin; for $r=0$ the correction term vanishes. We expect then that for r/d small the correction remains small; only when r/d becomes large compared with unity does it become relatively large, but by this time the potential is very small anyway. For this section then we just take the first term in (56),

$$v_{\text{eff}} \approx -b_{\text{eff}}^2e^{-r/d}, \quad (57)$$

where

$$b_{\text{eff}}^2 = \sqrt{3}\bar{g}b^2. \quad (58)$$

Now it is easy to write the algorithm for finding the three-body binding energy versus b. If (57) is a good approximation to the shape of the potential, then K_0 and b_{eff}^2 are connected by the relation between potential strength and binding energy for the two-body problem. From a numerical value of K_0, we get a numerical value of b_{eff}. But from this value of b_{eff} we get a value of b from Eq. (58) and so we can plot K_0 versus b, or since b always enters multiplied by d, K_0 versus bd. This is done in Fig. 1. In this figure we also compare our results with the variational results of Feshbach[7] on the same problem. We see that our result for the absolute magnitude of the energy is always *larger* (the energy itself, being negative, is always smaller) than the variational result. This is satisfactory in the sense that the *exact* magnitude of the energy is necessarily larger than the variational result, but of course it may be that the values we get with the present method are *too* large. Apropos of this we should remember that there is evidence that the Feshbach variational result gives the magnitude of the energy a few percent too small, for b_0 about 2.8, by comparison with the more extensive variational calculations of Rarita and Present.[8] But

[7] J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).
[8] W. Rarita and R. D. Present, Phys. Rev. **51**, 788 (1937).

even this comparison is not unambiguous, however, since the Rarita and Present calculations were done for spin-dependent forces. Of course for small b_0, where the variational calculation gives K_0 near zero, the ratio of our result to the variational one becomes infinite, but this again is a consequence of the fact that the variational calculation necessarily gives the result $K_0=0$ for a larger value of b_0 than the true one.

VII. ACCURACY OF THE APPROXIMATIONS

In this section we discuss the various approximations we have made above.

First, we dropped all the F_{lm} except F_{00} and F_{11} in expanding the integral in (43). To see what this involves, let us first discuss the quantities F_{lm} for a given l. The nonvanishing integrals are then $F_{l0}, F_{l2}, F_{l4}\cdots$ for l even and $F_{l1}, F_{l3}, F_{l5}\cdots$ for l odd. The reason that the magnitudes of these integrals decrease as m increases is that successive integrals have an additional factor

$$\left(\frac{3k'\kappa}{k'^2+3\kappa^2+K^2}\right)^2 \tag{59}$$

in the integrand. This factor is always less than unity as can be seen by writing it in the form $\{3k'\kappa/[k'^2+(9\kappa^2/4)+K^2+\frac{3}{4}\kappa^2]\}^2$ and noting that $3k'\kappa$ is always less than $k'^2+9\kappa^2/4$. Just how much less depends of course on all three variables, but we can get an idea of how this factor diminishes the integrals by calculating F_{02} and comparing it with F_{00}. We would then expect roughly that F_{04} is smaller compared to F_{02} by the same ratio that F_{02} is compared with F_{00}. In addition to this factor, the m dependence of the y integral in Eq. (47) tends to diminish successive integrals by a modest factor of m. So we begin by calculating F_{02}. If in the definition (47) we set $l=0$, $m=2$, do the y integration, use the expression (28) for W_{000} and change the orders of the resultant double integration, we get

$$F_{02}(k,\kappa)=-24\kappa_0^2\pi(bd)\int_0^\infty e^{-x/d}j_0(k_0x)\times j_0(\tfrac{3}{2}\kappa_0x)x^2dx\int_0^\infty\frac{j_0(xy')dy'}{[y'^2+\alpha^2(\kappa_0)]^3}.$$

