Liquid-Drop Nuclear Model with High Angular Momentum*

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(Received October 10, 1960)

The equilibrium shapes of rotating liquid drops are calculated in the spheroidal approximation. The solutions are prolate forms for high angular momenta such as are produced in compound nuclei resulting from heavy-ion bombardments. This is thought to account for the angular and energy distributions observed in heavy-ion-induced fission and light-particle evaporation.

NUCLEAR reactions induced by high-energy heavy ions differ in several ways from reactions with energetic protons or other light ions. Even at several hundred Mev the trajectory is classical up to the point of contact, the probability of formation of a compound nucleus is large,[1] the available energy per nucleon is only a few Mev, and the average angular momentum of the compound system is very large. In recent heavy-ion experiments on high-energy fission and particle evaporation, processes which are usually described with a compound-nucleus model, there is evidence for important effects due to large angular momentum. These effects are of two types: (a) The angular distributions of evaporation particles[2] and fission fragments[3] are sharply and symmetrically peaked ahead and behind ninety degrees in the center-of-mass system. (b) The barrier for charged particle evaporation[2] is lowered, and the fission barrier is probably reduced also.[4]

In an attempt to incorporate high angular momentum into the compound nucleus we have investigated a liquid-drop model with collective rotation. The model is similar to that of Pik-Pichak[4] who has studied the lowering of the fission barrier by high angular momentum. In this paper we are concerned with the bases of the model and a study of the equilibrium shapes of liquid-drop nuclei of high angular momentum. These shapes may be of interest as starting points for evaporation and fission calculations, but such calculations are not developed here.

THE MODEL

When an energetic heavy ion collides with a target nucleus the compound system can have very large angular momentum, typically 50 to 200 $\hbar$. If a compound nucleus is formed it must carry this angular momentum, and we assume that such large values can only be carried by collective rotation of the nucleons.

The available center-of-mass energy in such a collision is shared among several forms. Some fraction will excite the intrinsic motion of the nucleons, and the remainder will appear as collective energy: vibration, distortion, and rotation. Without a specific model the energy partition is not calculable, but with any model both the intrinsic and collective energies are appreciable fractions of the total.

The intrinsic and collective energies are related in some fundamental but obscure way. In classical language this relation is described by the hydrodynamic and thermodynamic properties of the nuclear matter. The most detailed considerations of the hydrodynamic properties of nuclei have been made in connection with the Bohr-Mottelson theory of low-energy collective rotations of deformed nuclei. The developments of this theory indicate that there is as yet no unambiguous way to introduce collective rotation into an arbitrary system of nucleons. The original theory[5] assumed a hydrodynamical model with irrotational flow and gave too small moments of inertia. Later calculations,[6] mostly with the Inglis model, have increased the moments to the range of the experimental values which lie midway between the irrotational and rigid-body values. The complexity of this low-energy problem arises from the interaction of nucleons in closed and unclosed shells. Closed shells rotate as irrotational fluid, but even small interactions with nucleons outside the shells cause large increases in the rotational inertia, even sufficient to cause the whole nucleus to assume the rigid-body value.

Recently it has been shown[7] that a Fermi gas rotates as a rigid body even when particle-particle interactions are present. Thus it seems plausible that in the absence of closed shells nuclei rotate as rigid bodies. At the high excitations of interest here it is unlikely that shell effects persist, and we assume rigid-body rotations.

* Supported in part by the U. S. Atomic Energy Commission.

[1] V. A. Druin, S. M. Polikanov, and G. N. Flerov, Soviet Phys.—JETP **5**, 1059 (1957).

[2] W. J. Knox, A. R. Quinton, and C. E. Anderson, Phys. Rev. Letters **2**, 402 (1959), W. J. Knox, *Proceedings of the Second Conference on Reactions Between Complex Nuclei, Gatlinburg, May, 1960* (John Wiley & Sons, Inc., New York).

