Neutron-Neutron Scattering Length*

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The final-state interaction of the two neutrons from the reaction $\pi^- + d \rightarrow 2n + \gamma$ has a pronounced and distinctive effect on the momentum spectrum of the outgoing particles. In particular, the neutron spectrum is sharply peaked in the neighborhood of 2 Mev, with a shape that is quite sensitive to the strength of the *nn* interaction. In the region of this peak, the relative neutron-neutron momentum is so small that the *nn* interaction is completely characterized by its scattering length. Hence it is proposed that a measurement of the shape of the neutron spectrum from this reaction may provide a convenient means of measuring the neutron-neutron scattering length. Neutron spectra are calculated in an impulse approximation, for several assumed values of the scattering length. It appears from their shapes that, in this way, present neutron-detection techniques should be capable of determining the scattering length to within 25%.

I. INTRODUCTION

BECAUSE the force between two nucleons is nearly strong enough to produce a bound singlet state, the singlet nucleon-nucleon scattering lengths are abnormally large, and hence very sensitive to slight differences between the *pp*, *np*, and *nn* interactions. It has long been known, for instance, that the nuclear singlet scattering lengths for *np* and *pp* scattering differ by 25% or more. The singlet *np* scattering length is about -23.7 fermis,[1] while the nuclear part of the *pp* scattering length is approximately -17 f,[2] corresponding to a difference in well depth of about 2%.

The cause of this distinct violation of charge independence has been the subject of considerable discussion. The simplest electromagnetic effects, such as the magnetic moment interaction,[2] seem to be too small to account for the discrepancy when handled realistically.[3] This suggests that the answer must be sought in an understanding of the mesonic origin of the nuclear force, and there is some indication[3] that the effect may be largely attributable to the $\pi^0 - \pi^\pm$ mass difference. If this is indeed the case, then the nucleon-nucleon force is to this extent inherently charge-dependent, but one can still ask whether the coupling of pions to nucleons is charge-independent. If effects such as the pion mass difference can be calculated accurately, it may be possible to employ the nucleon-nucleon scattering lengths in conjunction with pion-nucleon scattering data to check the charge independence of the pion-nucleon force with considerable accuracy.[4] The *nn* scattering length is one of the desirable pieces of information, and we wish to suggest what appears to be a feasible and accurate method of measuring it.

II. THEORY IN BRIEF

Our proposal for determining the *nn* scattering length is a special case of a more general technique discussed previously by Watson.[5] In fact, it is actually a modification of a very similar but less practical suggestion once made by Watson and Stuart.[6] The proposal is this: Of the two modes of π^- capture observed in deuterium,

$$\pi^- + d \rightarrow 2n,$$
$$\pi^- + d \rightarrow 2n + \gamma,$$

the latter occurs about 30% of the time. The final state contains two neutrons in both cases, but in the two-body mode the outgoing momenta are fixed by conservation laws and so not influenced by the final-state interaction. In the three-body mode, however, the conservation laws are not sufficient to determine the momenta uniquely. Each particle has a momentum spectrum, and the way in which the total momentum is distributed among the three outgoing particles (i.e., the spectrum shapes) is sensitive to the interactions between them. Of the three interactions, two are fortunately $n-\gamma$ interactions, very weak compared with the *nn* interaction; it is this which makes this reaction a "clean" one, preferable to several others one might consider. This leaves the desired *nn* interaction as the principal factor determining the spectra of final-state momenta. The neutron spectrum, in particular, is accessible to accurate measurement, and we believe that a measurement of this spectrum can be used to determine the *nn* scattering length with an accuracy of 25% or better.

Because the *nn* force is attractive, its qualitative effect on the spectrum of each neutron is easy to see. The neutrons have a range of recoil energies only be-

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[1] L. Hulthén and M. Sugawara, *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 5.

[2] J. Schwinger, Phys. Rev. **78**, 135 (1950).

[3] Riazuddin, Nuclear Phys. **7**, 217, 223 (1958).

[4] H. P. Noyes (private communication). D. Wong and H. P. Noyes (to be published).

[5] K. M. Watson, Phys. Rev. **88**, 1163 (1952).

[6] K.M. Watson and R. N. Stuart, Phys. Rev. **82**, 738 (1951). See also the more recent discussion by R. Karplus and L. S. Rodberg, Phys. Rev. **115**, 1058 (1959).

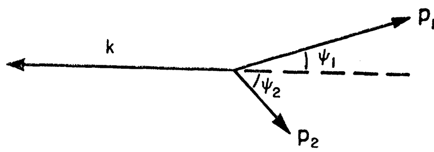

FIG. 1. Notation employed for momentum variables (k is the photon, and p_1 and p_2 the neutrons).

cause of the three-body final state; if the di-neutron were strongly bound, it would recoil with a unique energy, about 4.8 Mev. Although the force is presumably not quite strong enough to produce a bound state, its effect should still tend to equalize the neutron recoil velocities, and thus produce a peak in the energy spectrum of each neutron at about 2.4 Mev. The stronger the force, the sharper will be the peak, so we can expect its width to be the parameter most sensitive to the *nn* interaction.

The results of the detailed calculations to be described in Sec. III substantiate these expectations, as illustrated by Fig. 2. As an example of the type of measurement that would be required to determine the *nn* scattering length, we have plotted in this figure a portion of the neutron energy spectrum at an angle of 172° to the photon direction. We have chosen this restricted portion of the spectrum because it corresponds to the smallest values of the *relative* momentum p of the two neutrons. (The minimum relative momentum, of course, occurs at the peak of the spectrum.) Over this portion of the energy spectrum, p in fact remains so small that the *nn* interaction (and hence the shape of the spectrum) depends only on the *nn* scattering length. It is for this reason that a measurement of the spectrum shape can be used to determine the scattering length. To illustrate the sensitivity of this dependence, we have calculated the neutron spectrum for three assumed values of the *nn* scattering length, a, near the singlet *np* value of -24 fermis.[7] Recalling that when a is equal to $-\infty$ the force is just strong enough to produce a bound state, we see from the given curves that, as expected, the stronger the force, the more sharply is the spectrum peaked.

This particular spectrum is, as we shall see, only one of the many possible energy and angular distributions that one might measure, all of which can yield the same information on the *nn* interaction. Other distributions, such as that of Fig. 3, will be discussed later, after their inter-relationship has been made clear.