The integration over y' in this last integral gives the result $(\pi/16\alpha)(3-\alpha x)\exp(-\alpha x)$ and this enables us to do the integral over x. The result is somewhat lengthy so instead we simply write down the quantity that we really want for calculating $g_{\exp}$, namely $F_{02}(\frac{1}{2}k,k)$. For comparison we also write $F_{00}(\frac{1}{2}k,k)$ as derived from (49)

$$F_{00}(\tfrac{1}{2}k,k)=-2\pi^2(bd)^2\frac{2}{3k_0^2}\ln\left(\frac{[1+\alpha(k_0)]^2+4k_0^2}{[1+\alpha(k_0)]^2+k_0^2}\right),$$

$$F_{02}(\tfrac{1}{2}k,k)=-2\pi^2(bd)^2\frac{3k_0^2}{2\alpha}\left\{\frac{3[1+\alpha(k_0)]}{\{[1+\alpha(k_0)]^2+5k_0^2/2\}^2-(9/4)k_0^4}-\alpha(k_0)\frac{(9k_0^4/4)-[[1+\alpha(k_0)]^2+5k_0^2/2][5k_0^2/2-3[1+\alpha(k_0)]^2]}{\{[1+\alpha(k_0)^2+5k_0^2/2]^2-9k_0^4/4\}^2}\right\}.$$

From these formulas one can verify that F_{02} is at most a few percent of F_{00}, which means that its neglect makes an error of much less than a few percent in $g_{\exp}$ so we are safe in dropping it. From the remarks above, we can feel confident that F_{04} is small compared to F_{02} and so is also negligible for our purposes.

Now we consider the integrals F_{lm}, of which we have already calculated F_{11}. This is always smaller than F_{00} but must still be taken into account, as we have done. But, as we have discussed above, we expect F_{13} to be smaller than F_{11} for the same reasons that F_{02} is smaller than F_{00} and so we drop it and higher terms. As to the terms F_{lm} for higher l, the increasingly high order of the Bessel functions involved in the integrals (47) tends to make them smaller and smaller, at least for those values of k and κ which are important for our purposes, so we drop them all.

Now we discuss what was really the first approximation we made, namely, keeping only ϕ_0 in the expansion (15). To estimate the error involved, we proceed as follows. Equations (31) are a coupled set of integral equations that relate any ϕ_l to a sum over all other $\phi_{l'}$. If, however, it is true that ϕ_0 is large compared to the higher ϕ_l, then we can keep on the right-hand side only the term ϕ_0 (which we already know approximately), and so get an estimate of ϕ_l as an integral over ϕ_0.

There are various factors on the right-hand side of Eq. (31) which can reduce it identically to zero. Two of these factors are $[(-)^{l_2}+(-)^{l_3}]$ and $C(l,l_2,l_3)$: The first is zero if l_2+l_3 is odd and the second is zero if $l+l_2+l_3$ is odd. From this we see that ϕ_1 is identically equal to zero, for either l_2+l_3 or $1+l_2+l_3$ must be odd, and one of the two factors just mentioned necessarily vanishes. We turn to ϕ_2 then and estimate it in the same spirit. Thus we set $l=2$ in Eq. (31), and on the right-

hand side retain only ϕ_0, i.e., set $l_1=0$. We also set $\cos\gamma'=y$. Then we have

$$\phi_2(k,\kappa)=\frac{5}{8\pi^2(k^2+\frac{3}{4}\kappa^2+K^2)}\int_{-1}^{1}dy\int_0^\infty dk'\phi_0(\tfrac{1}{2}k',h(y))k'^2 \times\sum_{l_2=0}^{\infty}\sum_{l_3=0}^{\infty}(2l_2+1)(2l_3+1)i^{l_2+l_3}[(-)^{l_2}+(-)^{l_3}] \times C(2,l_2,l_3)W_{2l_2l_3}(k,k',\kappa)P_{l_2}(y). \quad (60)$$

For estimating ϕ_2 we follow the same procedure as for finding the iterated solutions for ϕ_0, namely we put the zeroth iterate for ϕ_0 on the right-hand side

$$\phi_0(\tfrac{1}{2}k',h(y))\approx\sum_{m=0}^{\infty}\frac{(3k'\kappa y)^m}{(k'^2+3\kappa^2+K^2)^{m+1}}. \quad (61)$$