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[4] G. A. Pik-Pichak, Soviet Phys.—JETP **7**, 238 (1958).

[5] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

[6] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 1 (1955); D. R. Inglis, Phys. Rev. **96**, 1059 (1954); S. A. Moszkowski, Phys. Rev. **103**, 1328 (1956); D. R. Inglis, Phys. Rev. **103**, 1786 (1956).

[7] R. D. Amado and K. A. Brueckner, Phys. Rev. **115**, 778 (1959); R. M. Rockmore, Phys. Rev. **116**, 469 (1959).

The collective properties of a nucleus are described by its shape. In the liquid-drop model two shape-dependent energies are used. These are the surface-tension energy and the Coulomb self-energy. Their shape dependence is usually derived by classical calculations for a distorted, uniformly charged liquid drop with constant density and surface tension, and the two energy coefficients are deduced from the Weizsäcker semiempirical model of nuclear masses.

Neither the assumption of uniform charge nor energy-independent surface tension is correct. The former conflicts with the Stanford electron-scattering data, and the latter with theories of the pressure of a statistical system. However, at the present time, it does not seem possible to incorporate these refinements, and we will ignore them. Our model is a liquid drop rotating as a rigid body with constant mass and charge densities and constant surface-tension described by the constants of the Weizsäcker model. The intrinsic energy of the nucleons is simply the remainder after the equilibrium collective energy is calculated with the classical minimum principles discussed below.

THE DYNAMICAL PRINCIPLES

The classical problem of an isolated rotating liquid held together by gravitation has received much attention as a model in cosmogony.[8] Although the forces in the nuclear problem are completely different, the general dynamical principles are the same and there are a number of analogies in detail.

Two dynamical principles have been developed to find the equilibrium shapes of an isolated liquid mass rotating as a rigid body.[9] One is an energy-minimum principle which is written,[8]

$$E=V+(J\hbar)^2/2\mathcal{I}=\text{minimum}, \tag{1}$$

where V is the algebraic sum of all potential energies, $J\hbar$ is the constant angular momentum, and $\mathcal{I}$ the rigid-body moment of inertia about the rotation axis. In the gravitational problem V is the total gravitational self-energy (negative). In the nuclear liquid drop the dominant (and negative) energy term is the volume energy which, for constant density, can be ignored in Eq. (1) since only volume-preserving shapes are considered. Even in processes where the number of nucleons changes, such as the initial impact leading to the compound nucleus, this volume energy can be ignored. Thus, in the nuclear liquid drop, V is the sum of a Coulomb self-energy (positive) and a surface-tension energy (also positive) which is the product of a surface-tension constant and the surface area. The energy minimum is to be sought by arbitrary shape variations conserving mass, volume, and angular momentum.

Since the Coulomb and surface energies can be integrated only for certain families of shapes, the generality of Eq. (1) is not immediately useful. One must assume some suitable family of trial shapes for which V can be calculated, find the minimum subject to this restriction, and proceed to the true minimum by expansions around this trial shape. Further, with Eq. (1) alone one never knows when the true minimum has been reached.

The second dynamical principle is derived from the hydrodynamical equation of motion and the constancy of pressure on the drop's surface. It reads[8]

$$-\phi+\tfrac{1}{2}\omega^2r^2=\text{constant on the surface}, \tag{2}$$

where ϕ is the total potential (per unit mass) at a surface point, ω the angular velocity, and r the radial distance from the rotation axis to the surface point. In the gravitational problem ϕ is the gravitational potential at the surface (negative). In the nuclear liquid drop[10] ϕ is the sum of a similar Coulomb potential (positive) and a surface-tension curvature potential or surface-tension pressure per unit mass (also positive). The practical difficulties of applying Eq. (2) are similar to those for Eq. (1) with one exception. If a solution to Eq. (2) is found then it is automatically a figure of stability.