At very little cost, the above argument can be sharpened enough to give a numerical estimate of the shape of the spectrum and explain why it depends on only one parameter of the *nn* interaction. Recall first a few salient points of the three-body decay kinematics. The capture occurs from the lowest mesonic Bohr orbit, i.e., essentially at rest in the laboratory system, so the momentum-conservation condition is $\mathbf{p}_1+\mathbf{p}_2+\mathbf{k}=0$. The total energy available is essentially the rest energy of the pion. Consequently, the spectrum of each neutron extends from zero to 70 Mev, and that of the photon from zero to about 131.5 Mev. The attractive *nn* interaction causes these spectra to be strongly peaked in the region of small relative *nn* momentum. This, as we noted earlier, corresponds to neutron energies in the laboratory system of a few Mev, which is a convenient range for using time-of-flight techniques.

The problem has, kinematically, two independent variables. That is, there are three outgoing momenta, hence three energies and three relative angles, and the four conservation conditions leave two of these six variables independent. The distribution of capture events will consequently be a two-dimensional distribution. Any two variables can be used to describe it; we have chosen to use E_1, the energy of one of the neutrons, and ψ_1, the angle its momentum makes with $-\mathbf{k}$, the *negative* photon direction. (This was used rather than its supplement merely to get an angle that is less than 90° for all the interesting cases.) Rather than $\mathbf{p}_1$ and $\mathbf{p}_2$, it will be especially convenient to introduce the momentum variables $\mathbf{p}=\frac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)$, the relative *nn* momentum in the center-of-mass system of the two neutrons, and $\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2$, the momentum of this c.m. system. The energy of the two neutrons is then $(p_1^2+p_2^2)/2M=(p^2+\frac{1}{4}P^2)/M=(p^2+\frac{1}{4}k^2)/M$, by momentum conservation. Since $k+(p^2+\frac{1}{4}k^2)/M$ must equal the initial-state energy,[8] p is uniquely determined by the photon energy. Consequently, we have found it convenient in Fig. 4 to plot, for illustration, a few contours of constant k, very near the high-energy end of the photon spectrum, i.e., near $p=0$. It is interesting to notice that they are restricted to fairly small values of ψ_1, and to E_1 values of a few Mev.

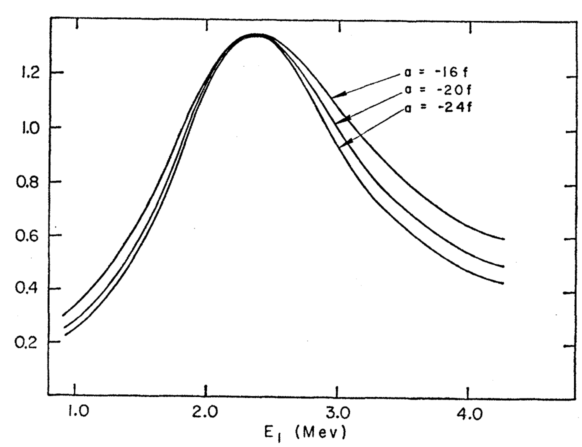

FIG. 2. Neutron spectrum at $\psi_1=8°$ for various assumed values of the *nn* scattering length. The curves have arbitrarily been normalized to the same height at $E_1=2.3$ Mev. For notation, see Fig. 1.

[7] The normalization of these curves is meaningless. For convenience of comparison, we have arbitrarily normalized them all to the same height at their maximum values.

[8] We employ units in which $\hbar=c=1$.

These contours are especially useful in analyzing the problem for two reasons. In the first place, we shall see shortly that p is the key variable of the problem, and that the matrix element for the process is very nearly a function of p alone (near $p=0$), so that these contours are also nearly contours of constant counting rate. Secondly, in order to be able to analyze the problem practically, the nn interaction must be limited to an interaction in relative S states. The nn phase shifts, however, are functions of p, and the S wave will be dominant only if p is sufficiently small—hence it is also important for this reason to know what region of the $E_1-\psi_1$ plane corresponds to small p values.

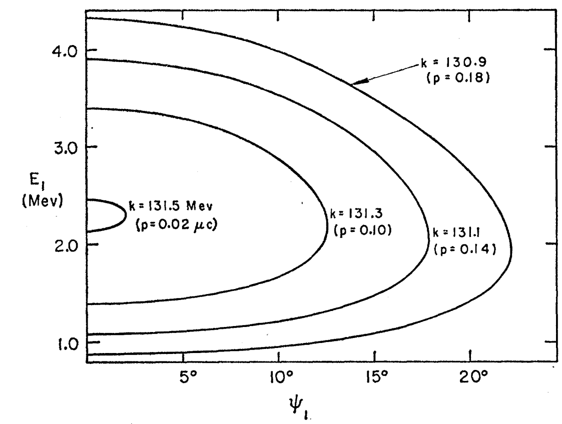

FIG. 4. Contours or loci of $k=$constant (and $p=$constant) in the $E_1-\psi_1$ plane, for values of k near the upper limit of the photon spectrum. As discussed in the text, these are also roughly contours of constant counting rate, the maximum being at $\psi_1=0$, $E_1=2.3$ Mev.

As Watson[5,6] has pointed out, one can estimate the energy dependence of the matrix element for the process in the $p\rightarrow 0$ limit by using the zero-range approximation for the nn interaction. (The justification for this is presented in Sec. III.) In the $p\rightarrow 0$ limit, only the S-wave interaction will be important; if $\psi_S(r)$ is the singlet S-state wave function for the nn system, the matrix element will have the form

$$M_S=\int \psi_S{}^*(r) r f(r) d^3r, \tag{1}$$

where $rf(r)$ represents all the other factors. All we need to know about it is that its p dependence is expected to be weak near $p=0$. In the zero range approximation, ψ_S is given by its asymptotic form all the way in to $r=0$,

$$\psi_S(r)=\sin(pr+\delta)/pr, \tag{2}$$

where $\delta(p)$ is the S-state nn phase shift. In this approximation,

$$M_S(p)=\frac{\sin\delta}{p}\left[\int f(r)\cos prd^3r+\cot\delta\int f(r)\sin prd^3r\right].$$

From the effective-range expansion, $\cot\delta$ approaches $-1/pa$ as p approaches 0, and clearly the integral involving $(\sin pr)$ will approach 0 at least as fast as p. We assume that $|a|\gg R_D$ (the radius of the deuteron); it is then readily seen that, for $pR_D\ll 1$, the p dependence of the integrals is negligible in comparison with that of $(\sin\delta)/p$. In this limit the p dependence of $M_S(p)$ is given by

$$M_S(p)\approx \sin\delta(p)/p. \tag{3}$$

The momentum dependence of $\delta(p)$ can be found from the effective-range expansion,

$$p\cot\delta=-1/a+\tfrac{1}{2}r_0p^2.$$

By suitably restricting the range of neutron energies detected, it is quite feasible in the present experiment to keep p so small that the effective-range term can be neglected. In this case,

$$\sin\delta=pa/(1+p^2a^2)^{\frac{1}{2}}. \tag{4}$$

The energy dependence of the cross section is then simply

$$|M_S|^2\Omega\approx[1/(1+p^2a^2)]\Omega, \tag{5}$$

where Ω is the appropriate phase-space factor. If neutrons are detected, Ω remains nearly constant near $p=0$. Consequently when p approaches 0, the cross section is principally a function of the relative nn momentum p, and is peaked, as expected, at $p=0$. Over an energy range such as that of Fig. 1, the more detailed calculation given below shows this approximation to be remarkably good.