Now we note that $C(2,l_2,l_3)$ is zero unless l_2 and l_3 satisfy the triangular equality, that is, unless l_3 is one of the values: l_2+2, l_2+1, l_2, l_2-1, l_2-2. But for $l_3=l_2+1$ or l_2-1 we see that $(-)^{l_2}+(-)^{l_2}$ vanishes, and we conclude that the only values of $C(2,l_2,l_3)$ we must consider are $C(2,l_2,l_2+2)$, $C(2,l_2,l_2)$, and $C(2,l_2,l_2-2)$. Now the higher the value of l_2, the smaller the integrals $W_{2l_2l_3}$ tend to be, because the integrands involve Bessel functions of higher orders. So we estimate the term in (60) with $l_2=0$ and see what a contribution it makes to ϕ_2. For $l_2=0$, the only nonvanishing coefficient is $C(2,0,2)=\frac{2}{5}$ and we can write, taking only the $m=0$ term in the sum (61),

$$\phi_2\approx-\frac{10(bd)^2}{(k^2+\frac{3}{4}\kappa^2+K^2)}\int_0^\infty j_2(k_0x)j_2(\tfrac{3}{2}\kappa_0x) \times\exp[-x(1+\alpha(\kappa_0))]xdx.$$

The integral can be done using Eq. (53), and then expanded in much the same way that led to Eq. (52) for F_{00}. We get

$$\phi_2\approx-\frac{12(bd)^2(k_0\kappa_0)^2}{(k^2+\frac{3}{4}\kappa^2+K^2)\{[1+\alpha(k_0)]^2+k_0^2+9\kappa_0^2/4\}^3}.$$

If we compare this with the approximate expressions (42) or (43) for ϕ_0, we see that ϕ_2 is always small compared with ϕ_0; it vanishes for either k_0 or κ_0 zero, but even at points where it does not vanish it is generally less than a few percent of ϕ_0.

Our third approximation was to use only the first term in the expression (56) for $v_{\text{eff}}(r)$. As we have indicated this approximation is best for small r; in fact the constant $\bar{g}$ was so defined that the correction terms vanish for $r=0$. Referring to Fig. (1), we also see that the deviation of $v_{\text{eff}}(r)$ from a pure exponential potential is largest for K_0 small, where g_{exp} deviates most from the horizontal straight line that corresponds to a pure exponential potential.

To get some idea of the effect of the correction terms in (56), we have calculated them numerically for the representative case $K_0=2$. We find that these terms strengthen the exponential potential, and in fact make its magnitude about three percent greater for $r/d=1$ and about ten percent greater for $r/d=2$. We have not taken this correction into account in calculating the results shown in Fig. 2 for the energy eigenvalue.

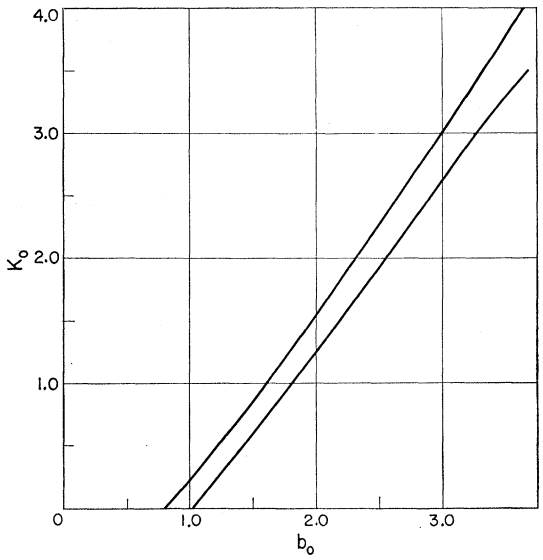

FIG. 2. The three-body energy parameter $K_0\equiv Kd$ plotted against $b_0\equiv bd$ for an exponential interparticle potential. The upper curve is the result of the method of this paper using the approximations represented by (54) and (57). The lower curve gives values obtained with a simple variation function due to Feshbach. In comparing these, one should note that it is known that the Feshbach result for K_0 is too small by a few percent for b_0 around 2.8, by comparison with a result of Rarita and Present, which result is itself based on a variational calculation.