Discussions of stability in cosmogony[8] distinguish between ordinary and secular stability. A secularly stable shape is stable for all small oscillations even when there is internal dissipation of dynamical energy, whereas if this "friction" is zero the stability is said to be "ordinary." This distinction only occurs in problems with collective rotation and the rules are rather unfamiliar. A secularly stable shape is also ordinarily stable, but the reverse does not hold. Ordinary instability implies secular instability, but again the reverse does not hold. The condition of *absolute* minimum of Eq. (1) insures secular (and hence ordinary) stability. Equation (2) is less stringent; both kinds of stability are included in its equilibrium shapes.

In the gravitational problem simple trial shapes (at least in retrospect) are also exact solutions. They are oblate ellipsoids of revolution about the rotation axis (Maclaurin spheroids), and nondegenerate ellipsoids with their shortest axis along the rotation axis (Jacobi ellipsoids). As the angular momentum increases from zero (sphere) the secularly stable shapes are a sequence of Maclaurin spheroids of increased flattening. At a certain angular momentum this sequence crosses the sequence of Jacobi ellipsoids; that is, there is one shape common to the two sequences and at this point secular

[8] Reviews of the long history of this problem appear in: J. H. Jeans, *Problems of Cosmogony and Stellar Dynamics* (Cambridge University Press, New York, 1919); R. A. Lyttleton, *The Stability of Rotating Liquid Masses* (Cambridge University Press, New York, 1953); W. S. Jardetzky, *Theories of Figures of Celestial Bodies* (Interscience Publishers, New York, 1958).

[9] Rotation as a rigid body in no way implies rigidity in the elastic sense, but simply means that the local transverse, *collective* velocity is proportional to the radial distance from the axis of rotation.

[10] W. J. Swiatecki, Phys. Rev. **104**, 993 (1956), has used Eq. (2) without the rotation term to find the saddle-point shapes in liquid-drop fission theory.

stability is transferred from the Maclaurin to the Jacobi shapes. This is an example of Poincaré's organization of "linear sequences" of secular stability and their "points of bifurcation."

The dynamical assumptions of Eqs. (1) and (2) are sufficiently general for the nuclear liquid drop model. Of course, the solutions will not be Maclaurin spheroids and Jacobi ellipsoids. However, they might be expected to be somewhat similar. With small angular momentum a spherical drop will become oblate, and for high angular momentum prolate forms will minimize Eq. (1).

A principal difference between the cosmogonical and the nuclear problem involves the relevance of secular stability. In the gravitational problem the system begins in a secularly stable shape, and the question is to trace its evolution as the density slowly increases. Evolution along a sequence of shapes only occurs if they are secularly stable because of the irreversible transfer of collective dynamical energy into heat. In the nuclear problem the true relation between collective and intrinsic energies may permit reversible interchanges. However, the constant-volume model used here cannot treat such interchanges since no thermodynamic work can be performed at constant volume. This is a rather unsatisfactory feature of the model since it separates the intrinsic and collective energies permanently, even in the initial impact and the evolution closely following it. This cannot be true. Thermal expansion and lowering of the surface tension must occur at high intrinsic excitation but, as we have noted, we are unable to include these effects quantitatively at this time.

We ignore these difficulties and assume that the system after impact evolves along a sequence of unstable states during which collective energy is irreversibly transferred to intrinsic energy until the absolute minimum of Eq. (1) is reached. We also ignore collective vibrations or other processes which could interrupt this evolution. If the energy minimum of Eq. (1) is deep and prolate we feel that this is reasonable, since the system is already elongated at impact. If there are several minima separated by low barriers the assumption is very questionable.

THE SPHEROIDAL APPROXIMATION

For analytical calculation of the stable shapes of Eq. (1) it is necessary to have explicit formulas for the energies. The simplest shapes for this purpose are spheroids (ellipsoids of revolution), and we have investigated these shapes in some detail.