The condition that the effective-range term be negligible is $\frac{1}{2}r_0ap^2\ll 1$. If we insert the singlet np values $r_0=2.65$ f and $|a|=24$ f, this means

$$p/\mu\ll 0.27,$$

μ being the pion mass. More detailed calculations show that

$$p/\mu<0.19 \tag{6}$$

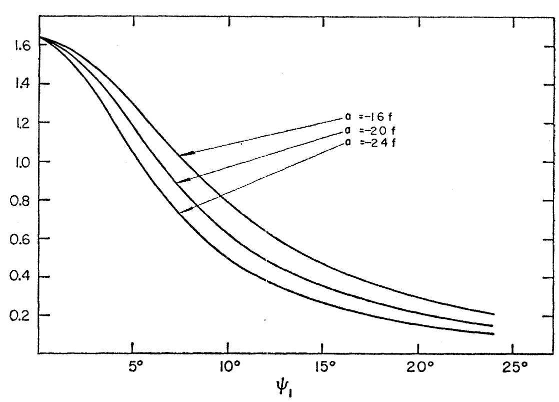

FIG. 3. The angular distribution (relative to the photon direction) of 2.4-Mev neutrons. The curves for different assumed values of the nn scattering length have been normalized arbitrarily to the same value at $\psi_1=0$.

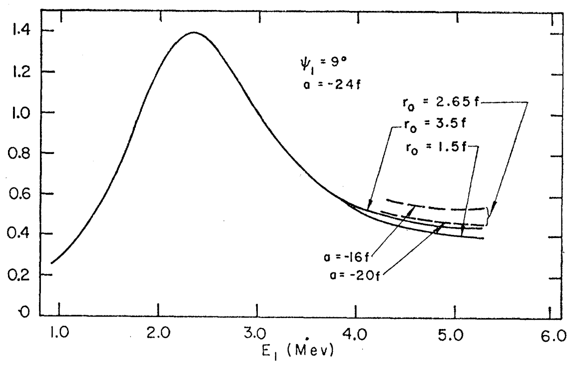

FIG. 5. Neutron energy spectrum at $\psi_1=9°$, showing the effect of varying r_0, the nn effective range. For the solid curve, the nn scattering length was taken to be -24 f. Note that the spectrum is very insensitive to r_0 for E_1 between 1 Mev and 3.5 Mev, where p is small. For comparison, the dashed curves show the dependence of the spectrum on the scattering length when r_0 is held at 2.65 f (the value used in Figs. 2 and 3).

provides a "safe" upper limit. This is approximately the value of p on the largest of the contours in Fig. 3, so the region inside this contour is the usable region of the $E_1-\psi_1$ plane. (The exact effect of varying r_0 is illustrated by the curves shown in Fig. 5.) The corresponding range in the relative energy $E=p^2/M$ of the two neutrons in their c.m. system is

$$E<750 \text{ kev}. \tag{7}$$

From this discussion, it is clear that the essential variable of the problem is p. Whatever distribution one actually measures, its shape will be merely a reflection of the simple function $M_S(p)$. Since a two-variable distribution is inconvenient and time consuming to measure experimentally, one might reduce it to a one-dimensional distribution by holding some convenient variable constant. One possibility is that employed in Fig. 2, the neutron energy spectrum at $\psi_1=$constant. Another is the angular distribution of neutrons at $E_1=$constant, such as that plotted in Fig. 3 for $E_1=2.4$ Mev. (The approximate shapes of these distributions can be obtained by thinking of them as vertical and horizontal slices of the contour plot in Fig. 4.) Clearly there are an infinite number of ways of extracting a one-dimensional distribution from a two-dimensional one, and these two were merely chosen to illustrate the method. If others prove more feasible for experimental reasons, we have the necessary functions coded for 650 computation, and can readily supply whatever alternative data are needed.

There is still another way of deriving a one-dimensional distribution. This is the method used by Watson and Stuart,[6,9] who integrated over one of the variables the angle between $\mathbf{p}$ and $\mathbf{k}$, and were left with a distribution over k. Since k is related directly to p, this looks like an attractive thing to do, especially since it involves only the detection of single photons rather than the two-variable coincidence experiments needed for the above method. In spite of this, we believe it to be a less practical method. The reason is that the γ-ray spectrum, which peaks at about 130 Mev (see Fig. 2, Watson and Stuart[6]), has a width about equal to that of the neutron spectrum given in Fig. 2, approximately 2 Mev. Consequently, whichever particle is detected, the neutron or the gamma ray, the same 2-Mev width must be measured (to about 200 kev). The advantage to be gained by looking at the neutrons is that this is a far easier thing to do with 3-Mev neutrons than with 130-Mev γ rays.[10]

However, if the di-neutron were bound, the peak of the photon spectrum would be shifted considerably. By looking for this effect, Phillips and Crowe[11] were able to conclude with considerable certainty that the scattering length is negative. Note that when the effective-range term is negligible then Eq. (5), the cross section for the production of unbound neutrons, is insensitive to the sign of the scattering length; thus it is important to have this independent determination of it.

III. CALCULATION OF THE SPECTRA

We shall assume that the impulse approximation provides an accurate estimate of the momentum distributions we wish to calculate. By this we mean that the recoil momentum from the photon emission is initially absorbed solely by the proton, and that the matrix element describing the process is the same for the proton in the deuteron as it is for a free proton. This means that the momentum with which the spectator neutron emerges is transferred to it entirely via its interaction with the other nucleon. In Sec. IV we shall estimate the effect of "exchange currents," which enable the spectator neutron to participate directly in the photon emission process; it does not appear to modify seriously the results obtained within the impulse approximation.