Now we discuss the last approximation we made, which was to use the first iterate $\phi_0^{(1)}$ in evaluating $g_{\text{exp}}(k_0,K_0)$. Here it is rather more difficult to estimate the errors involved, and we must content ourselves with qualitative remarks.

Hopefully, we can get some idea of the accuracy of the first iterate by investigating the analogous iteration procedure for the *two*-body problem, for which one can find the exact solution to compare with successive iterates. This is done in the Appendix, and the results are plotted in Fig. 3. We see that the first iterate makes an appreciable correction to the zeroth order functions, but even so the error that remains is appreciable for the larger values of the variable. There is no reason to expect that the same iteration procedure as applied to the three-body problem will give much better results (and in fact no reason to suspect it will give much worse results). It should be remembered, however, that in calculating the eigenvalue for the three-body problem we use not the wave function $\phi_0(k,\kappa)$ itself, but the characteristic ratio $\phi_0(\frac{1}{2}k,k)/\phi_0(k,0)$. To repeat an earlier remark, it seems likely that this *ratio* is rather insensitive to errors in the wave function itself and may be given relatively accurately even by the first order iterate. One piece of evidence that points in this direction is the fact that although the zeroth and first iterates differ appreciably, especially for large values of the arguments,

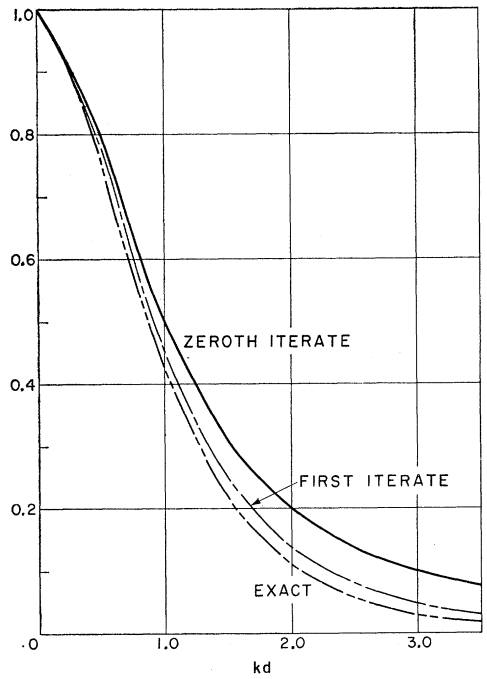

FIG. 3. Comparison of the zeroth order approximation $\phi_{00}^{(0)}(k_0)$ and the first iterate $\phi_{00}^{(1)}(k_0)$ with the exact wave function for the two-body problem with exponential potential for $2Kd=1$.

the change in g_{exp} as calculated with these two iterates is relatively small.

VIII. DISCUSSION

In this section we make some final remarks. We would emphasize that the paper is really divided into two parts. First is the derivation of the equations which determine the wave function when it is written in the special form (10) and the discussion which shows that it can be expanded and truncated, leading to the basic equation (32). The second part of the paper is an approximate method for solving this equation and extracting its eigenvalue. But the merits of writing the wave function as we have done should be weighed in their own right, independently of this approximation method, for it is not likely that it is the last word on the subject. We have simply tried to see how far we could get using the simplest analytic techniques. If one wanted to resort to machine computation, possibly even just straightforward machine iteration of Eq. (32), it seems likely that one could solve it to any desired accuracy.

There is some resemblance between this paper and the work of Svartholm.[9] The points of resemblance are that we both use momentum space and both use an iteration technique. The essential differences are in the way we choose to write the wave function, which has the all-important consequence that the expansion (15) converges well enough that we can limit it to its first term, and in the fact that Svartholm's method is a variational one whereas ours is not.

[9] N. Svartholm, thesis, Lund, 1945 (unpublished).

The point of view presented in this paper can be generalized to more complicated problems. For example, for the symmetric four-body problem which is the analog of the one we treat, we can generalize the wave functions (10) in an obvious way. The same remark holds for the symmetric N-body problem. It remains to be seen whether the transformations of coordinates and integrations that would be involved in reducing these problems to an "equivalent two-body problem" can in fact be worked out.