It is necessary to remember that shapes which minimize Eq. (1) are not the true stable shapes unless all distortions are permitted. With only spheroidal distortions the minimum shapes may be poor approximations. Indeed, it can be shown with Eq. (2) that the rotating nuclear liquid drop shapes are *never* spheroids. However, the spheroidal minima of Eq. (1) might be expected to be fair approximations in most cases and should be good starting points for further approximations using more general distortions.

The calculation proceeds as follows. We assume an incompressible liquid with surface tension and uniform volume charge and calculate the energy of Eq. (1) for two classes of spheroids: (a) oblate spheroids with the rotation axis along the symmetry axis, (b) prolate spheroids with the rotation axis through the center and perpendicular to the symmetry axis. Taking z as the rotation axis and a, b, c as the semiaxes of the ellipsoids, the energy of Eq. (1) for fixed angular momentum is a function of the shape described by a single parameter η: $\infty>(a/c=b/c=\eta)>1$ for the oblate case and $0<(c/a=b/a=\eta)<1$ for the prolate case.[11] We call the axis ratio η in both cases. Thus η ranges from zero to infinity, passing through the sphere at $\eta=1$. The numerical coefficients for the Coulomb and surface energies are taken from the semiempirical mass formulas of Green.[12] For spherical nuclei, denoted by superscript 0, Green's formulas can be written

$$E_s^0=4\pi r_0^2 \mathcal{O} A^{\frac{2}{3}}=17.80A^{\frac{2}{3}} \text{ Mev},$$
$$E_c^0=\tfrac{3}{5}e^2 r_0^{-1}Z^2A^{-\frac{1}{3}}=0.710Z^2A^{-\frac{1}{3}} \text{ Mev},$$
$$r_0=1.216\times10^{-13} \text{ cm},$$
$$r=r_0A^{\frac{1}{3}},$$
$$\tfrac{1}{2}\hbar^2\mathcal{J}_0^{-1}=35.34A^{-5/3} \text{ Mev},$$

where A, r, and Z refer to the compound nucleus. The collective energy of Eq. (1) is

$$E=E_c^0\left(\frac{E_c}{E_c^0}\right)+E_s^0\left(\frac{E_s}{E_s^0}\right)+\frac{1}{2}\frac{(J\hbar)^2}{\mathcal{J}_0}\left(\frac{\mathcal{J}_0}{\mathcal{J}}\right), \qquad (3)$$

where the dimensionless terms in parentheses depend only on the shape. For prolate spheroids

$$\frac{E_s}{E_s^0}=\tfrac{1}{2}\eta^{\frac{2}{3}}\left\{1+\frac{\sin^{-1}(1-\eta^2)^{\frac{1}{2}}}{\eta(1-\eta^2)^{\frac{1}{2}}}\right\},$$
$$\frac{E_c}{E_c^0}=\tfrac{1}{2}\eta^{\frac{2}{3}}(1-\eta^2)^{-\frac{1}{2}}\ln\left\{\frac{1+(1-\eta^2)^{\frac{1}{2}}}{1-(1-\eta^2)^{\frac{1}{2}}}\right\},$$
$$\frac{\mathcal{J}_0}{\mathcal{J}}=2\eta^{4/3}/(1+\eta^2),$$

and for oblate spheroids

$$\frac{E_s}{E_s^0}=\tfrac{1}{2}\eta^{\frac{2}{3}}\left\{1+\frac{\ln[\eta+(\eta^2-1)^{\frac{1}{2}}]}{\eta(\eta^2-1)^{\frac{1}{2}}}\right\},$$
$$\frac{E_c}{E_c^0}=\eta^{\frac{2}{3}}(\eta^2-1)^{-\frac{1}{2}}\tan^{-1}[(\eta^2-1)^{\frac{1}{2}}],$$
$$\frac{\mathcal{J}_0}{\mathcal{J}}=\eta^{-\frac{2}{3}}.$$

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[11] There is also an identical prolate form with the long axis along y.