In coordinate space the transition operator for a process in which momentum $-\mathbf{\Delta}$ is transferred to the proton has the form $\exp(-i\mathbf{\Delta}\cdot\mathbf{r}_1)T(\mathbf{q}_i,\mathbf{q}_f)$, where $\mathbf{r}_1$ is the position of the proton and $\mathbf{q}_i$ and $\mathbf{q}_f$ are the center-of-mass relative momenta in the initial and final states, respectively. In the present case the pion carries no momentum, so $\mathbf{\Delta}$ is equal to $\mathbf{k}$, the momentum of the photon. In terms of T, the matrix element for radiative absorption in deuterium is

$$M=\int [e^{-i\mathbf{k}\cdot(\mathbf{r}_1+\mathbf{r}_2)/2}\psi_p(\mathbf{r}_1-\mathbf{r}_2)]^* \times e^{-i\mathbf{k}\cdot\mathbf{r}_1}T\phi(\mathbf{r}_1-\mathbf{r}_2)\xi d^3r_1d^3r_2. \tag{8}$$

[9] Also, more recently, by D. W. Joseph, Phys. Rev. **119**, 805 (1960), who considered the alternative process, $\pi^-+d\rightarrow 2n+e^++e^-$.

[10] Additional reasons for imposing this stringent restriction on k are that certain of the approximations used become less good as k is decreased, and at the same time the contributions from poorly-known sources, such as the deuteron D state, become quite significant. See the detailed discussion in Sec. IV.

[11] R. Phillips and K. Crowe, Phys. Rev. **96**, 484 (1954).

Here $\mathbf{r}_1$ and $\mathbf{r}_2$ are, respectively, the proton and neutron position vectors, and we have used $\mathbf{p}_1+\mathbf{p}_2=-\mathbf{k}$; $\psi_\mathbf{p}$ is the wave function for the relative motion of the two neutrons in their center-of-mass system, ϕ is the deuteron wave function, and ξ the wave function of the pion in its Bohr orbit.

Two approximations must be made in order to proceed. First, since the radius of the pion Bohr orbit is some 50 times larger than the deuteron radius, it is a very good approximation to call ξ a constant over the "inside" of the deuteron, and remove it from under the integral sign. (Since we are not interested in the magnitude of the matrix element but only its energy dependence, we shall neglect ξ altogether.) Secondly, since this is a three-body process, to do the integral correctly we would have to know $T(\mathbf{q}_i,\mathbf{q}_f)$ off the two-body mass shell, but in fact very little is known about it, even on the mass shell in the energy range we used. However, the range of $\mathbf{q}_i$ and $\mathbf{q}_f$ values that appear in $T\phi$ is due entirely to the motion of the proton in the deuteron. This range of $\mathbf{q}_i$ and $\mathbf{q}_f$ values is in fact very small, and the best evidence available indicates that T should remain very nearly constant over this range. We shall give this argument in detail below. Its conclusion is that we may safely neglect the momentum dependence of T altogether, so that it becomes a constant rather than a momentum operator. It is, however, still an operator on the spin variables, which of course can always be written in the form $T=A+\mathbf{B}\cdot\boldsymbol{\sigma}$; T is the pion photoproduction operator, which near threshold has the property $A\ll B$.[12] Since the absorption process corresponds to photoproduction approximately at threshold, this must also be the case for absorption, and so we shall take $T=\mathbf{B}\cdot\boldsymbol{\sigma}$. This has the effect of weighting the triplet *nn* states twice as heavily as the singlet ones, but for the range of momentum values in which we shall be interested, the calculations show the triplet contribution to be small anyway. Thus the neglect of the A term has little effect on the results.

Then if all energy-independent factors are neglected, the singlet matrix element is given in terms of an integral over the relative coordinate $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ only:

$$M^{\text{sing}}(\mathbf{k},\mathbf{p})=\int e^{-\frac{1}{2}i\mathbf{k}\cdot\mathbf{r}}\psi_{\mathbf{p},s}{}^*(\mathbf{r})\phi(\mathbf{r})d^3r, \tag{9}$$

where $\psi_{\mathbf{p},s}(\mathbf{r})$ is the singlet spatial part of the c.m. wave function for the two neutrons with relative momentum $\mathbf{p}$. The triplet matrix element has an exactly analogous form.

In order to evaluate the integral, we must know the two wave functions involved: $\psi_\mathbf{p}$ for the *nn* system, and ϕ for the deuteron.

By restricting ourselves to events for which the relative *nn* energy is less than 1 Mev, we guarantee that their interaction occurs only in the S state, so that $\psi_{\mathbf{p},s}$ differs from a properly symmetrized plane wave only in the S state. Watson and Stuart, however, have argued that in fact M_S is nearly independent of the exact form of ψ_S, the singlet-S part of $\psi_{\mathbf{p},s}$, for small values of p. The reason is as follows. We saw above that if the asymptotic form of ψ_S is used, $M_S(p)\sim\sin\delta(p)/p$. Let the exact ψ_S differ from its asymptotic form by $\Delta\psi_S$, and let $f(p,r)=\Delta\psi_S(p/\sin\delta)$, or $\Delta\psi_S=f(p,r)(\sin\delta/p)$. We note that $f(p,r)$ differs from zero only inside the range of the *nn* force, where the depth of the potential is much greater than p^2/M for the range of p values we are considering. This, however, is just the condition that guarantees that the integral appearing in the effective-range expansion is nearly constant except for the normalization factor of the wave function. As Blatt and Jackson[13] have shown in detail, if the customary normalization is used, this integral in the *np* case changes by only about 1% as p varies over the range 0 to 0.2 μ which we are considering.[14] It is seen, however, that $f(p,r)$ has just this normalization. Therefore, its integral should also be very nearly p independent over this range, showing that the correction to $M_S(p)$ has the same $(\sin\delta)/p$ dependence as found in Eq. (3). Consequently, any reasonable wave function may be used to calculate the correction ΔM_S. We have used a square-well function because of its simplicity. As an indication of the validity of the Watson-Stuart argument, we might mention that in the neutron spectrum shown in Fig. 2, the singlet-S contribution (the integral I_2 given in the Appendix) differs from a $\sin\delta(p)/p$ dependence by less than 1% over the entire energy range shown.