APPENDIX. TWO-BODY PROBLEM

For reference we collect here some general results for the two-body problem, viz., two particles of mass m bound by a potential $V(r)$. It is convenient to have a notation as similar as possible to the three-body notation, and to this end we use the same symbols for the two-body functions as for their three-body analogs, but with a tilde over the two-body functions. Thus we call the wave function for relative motion $\tilde{\psi}(\mathbf{r})$ ($\mathbf{r}$ is of course the relative coordinate). It satisfies the Schrödinger equation

$$(\nabla^2-K^2)\tilde{\psi}(\mathbf{r})=v(r)\tilde{\psi}(\mathbf{r}), \tag{A.1}$$

where, as in the text,

$$v(r)=mV(r)/\hbar^2, \quad K^2=m|E|/\hbar^2,$$

and we have assumed that E is negative, corresponding to a bound state. We define the Fourier transform $(\tilde{\phi}\mathbf{k})$ of the wave function by

$$\tilde{\phi}(\mathbf{r})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\tilde{\phi}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{k}. \tag{A.2}$$

$\tilde{\phi}(\mathbf{k})$ satisfies the integral equation [Schrödinger equation in momentum space]

$$\tilde{\phi}(\mathbf{k})=-\frac{1}{(2\pi)^3(k^2+K^2)}\int(\tilde{\phi}\mathbf{k}')w(\mathbf{k}-\mathbf{k}')d\mathbf{k}', \tag{A.3}$$

where, as in the text [Eq. (21)],

$$w(\mathbf{k}-\mathbf{k}')=\int v(r)\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]d\mathbf{r}. \tag{A.4}$$

Again as in the text, we write

$$w(\mathbf{k}-\mathbf{k}')=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}w_l(k,k')Y_{lm}^*(\Omega_k)Y_{lm}(\Omega_{k'}), \tag{A.5}$$

where

$$w_l(k,k')=4\pi\int_0^{\infty}v(r)j_l(kr)j_l(k'r)r^2dr. \tag{A.6}$$

Now we expand $\tilde{\phi}(\mathbf{k})$ in spherical harmonics,

$$\tilde{\phi}(\mathbf{k})=\sum\tilde{\phi}_{lm}(k)Y_{lm}(\Omega_k),$$

and put this and Eq. (A.5) into Eq. (A.3) to get a set

of uncoupled equations for the functions $\tilde{\phi}_{lm}$:

$$\tilde{\phi}_{lm}(k)=-\frac{1}{2\pi^2(k^2+K^2)}\int_0^\infty \tilde{\phi}_{lm}(k')w_l(k,k')k'^2dk'. \quad (A.7)$$

Finally, we remark that by using Eq. (A.6) and the Fourier-Bessel inversion theorem we can express $v(r)$ in terms of $w_0(k,0)$:

$$v(r)=\frac{1}{2\pi^2}\int_0^\infty w_0(k,0)j_0(kr)r^2dr. \quad (A.8)$$

Now we turn to the second subject of this Appendix: some results on an iteration procedure[10] as applied to the S-state two-body equation,

$$\tilde{\phi}_{00}(k)=-\frac{1}{2\pi^2(k^2+K^2)}\int_0^\infty \tilde{\phi}_{00}(k')w_0(k,k')k'^2dk'. \quad (A.9)$$

This is an equation which resembles the three-body equation (32) of the text, and our hope is that it will serve as at least a rough guide to the question of the convergence of the iteration procedure used there. We shall discuss the exponential

$$v(r)=-b^2e^{-r/d},$$

since that is one for which the Schrödinger equation can be solved exactly (for states of zero angular momentum), and we begin by summarizing the exact results. The solution of the radial wave equation for $l=0$ is expressed in terms of Bessel functions, and the condition that the wave function be finite at the origin leads to the equation

$$J_{2Kd}(2bd)=0,$$

which for a given bd determines Kd or vice versa. For the numerical work it is convenient to choose values of $2bd$ which make the Bessel function have integral order, for example,

$$2Kd=1 \quad \text{for} \quad 2bd=3.832\cdots;$$
$$2Kd=2 \quad \text{for} \quad 2bd=5.136\cdots.$$