[12] A. E. S. Green, *Nuclear Physics* (McGraw Hill Book Company, Inc., New York, 1955).

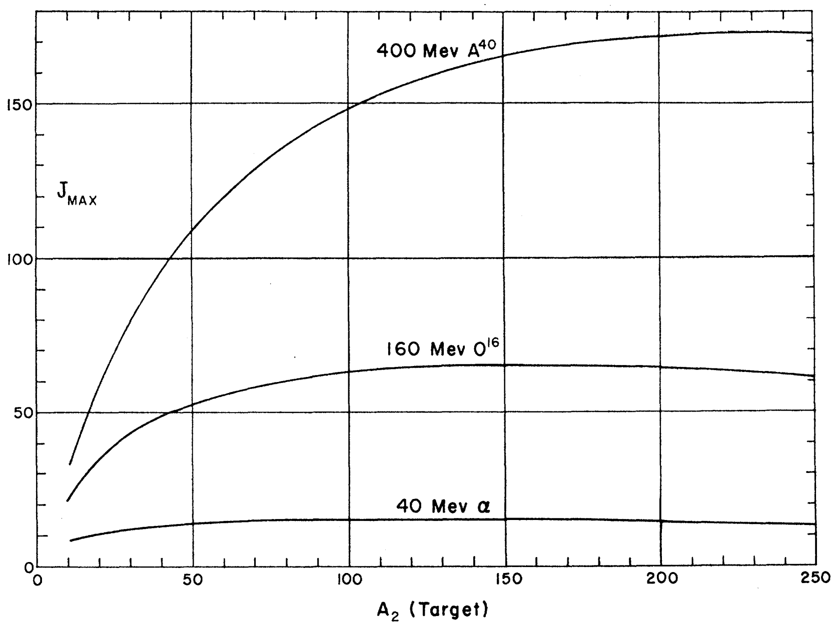

FIG. 1. Maximum angular momentum in compound-nucleus formation assuming classical trajectories and tangential contact of ion and target. The angular momentum is sensitive to the nuclear radii which are assumed to be $1.216\times10^{-13}A^{\frac{1}{3}}$ cm. The assumed relation[12] between the target Z_2 and A_2 is $Z_2=\frac{1}{2}A_2-0.2A_2^2/(A_2+200)$.

To study the shapes of nuclei throughout the periodic table we must assume some relation between Z and A for the compound nucleus. In these calculations we assume that the *target nuclei* lie on Green's line[12] of beta stability and choose compound nuclei made from these by oxygen-16 capture. However, this does not mean that our compound nuclei apply only to this mode of formation.