Finally, there is the choice of the wave function ϕ for the deuteron. However, again the exact wave function used does not critically affect the functional form of $M(p)$ for small p, provided it has the correct asymptotic behavior, $e^{-\alpha r}/r$. This can be inferred from the fact that any momentum parameters that describe the spatial dependence of $\phi(r)$ within the range of the *np* force [such as β in the Hulthén function $(e^{-\alpha r}-e^{-\beta r})/r$] must necessarily be larger than α, for they correspond to smaller spatial dimensions.[15] If, however, such parameters are called β_1, β_2, etc., then $M(p)$ can only depend on p through p/k, p/α, p/β_1, etc. Now we are only concerned with p in the range $0<p<0.2\,\mu$, whereas $k\sim\mu$, $\alpha\sim 0.33\,\mu$, and $\beta_i>\alpha$. The β of the Hulthén function, which should represent something of an average of the β_i, is about 1.5 μ. Thinking of $M(p)$ as a power series in p/k, p/α, and p/β_i, we see that the restriction of p to the range $p\leq 0.2\,\mu$ will permit only a very weak dependence of $M(p)$ on the β_i, i.e., on the exact interval form of the deuteron wave function. Consequently, we have employed the Hulthén function for simplicity; as noted

[12] See, e.g., M. J. Moravcsik, Phys. Rev. **104**, 1451 (1956).

[13] J. M. Blatt and J. D. Jackson, Revs. Modern Phys. **22**, 77 (1950); Phys. Rev. **76**, 18 (1949).

[14] Note that $p^{-1}\sin\delta$ varies by a factor of three over this range of p.

[15] For example, see the values of the parameters used in an approximate wave function by M. J. Moravcsik, Nuclear Phys. **7**, 113 (1958) all of which are $\geq 1.5\beta$.

above, its influence on $M(p)$ is so small that $M(p)$ is equal to $\sin\delta(p)/p$ to within 1% over the range $p\leq 0.2\,\mu$. [This does not imply that the contribution to the magnitude of $M(p)$ comes almost entirely from outside the range of the forces. The internal integral contributes substantially, but its p dependence, like that of the external integral, comes primarily from the normalization of the *nn* wave function.]

Since all the integrals to be calculated have been stated explicitly by Watson and Stuart,[6] we shall not list them here, but for reference we have compiled the results of the integrations, in their notation, in the Appendix.

IV. ESTIMATES OF THE ACCURACY OF THE THEORY

The approximations we have employed are the following.

(1) Assumed validity of the impulse approximation.
(2) Assumed constancy of meson bound state over the volume of the deuteron.
(3) Approximation to $\psi_p(\mathbf{r})$.
(4) Approximation to $\phi(\mathbf{r})$, both S and D waves.
(5) Neglect of the momentum dependence of $T(\mathbf{q}_i,\mathbf{q}_f)$.

We shall attempt to make a numerical estimate of the effect of each of these approximations on the shape of the energy spectrum of Fig. 2. We choose this particular spectrum for our estimate merely because it is convenient; it is, however, quite representative of other possible spectra, for they are all merely different reflections of the central function $M(p)$. One might use this spectrum to determine the scattering length by, for instance, measuring the ratio of the counting rate at the peak to that at, say, $E_1=4$ Mev, and we shall choose this ratio as our figure-of-merit. The "effects" of the various approximations will be given in terms of their effect on this ratio. To be quite specific, two points at which we have convenient data available to make the estimate are $E_1=2.39$ Mev ($p=0.066\,\mu$) and $E_1=4.25$ Mev ($p=0.19\,\mu$). Consequently, if $R(p)$ is the counting rate at a point on this spectrum, we define

$$\rho=R(0.066)/R(0.19), \tag{10}$$

and we shall attempt to estimate the effect of each of the approximations on ρ.

In quoting the possible errors in this way, it must of course be borne in mind that they are defined relative to this specific momentum range, $p\leq 0.19$. Since this covers the entire range we suggest as being "usable," it gives a maximum estimate of the error to be expected over this range. However, if a larger range of p is employed, it is to be expected that the error encountered will increase. This, as well as the desire to eliminate the influence of the effective range, strongly suggests a restriction of the measurements to as small values of p as possible.

It is also well to keep the following points in mind during the error discussion. We may conveniently divide the *nn* wave function into an S-wave part, a "non S-wave singlet part," and a triplet part. In the last two parts we have assumed all phase shifts for $l>0$ to be zero, which should be extremely good for neutrons with a relative c.m. energy of less than 1 Mev. Consequently, these parts of the wave function are "exact." The major contribution to the cross section in all cases considered comes from the S wave, which we have taken as a square-well function, but the other contributions are not completely negligible. The non S-wave singlet contributions are always quite small, and at $p=0.066\,\mu$ the triplet contribution is also negligible. (It is noted in the Appendix that $I_t=0$ whenever $E_1=E_2$.) However, the triplet contribution increases rapidly as p increases, and at $p=0.10$ it is about 9% of the cross section.

Estimate (1). The impulse method employed in Sec. III assumes that the spectator neutron has no "distorting influence" on the proton during the absorption and emission process; this means that this neutron received none of its momentum directly from the recoiling photon, but only from the other recoiling nucleon. This would seem to be an eminently reasonable approximation in view of the large size and weak binding of the deuteron, and there is some experimental evidence to this effect. White *et al.*[16] have measured π^+ photoproduction from the deuteron at about 300 Mev, and found it to be some 25% smaller than the corresponding free-proton cross section; this agrees well with a calculation by Chew and Lewis[17] which employs this same impulse approximation.

Not only are the violations of the impulse assumption thus likely to be small, they will also alter the momentum distributions in a predictable direction, which we can see as follows. The essential part of the impulse approximation was the appearance of the factor $e^{-i\mathbf{k}\cdot\mathbf{r}_1}$, representing the recoil momentum ($-\mathbf{k}$) absorbed by the proton alone. Since the proton is not free, there are, presumably, mechanisms by which this recoil momentum can be shared by both the nucleons; Fig. 6 indicates one such diagram, in which the photon is emitted by a charged meson being exchanged between the nucleons. Although, of course, the exact effect of this strong interaction cannot be calculated, it would appear quite reasonable to make the *ansatz* that, by analogy with the $e^{-i\mathbf{k}\cdot\mathbf{r}_1}$ factor, the general form of the operator describing the emission of a photon by a two-nucleon system is

$$e^{-i[f\mathbf{k}\cdot\mathbf{r}_1+(1-f)\mathbf{k}\cdot\mathbf{r}_2]}T', \tag{11}$$

where T' is a momentum operator analogous to T, and f is a number between 0 and 1 which gives the fraction of the recoil momentum absorbed by the proton.

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[16] R. S. White, M. J. Jakobson, and A. G. Schultz, Phys. Rev. **88**, 836 (1952).
[17] G. F. Chew and H. W. Lewis, Phys. Rev. **84**, 779 (1951).