Once the eigenvalues have been determined, the momentum wave function $\phi_{00}(k)$ is given by (dropping irrelevant normalizing factors)

$$\tilde{\phi}_{00}(k)\propto\int_0^\infty j_0(kr)J_{2Kd}(2bde^{-r/2d})rdr.$$

If we expand the Bessel function in this last integral and integrate term by term, we get a series representation for ϕ_{00}. For example, for $2Kd=1$ we get

$$\tilde{\phi}_{00}\propto\left(\frac{1}{0!1!(1+4k_0^2)}-\frac{(1.916)^2}{1!2!(9+4k_0^2)}+\frac{(1.916)^4}{2!3!(25+4k_0^2)}-\cdots\right), \quad (A.10)$$

where $k_0=kd$.

[10] Also compare E. E. Salpeter, Phys. Rev. 84, 1226 (1951).

Now we turn to the approximate, iterated solutions of Eq. (A.9). As we have mentioned in the text we expect that, at least for short-range forces, the function $1/(k^2+K^2)$ is a good approximation to $\tilde{\phi}_0(k)$ for small k. We take this as our zeroth iterate,

$$\tilde{\phi}_{00}^{(0)}=1/(k^2+K^2),$$

and define an nth iterate by

$$\tilde{\phi}_{00}^{(n)}(k)=-\frac{1}{2\pi^2(k^2+K^2)}\int_0^\infty \phi_{00}^{(n-1)}(k)w_0(k,k')k'^2dk'.$$

We consider the first iterate. It is

$$\tilde{\phi}_{00}^{(1)}(k)\propto\frac{1}{k^2+K^2}\int_0^\infty\frac{w_0(k,k')k'^2dk'}{k'^2+K^2}$$
$$=-\frac{4\pi b^2}{k^2+K^2}\int_0^\infty\int_0^\infty\frac{e^{-r/d}j_0(kr)j_0(k'r)r^2k'^2drdk'}{k'^2+K^2}.$$

In terms of $k_0=kd$, $K_0=Kd$,

$$\tilde{\phi}_{00}^{(1)}(k_0)\propto\frac{1}{k_0^2+K_0^2}\int_0^\infty\int_0^\infty\frac{e^{-y}j_0(k_0y)j_0(k_0'y)k_0'^2y^2dk_0'dy}{k_0'^2+K_0^2}.$$

The double integral can be done, using first

$$\int_0^\infty\frac{j_0(k_0'y)k_0'^2dk_0'}{k_0'^2+K_0^2}=\tfrac{1}{2}\pi\frac{e^{-K_0y}}{y},$$

and we get

$$\tilde{\phi}_{00}^{(1)}\propto\frac{1}{(k_0^2+K_0^2)[k_0^2+(1+K_0)^2]}.$$

It is easy to see that the higher iterates lead to essentially the same integrals as those above, and we can immediately write down the results:

$$\tilde{\phi}_{00}^{(2)}\propto\frac{1}{[k_0^2+K_0^2][k_0^2+(1+K_0)^2][k_0^2+(2+K_0)^2]},$$

and

$$\tilde{\phi}_{00}^{(n)}\propto\frac{1}{[k_0^2+K_0^2][k_0^2+(1+K_0)^2]\cdots[k_0^2+(n+K_0)^2]}.$$

To give an idea of the accuracy one gets by this iteration technique, we compare the correct wave function, as given by (A.10), with successive iterates for the case $2Kd=1$. The results are plotted in Fig. 3.

We note, as expected, that the error in the iterated functions gets large for large values of the variables. We also note that the zeroth iterate is quite far off, which is not unexpected since it is independent of the potential shape. For most values of the variables we also see that the first iteration does much of the correction, i.e., it differs much more from the zeroth iterate than the second does from the first. We remark finally that one gets similar results for other values of Kd; they are slightly worse for $2Kd$ greater than unity and somewhat better for $2Kd$ less than unity.