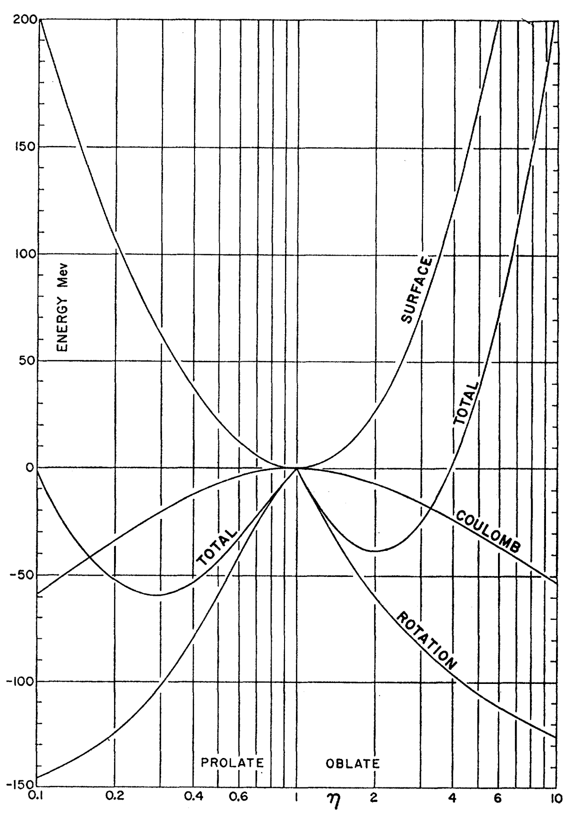

FIG. 2. Variation of the surface, Coulomb, and rotation energies, of Eq. (3) and their sum, the total collective energy for the compound nucleus $A=66$, $Z=31$ with $J=70$. The energies $E_c^0=168.64$ Mev, $E_s^0=290.70$ Mev, $\frac{1}{2}\hbar^2\mathcal{J}_0^{-1}J^2=160.60$ Mev of the spherical shape are taken to be zero in the figure.

In the calculations J is treated as a parameter and a different energy vs shape curve results for each J. In order to use these curves, the possible J in a given collision are investigated separately. We assume that the maximum J, $J_{\max}$, corresponds to tangential contact of ion and target in a classical Coulomb scattering with *no distortion* of either nucleus. On this assumption

$$J_{\max}\hbar=(r_1+r_2)[2M(E_{\mathrm{c.m.}}-E_{\mathrm{Cb}})]^{\frac{1}{2}},$$

where r_1 and r_2 are the radii of ion and target nuclei, M is the reduced mass, $E_{\mathrm{c.m.}}$ is the *total* kinetic energy in the center-of-mass system at infinite separation, and E_{Cb} is the Coulomb barrier energy defined as

$$E_{\mathrm{Cb}}=\frac{e^2Z_1Z_2}{r_1+r_2}.$$

Figure 1 shows $J_{\max}$ for several ions and targets. Although each impact yields a different J and hence a different shape, certain averages are of interest. The arithmetic average J for a uniform flux of ions is $\frac{2}{3}J_{\max}$, and the J which divides all collisions into equal numbers with greater and lesser J is $2^{-\frac{1}{2}}J_{\max}$.

Figure 2 shows the variations of the surface, Coulomb, and rotational energies with the distortion parameter η for a particular J and compound nucleus. Figure 3 is a set of energy curves for the same nucleus. Figure 4 gives the value of η for which the energy is a minimum; these curves were calculated from $dE/d\eta=0$.

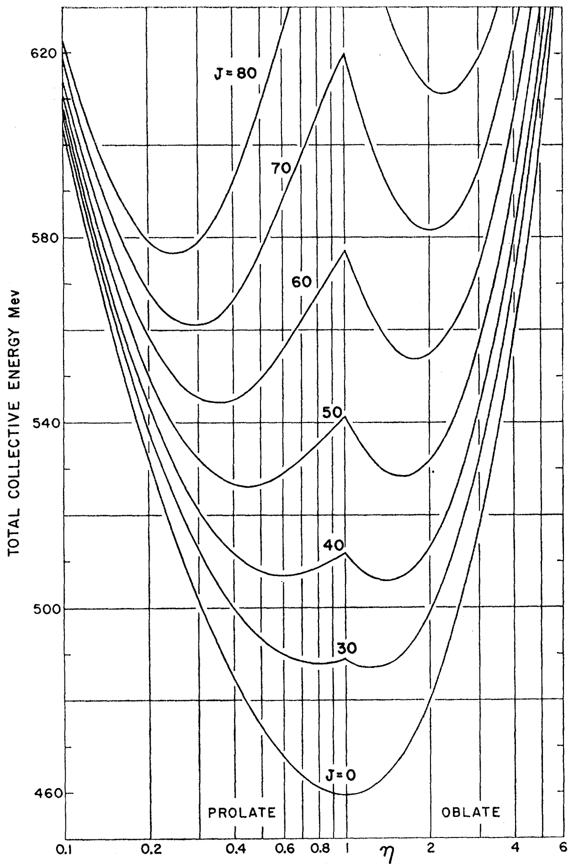

FIG. 3. The total collective energy E of Eq. (3) as a function of the distortion parameter η for the compound nucleus: $A=66$, $Z=31$.