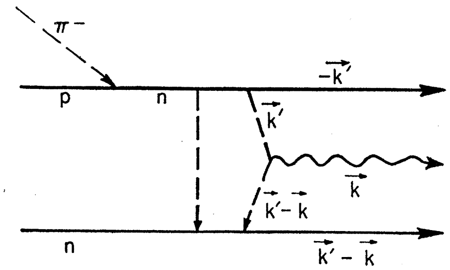

FIG. 6. Feynman diagram for a matrix element which violates the impulse approximation. Note that the recoil momentum $-\mathbf{k}$ is shared between the two neutrons even without any final-state interaction.

If this conjecture is correct, the matrix element is, as before,

$$M=\int e^{i\mathbf{k}\cdot(\mathbf{r}_1+\mathbf{r}_2)/2}\psi_p^*(\mathbf{r}) \times e^{-i[f\mathbf{r}_1+(1-f)\mathbf{r}_2]\cdot\mathbf{k}}T'\phi(\mathbf{r})d^3r_1d^3r_2 = \int e^{-i(f-\frac{1}{2})\mathbf{k}\cdot\mathbf{r}}\psi_p^*(\mathbf{r})T'\phi(\mathbf{r})d^3r. \quad (12)$$

Thus, it reduces as before to an integral over the relative coordinate $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$. At $f=1$ we of course recover the previous matrix element, and at $f=\frac{1}{2}$ the exponential factor is just unity. This is because the nucleons then recoil as a unit, and it merely expresses over-all momentum conservation for this "elastic" case.

Of course, nothing has really been gained if the form of the operator T' is not known. However, we shall find below that the operator T corresponding to $f=1$ can be well approximated as a constant. The reason is essentially that T, being a meson-absorption operator, affects principally the high-momentum components of $\phi(r)$, which do not strongly affect the resulting matrix element, $M(p)$. Since the same argument would appear to apply to T', we shall also assume that T' is simply a constant.

If this is the case, the matrix element is altered in a very simple way and can be obtained from the previous result simply by replacing k by $(2f-1)k$. The number f need only range from 0.5 to 1, for the other half of its range repeats the same results. Thus, the effect is always to decrease k, and it is found that this always causes the neutron spectra to become *more peaked*, i.e., increases the quantity ρ. If we take the extreme case, $f=\frac{1}{2}$, we find that ρ is increased by about 18%.

This would roughly correspond to increasing the magnitude of the (negative) scattering length by about 20%, i.e., shifting one of the curves of Fig. 2 into the next lower curve. However, this is assuming that the impulse approximation is entirely incorrect. As a liberal estimate, it would seem unlikely that effects of this kind should change ρ by more than perhaps 5%. Also, the direction of the error is known in this case. If the scattering length is obtained by using Fig. 2 or Fig. 3, in which the impulse approximation was employed, then, to the extent that the impulse approximation is incorrect, the scattering length so obtained would be too large in absolute magnitude.

Estimate (2). The bound-state wave function $\xi(r)$ of the pion should be given fairly well by e^{-r/r_0}, r_0 being about 200 fermis in this case. Since the deuteron wave function effectively restricts the integration to distances of less than 4 f, ξ varies only from 1 to $1-1/50$, or something like 2%. As we have seen, however, the functional form of $M(p)$ comes almost entirely from the normalization of the *nn* wave function rather than from the integral itself, so that this slight change in the integrand should produce a completely negligible change in the form of $M(p)$.

Estimate (3). The S-state part of ψ_p, the *nn* wave function, was approximated by a square-well function with the correct scattering length and effective range. (The effective range was chosen as 2.65 f, the same as for the *np* singlet state. The influence of the effective range in this way is unavoidable but very small.) Since the correct wave function is unknown, we cannot say by how much the square-well function is in error, but we can get a gross idea of the influence of the shape of the wave function by asking how much ρ would be changed if we used the asymptotic form of the wave function instead of the square-well function, i.e., the zero-range approximation. As noted above, this changes the p dependence of the S-wave integral by much less than 1%. It does increase its magnitude by about 15%, though, and since the triplet contribution does not change, ρ is increased by 3.6%. Since one would not expect the exact wave function to differ from the square-well function as radically as does the zero-range function, we might estimate the possible error in ρ from this source to be about 2%.

Estimate (4). The fact that the Hulthén wave function for the deuteron S state may be somewhat inaccurate is insignificant, for, as we noted above, the p dependence of the main part of the cross section (that due to the *nn* S wave, I_2 of the Appendix) differs from $\sin\delta(p)/p$ by less than 1% over the whole usable range.

The D state of the deuteron is more troublesome, however. Since it is very poorly known, it is unfortunate that it contributes significantly to the process. We have used the simplest possible approximation to it, and in this approximation it contributes 12.7% of the cross section at $p=0.66\,\mu$ and 9.5% at $p=0.19\,\mu$.[18] Because its fractional contribution does not change much over this range, though, its effect on the spectrum shape is considerably less. It is impossible to estimate accurately how far wrong our approximation to the D state is, but if it is neglected altogether, ρ is increased by 4.2%. Since

[18] Watson and Stuart estimated that the D-state integral contributed only about 3% to their cross section. This included an integration over angles we have not considered, but even allowing for this the difference between 3% and 9% does not seem easy to understand.

we know so little about what this function should be, it seems quite possible that this could contribute an error of 4% in ρ.

Estimate (5). Finally, perhaps the most serious source of uncertainty is the assumption that the meson absorption operator, $T(\mathbf{q}_i,\mathbf{q}_f)$, remains constant over the range of momenta present in the deuteron. [More precisely, the assumption is that the shape of $T\phi(\mathbf{r})$ is not significantly different from that of $\phi(\mathbf{r})$ for large values of r.] Nothing at all is known experimentally about this operator in the energy range employed here (which would correspond to π^- photoproduction from neutrons just at threshold). In fact very little can reliably be said about it at all, but we wish to point out a few kinematic features which tend to strengthen considerably one's confidence in the assumption that it remains constant.

The most important consideration is merely the fact that the pion is captured predominantly from an S state. Since the deuteron is also largely S state, the $\pi^- - p$ relative angular momentum is zero, and one knows from general considerations[19] that the matrix element is constant for small relative momenta in this case.

The more detailed kinematic facts are these. The relative $\pi - p$ momentum comes from the motion of the proton in the deuteron. Assuming an average kinetic energy of about 15 Mev for the bound proton, we estimate the significant range of momenta in the Fourier transform of its wave function to be from zero to about 1.5 μ.

In $T(\mathbf{q}_i,\mathbf{q}_f)$, $\mathbf{q}_i$ is the relative $\pi^- - p$ momentum and $\mathbf{q}_f$ the relative $n-\gamma$ momentum, both in the c.m. system. When two-body kinematics do not hold, $\mathbf{q}_i$ and $\mathbf{q}_f$ are independent quantities; in the present case the range of values they assume is determined entirely by the range of proton momenta in the deuteron. In the initial state, the c.m. system nearly coincides with the proton, so although the range of proton momenta in the laboratory is 1.5 μ, in the c.m. system it is reduced by $\mu/(M+\mu)$ and extends only from zero to 0.2 μ. The range of π^- momenta in the c.m. is of course the same, so $q_i \leq 0.4\,\mu$. The range of c.m. photon momenta in the final state depends on the relative proton-photon direction. If all directions are assumed possible, the maximum range of c.m. photon momenta is from 0.75 μ to 1.13 μ, and so 1.5 $\mu < q_f < 2.26\,\mu$.

Although we expect T to be constant for q_i "small," we would like to know how small. The only available information on T near the photoproduction threshold is that derivable from the dispersion-theory arguments of Chew *et al.*[20] From their complete photoproduction amplitude near threshold the $(\pi - p)$ S-wave part can be projected out. For small values of q_i/μ, this expression is given approximately by[21]

$$T_s \approx 1+0.15 i q_i^2/q_f\mu. \tag{13}$$

This is, of course, only the expression on the two-body energy shell, but for the above ranges of q_i and q_f it indicates that $|T|^2$ remains extremely constant, in fact constant to within 0.02%. Thus, if the expression for T at nearby points off the energy shell is any kind of reasonable continuation of Eq. (13), the error incurred by considering T to be constant is entirely negligible.

V. CONCLUSION

The effect of the final-state nn interaction on the momentum spectrum of each neutron is found to be pronounced and distinctive when the relative nn momentum is small. (The spectra of Fig. 2 would be nearly flat if this interaction were not present.) Consequently, a measurement of one of these spectra appears to provide a feasible means for determining the nn scattering length. From the examples given in Figs. 2 and 3, it would appear that the different types of spectra are all about equally sensitive to the scattering length, so that the choice can best be made on grounds of experimental convenience.

ACKNOWLEDGMENTS

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APPENDIX

The integrals that appear in the calculation are listed here for reference. Since we employed the same wave functions as Watson and Stuart,[6] we shall use their notation and refer the reader to their paper for further details.

Since the following two integrals occur frequently, it is useful to introduce a notation for them. If $j_0(x) = \sin x/x$, we define

$$S(\alpha,p,q) = q\int_0^\infty e^{-\alpha r}\sin pr\cdot j_0(qr)dr = \tfrac{1}{4}\ln\frac{\alpha^2+(p+q)^2}{\alpha^2+(p-q)^2}, \tag{A-1}$$

and

$$C(\alpha,p,q) = q\int_0^\infty e^{-\alpha r}\cos pr\cdot j_0(qr)dr = \frac{1}{2}\left[\tan^{-1}\left(\frac{2\alpha q}{\alpha^2+p^2-q^2}\right)+\epsilon\pi\right], \tag{A-2}$$

where

$$\epsilon=0 \quad \text{if} \quad \alpha^2+p^2-q^2\geq 0,$$
$$\epsilon=1 \quad \text{if} \quad \alpha^2+p^2-q^2< 0.$$

[19] See, e.g., H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, Illinois, 1955), Vol. II, p. 143.

[20] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[21] M. J. Moravcsik (private communication).

(In both integrals it is assumed that α, p, and q are all positive.)

Another pair of integrals that arose may perhaps be of use to others. If $j_2(x)$ is the second spherical Bessel function, the integrals are

$$S_2(\alpha,p,q)=\int_0^\infty e^{-\alpha r}\sin pr\cdot j_2(qr)dr=\frac{3}{2}\frac{p}{q^2}+\frac{3\alpha^2-3p^2+q^2}{2q^3}S(\alpha,p,q)-\frac{3}{2}\frac{\alpha p}{q^3}C(\alpha,p,q),\quad\text{(A-3)}$$

and

$$C_2(\alpha,p,q)=\int_0^\infty e^{-\alpha r}\cos pr\cdot j_2(qr)dr=-\frac{3}{2}\frac{\alpha}{q^2}+\frac{3\alpha^2-3p^2+q^2}{2q^3}C(\alpha,p,q)+3\frac{\alpha p}{q^3}S(\alpha,p,q).\quad\text{(A-4)}$$

In calculating the integrals I_t, I_1, and I_2, we use the Hulthén wave function for the S state of the deuteron,

$$\phi_0(r)=(e^{-\alpha r}-e^{-\beta r})/r.\quad\text{(A-5)}$$

Consequently, these three integrals will have the form

$$I=I(\alpha)-I(\beta),$$

so we need only give $I(\alpha)$. The integrals are

$$I_t(\alpha)=\int d^3r[e^{-i\mathbf{p}\cdot\mathbf{r}}-e^{i\mathbf{p}\cdot\mathbf{r}}]e^{-i\mathbf{k}\cdot\mathbf{r}/2}e^{-\alpha r}/r=4\pi[(\alpha^2+q_+^2)^{-1}-(\alpha^2+q_-^2)^{-1}],\quad\text{(A-6)}$$

where $\mathbf{q}^\pm=\mathbf{p}\pm\frac{1}{2}\mathbf{k}$. Note that when $E_1=E_2$, then $q_+^2=q_-^2$, and $I_t=0$.

$$I_1(\alpha)=\int d^3r[e^{i\mathbf{p}\cdot\mathbf{r}}+e^{-i\mathbf{p}\cdot\mathbf{r}}-2(pr)^{-1}\sin pr]e^{-i\mathbf{k}\cdot\mathbf{r}/2}e^{-\alpha r}/r=4\pi[(\alpha^2+q_+^2)^{-1}+(\alpha^2+q_-^2)^{-1}-4(pk)^{-1}S(\alpha,p,\tfrac{1}{2}k)].\quad\text{(A-7)}$$

$$I_2=I_2'+I_2'',$$

$$I_2'(\alpha)=2\int d^3r(pr)^{-1}\sin(pr+\delta)\cdot e^{-i\mathbf{k}\cdot\mathbf{r}/2}e^{-\alpha r}/r=\frac{16\pi}{k}\frac{\sin\delta}{p}[C(\alpha,p,\tfrac{1}{2}k)+\cot\delta\cdot S(\alpha,p,\tfrac{1}{2}k)].\quad\text{(A-8)}$$

$$I_2''(\alpha)=L_2(\alpha)-L_{20}(\alpha),$$

where

$$L_{20}(\alpha)=2\int_{r\leq R}d^3r(pr)^{-1}\sin(pr+\delta)\cdot e^{-i\mathbf{k}\cdot\mathbf{r}/2}e^{-\alpha r}/r,$$

and where the integration is only over the volume of a sphere of radius R. This is because we are using a square-well wave function for the $n-n$ system, so that the integrands of L_2 and L_{20} are equal outside the well and these parts of the integrals cancel. If the well depth is V and its radius R, we define $\gamma=(M^2V^2+p^2)^{\frac{1}{2}}$ and note that the boundary conditions at $r=R$ yield the conditions $p\cot(pR+\delta)=\gamma\cot\gamma R$. This expression can be used to express the well parameters in terms of the effective range and scattering length, and can also be used to condense the L_{20} integral into compact form. The result is

$$L_{20}(\alpha)=\frac{8\pi}{\alpha^2+p^2}\left\{\frac{\sin\delta}{p}(\alpha+p\cot\delta)-\frac{\sin(pR+\delta)}{p}e^{-\alpha R}(\alpha+\gamma\cot\gamma R)\right\}.\quad\text{(A-9)}$$