The intrinsic excitation energy of the nucleus can be obtained by subtracting from the total energy available in the reaction the sum of the rotational, surface, and

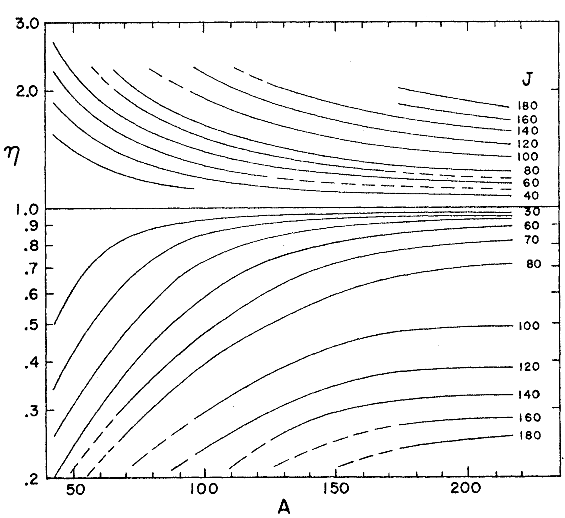

FIG. 4. The spheroidal distortion parameter at the prolate and oblate energy minima as a function of the compound nucleus A for various angular momenta.

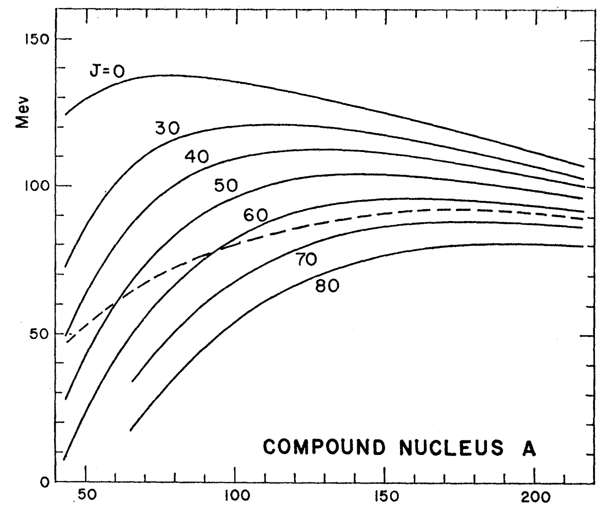

FIG. 5. Intrinsic excitation energy at the prolate energy minimum of compound nuclei formed by bombardment with 160-Mev O^{16} ions. Curves for different angular momenta are shown. The dashed curve corresponds to J_{max}.

Coulomb energies. The excitation energies for the shapes of minimum collective energy are shown in Figs. 5 and 6 for the cases of bombardment with 160-Mev oxygen ions and 400-Mev argon ions.

If the spheroidal shapes give absolute minimum for Eq. (1), then for these shapes Eq. (2) would be satisfied. This is not the case. The total curvature of spheroids varies too rapidly[13] near the outer edges of the figures

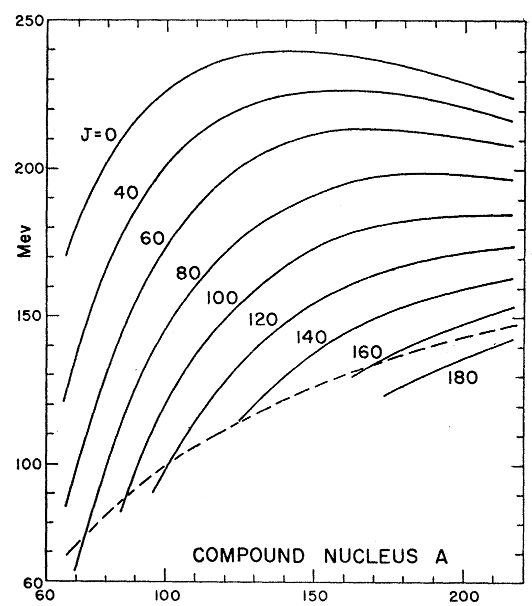

FIG. 6. Intrinsic excitation energy at the prolate energy minimum of compound nuclei formed by bombardment with 400-Mev A^{40} ions. Curves for different angular momenta are shown. The dashed curve corresponds to J_{max}.