The other half of $I_2(\alpha)$ is the integral of the square-well function,

$$L_2(\alpha)=2\int_{r<R}d^3r\left[\frac{\sin(pR+\delta)}{\sin\gamma R}\frac{\sin\gamma r}{pr}\right]e^{-i\mathbf{k}\cdot\mathbf{r}/2}e^{-\alpha r}/r.$$

It appears that this integral cannot be expressed in terms of elementary functions. After the angular integral is done, however, a factor $\sin\frac{1}{2}kr$ remains in the integrand. Now $k\sim 0.93\ \mu$, and for $r_0=2.65$ f, $R\sim 1.76/\mu$, so $\frac{1}{2}kR=0.83$. Sin(0.83), however, is equal to 0.76. Consequently, even at the upper limit of integration we make less than a 10% error in the integrand by using $\sin\frac{1}{2}kr=\frac{1}{2}kr$, and this should have a negligible effect on the accuracy of the result. With this approximation, we find

$$L_2(\alpha)=8\pi\frac{\sin(pR+\delta)}{p}\frac{1}{\alpha^2+\gamma^2}\times\left\{\frac{\gamma}{\sin\gamma R}-e^{-\alpha R}(\alpha+\gamma\cot\gamma R)\right\}.\quad\text{(A-10)}$$

Finally, there is the integral over the D state of the deuteron,

$$I_3=2\int d^3r(pr)^{-1}\sin(pr+\delta)j_2(\tfrac{1}{2}kr)\phi_2(r).\quad\text{(A-11)}$$

Because $j_2(\frac{1}{2}kr)$ is small near the origin, we have simply used the asymptotic form of the $n-n$ wave function. For the same reason we shall simply use the asymptotic form of the D state,

$$\phi_2=ne^{-\alpha r}.\quad\text{(A-12)}$$

Hulthén and Sugawara[1] suggest that n should be taken as 0.2, which is the value we have used. In this case

$$I_3=2\int d^3r(pr)^{-1}\sin(pr+\delta)j_2(\tfrac{1}{2}kr)(0.2)e^{-\alpha r}/r=(0.2)8\pi\frac{\sin\delta}{p}\{C_2(\alpha,p,\tfrac{1}{2}k)+\cot\delta\cdot S_2(\alpha,p,\tfrac{1}{2}k)\}.\quad\text{(A-13)}$$

In terms of these functions, the counting rate R is given as

$$R=\{\tfrac{2}{3}|I_t|^2+\tfrac{1}{3}(|I_1+I_2|^2-\sqrt{2}I_2I_3)\}\Omega, \quad \text{(A-14)}$$

keeping only the leading term in I_3 and dropping all factors of 2π, etc., since we are not concerned with the absolute normalization of R. Factor Ω is the phase-space factor which depends on the pair of variables observed. If E_1 and ψ_1 are observed, as in Figs. 1 and 2,

$$\Omega=\frac{2\pi M^2k^2p_1}{M+k-p_1\cos\psi_1}dE_1d\Omega_1. \quad \text{(A-15)}$$

The following constants were employed in the numerical evaluation of these functions:

$$\mu=139.63 \text{ Mev},$$
$$M_n/\mu=6.2786,$$
$$\alpha/\mu=0.3274,$$
$$\beta/\mu=1.54,$$
$$[\mu-(M_n-M_p)-B_0]/M_n=0.1449.$$

In employing the effective-range expansion, we used the first two terms, with $r_0=2.65$ f.

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Charge Distribution in the Fission of Uranium Isotopes Induced by 20–40 Mev Helium Ions*

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The primary yields of Br^{82}, I^{130}, La^{140}, Pr^{142}, and I^{135} have been accurately determined for the medium-energy (20–40 Mev) helium-ion-induced fission of U^{233}, U^{235}, and U^{238}. These accurate primary yield data have been correlated with the constant-charge-ratio rule for nuclides away from the neutron shells and give a smooth, but different, distribution curve for nuclides of 83 neutrons.

INTRODUCTION

THE details and mechanism of the primary nuclear charge distribution of fission fragments in higher energy fission have largely been inferred from studies involving the determination of primary yields of various fission products. In principle, if primary yield data for enough of the nuclides of a given mass chain could be obtained, a charge distribution curve could be constructed, the maximum of which would define Z_p, the most probable charge for that mass. A comparison of Z_p with that predicted by the various theories describing the nuclear arrangement at the moment of fission can yield valuable information on this important phenomenon. Therefore, the construction of such distribution curves based on primary yield data has been the goal of the majority of previous investigations. Since the necessary experimental conditions for the determination of the charge distributions of every fission product mass do not exist, it becomes necessary to assume that the charge distribution is essentially independent of mass to correlate data for different masses.

Charge distributions in fission were first considered for low-energy (thermal neutron) fission of U^{235} by Glendenin, Coryell, and Edwards[1] (and later modified by Pappas[2]). They obtained the most probable charge, Z_p, by postulating equal beta-decay chain lengths for the light and heavy fragments. This postulate is usually referred to as the equal-charge-displacement rule (E.C.D.).

Another hypothesis was proposed by Goeckermann and Perlman[3] to obtain Z_p values which would best correlate their primary yield data obtained from bismuth fission induced by 190-Mev deuterons. This hypothesis assumes that fission at high energies is so rapid that the charge distribution in the fragments is essentially the same as in the fissile nuclide, i.e., a constant-charge ratio (C.C.R.).

Steinberg and Glendenin[4] have adequately discussed these rules in a summary concerned with the radiochemical data on the fission process. Additional primary yield data obtained by mass spectrometric and radiochemical methods have appeared for neutron fission of

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