[13] W. J. Swiatecki, reference 10, finds this also for nonrotating liquid drop nuclei.

to allow Eq. (2) to be satisfied. Typical errors in fitting the surface pressure with spheroidal figures are ten percent. Another less serious point regarding prolate spheroidal shapes should be mentioned. No prolate spheroid can satisfy Eq. (2) since all points on a circular trace which lies in plane parallel to the rotation axis have the same total curvature and Coulomb potential but are at different distances from the rotation axis and hence cannot satisfy Eq. (2). This is also true in the gravitational problem and is responsible for the flattening of the Jacobi ellipsoids.

DISCUSSION

Aside from questions regarding the validity of the classical model for nuclei of high angular momenta, the calculations themselves are quite incomplete. The spheroidal shapes are thought to be fairly good first approximations but this has not been proven. Calculations of distorted spheroids would test this. We have begun such calculations using the methods of spheroidal harmonic analysis,[14] but the calculations are incomplete.

If, however, we accept the spheroidal shapes as reasonable approximations to the true equilibrium shapes, we may discuss qualitatively the effect of high angular momentum in compound nucleus reactions. We note that the distortions obtained are large and that the prolate minima are deeper than the oblate minima for large angular momenta. The Coulomb potential barrier for the emission of charged particles from the ends of the prolate shapes (or the equators of the oblate shapes) are significantly lower than the barriers for the spherical shapes. For example, the Coulomb potential at the outer ends of the minimum-energy prolate shape for the case of 160-Mev oxygen ions on Ni and $J=37$, is about 2 Mev lower than for the corresponding spherical shape. Furthermore, as compared with the spherical shape, the nuclear temperature will be significantly decreased because of the energy held in rotation and distortion.

For heavy compound nuclei the prolate shapes indicate a lower fission barrier as compared with the sphere.[4] It can be shown that for any sequence of shapes with increasing moment of inertia leading to fission, the effect of angular momentum will be to lower the fission barrier. The location of fission barriers and saddle point shapes remains to be investigated in detail although some work has been done on this problem.[4]

In the interaction of light elements the minimum energy figure may be greatly elongated. For example, in the collision of maximum angular momentum of 160-Mev oxygen ions with Al the prolate equilibrium figure has an axis ratio of 3 to 1. For such small compound systems and very high angular momentum it is unlikely that compound nuclei are formed. Such collisions have been discussed as grazing contacts.[15] Even disregarding these physical questions: of the shapes calculated here, those of highly distorted light nuclei are probably the poorest approximations to the true minimum energy shapes. We surmise that calculations done with more general distortion parameters would show these highly distorted shapes to be unstable to fission. This may be related to the high probability of fragmentation in the interaction of Al with 160-Mev oxygen ions.[16]

[14] W. J. Swiatecki, reference 10; and *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, June, 1958* (United Nations, Geneva, 1958); U. L. Businaro and S. Gallone, Nuovo cimento **1**, 629 and 1277 (1955).

[15] R. Kaufman and R. Wolfgang, Phys. Rev. Letters **3**, 232 (1959); and *Proceedings of the Second Conference on Reactions Between Complex Nuclei, Gatlinburg, May, 1960* (John Wiley & Sons, Inc., New York).

[16] C. E. Anderson, W. J. Knox, A. R. Quinton, and G. R. Bach, Phys. Rev. Letters **3**, 557 (1959).