characteristics but the spin of F^{20} is not yet sure ($J=2$ or 3) although the parity is established to be even, as expected.[1] However, the observation[7] that $J\neq 0$ for the 4.97-Mev state taken together with this large log ft value suggests it has odd parity, or if even parity, then $J=1$. The relative weakness of the ground-state transition argues, though not powerfully, against $J=1$. It would clearly be of considerable value to improve our knowledge both of the ground-state transition from this state and also of the F^{20} beta decay.

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Extension of the Shell Model for Heavy Spherical Nuclei

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The Bardeen-Bogoliubov-Belyaev treatment of the pairing correlations is applied to spherical nuclei with a general nuclear force. The interaction between quasi-particles is treated by the method of linearized equations of motion. An advantage of this treatment is that the same equations describe single-particle excitations and collective excitations, so that the former are orthogonal to the latter and the total number of states is correct. Another advantage is that the spurious states due to the fluctuations in the number of particles are automatically eliminated. The equations to be solved resemble those for a two-body shell model calculation. Simple estimates, based on delta-function or quadrupole forces, are made for the vibrational frequencies in various modes and transition matrix elements. It is concluded that the method is as powerful as other known methods for dealing with collective states by the shell model, and that the same order of magnitude for the effective nuclear force seems capable of fitting all the data.

1. INTRODUCTION

THE past two years have seen some important developments in the theory of nuclear structure. The recent success in the theory of superconductivity[1] stimulated the application of the same ideas to nuclear physics.[2–5] According to the new point of view, the pairing correlations and the energy gap must play a fundamental role in our understanding of many nuclear properties. Belyaev[6] has discussed the influence of pairing correlations on the collective behavior of nuclei; and Kisslinger and Sorensen[7] have obtained good agreement with many detailed properties of single-closed-shell spherical nuclei, by using a simple interaction composed of a pairing force and a quadrupole force and treating it by the new methods. In a different line of research, there has been increasing success in accounting for collective effects starting from the ideas of the shell model. Here, we mention the work of Brown and Bolsterli[8] who showed that the location of the giant photoresonance could be explained by taking into account particle-hole interactions.

The present work represents another extension of these ideas. The aim is to develop an approximation suitable for calculating the properties of all low-lying levels of heavy spherical even-even nuclei, starting from a general shell-model Hamiltonian. To do this, we first perform the Bogoliubov-Valatin transformation[9] on the Hamiltonian (Sec. 2). The result can be interpreted in terms of a Hamiltonian of "quasi-particles" and an interaction between these quasi-particles. It is the existence of a gap in the spectrum of quasi-particles which restricts the low excited levels to two quasi-particles and makes possible a simple shell-model type of calculation. This is not quite true, however, because a few levels containing many quasi-particles may be brought down by collective effects. Fortunately, there is a well-known method which was devised to deal with this difficulty in other many-body problems, the method of linearized equations of motion. We use it (Sec. 3), and the resulting equations apply equally well to collective states and to noncollective states of two quasi-particles. This is a great advantage, as in the past one has had to treat the two kinds of states by different methods, with the result that one ended up with too many states and that often they were not mutually orthogonal. Also, one can now treat states which are only weakly collective, and for which the standard methods of dealing with collective states are not valid. Finally, we shall see that the spurious states due to the

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957), referred to in the following as BCS.
[2] A. Bohr, B. R. Mottelson, and D. Pines, Phys. Rev. **110**, 936 (1958).
[3] A. Bohr, *Comptes Rendus du Congrès International de Physique Nucléaire, Paris, 1958* (Dunod, Paris, 1959).
[4] B. R. Mottelson, in *The Many-Body Problem* (John Wiley & Sons, Inc., New York, 1959).
[5] V. G. Soloviev, Nuclear Phys. **9**, 655 (1958).
[6] S. T. Belyaev, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31**, No. 11 (1959). Some related work is due to A. Kerman (to be published).
[7] L. S. Kisslinger and R. A. Sorensen, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. (to be published), referred to in the following as KS.
[8] G. E. Brown and M. Bolsterli, Phys. Rev. Letters **3**, 472 (1959).
[9] N. N. Bogoliubov, Nuovo cimento **7**, 794 (1958); J. G. Valatin, Nuovo cimento **7**, 843 (1958).

nonconstancy of the number of particles, a difficulty introduced by the Bogoliubov-Valatin transformation, can be easily eliminated. The equations resemble those for a two-body shell model calculation and should be easy to solve with a realistic nuclear force, except for the rather formidable problem of the choice of parameters. Here, we shall content ourselves with rough estimates based on quadrupole forces and δ-function forces. These are sufficient to show that the new method is as effective as older ones in dealing with quadrupole vibrations (Sec. 4A). We also apply it to collective vibrations of closed shells, in which case it is not necessary to perform the Bogoliubov-Valatin transformation, as the spacing between major shells already plays the role of an energy gap (Sec. 4B). One finds that the strength of the nuclear force, which is needed to bring the various pairing and collective effects into agreement with experiment, is of the same order of magnitude in all cases. Finally we show how one can estimate the enhancement of inelastic cross sections (Sec. 4C).

Many of the manipulations that we need to perform require the use of the algebra of angular momentum.[10] We shall not reproduce the details, as they are straightforward. We use the Condon and Shortley choice of phases[11] in all cases.

2. THE HAMILTONIAN

A. The Shell-Model Hamiltonian

Since we are concerned with heavy nuclei, we shall use j-j coupling, but we shall not use the isotopic spin formalism. The single-particle shell model states will be specified by various quantum numbers: the charge q, n, l, j, m. These states will be designated by Greek subscripts, and the corresponding creation and absorption operators will be called, for example, c_α^* and c_α. They satisfy the usual Fermion anticommutation rules,

$$c_\alpha c_\beta + c_\beta c_\alpha = c_\alpha^* c_\beta^* + c_\beta^* c_\alpha^* = 0, \tag{1a}$$

$$c_\alpha^* c_\beta + c_\beta c_\alpha^* = \delta_{\alpha\beta}. \tag{1b}$$

In association with the subscript α, we shall use the Roman subscript a, which stands for all quantum numbers above except the magnetic quantum number m. The starting Hamiltonian H has two parts. One is the sum of the single-particle energies,

$$H_{sp} = \sum_\alpha \epsilon_a c_\alpha^* c_\alpha, \tag{2}$$

which runs over all values of the quantum numbers. The second part is the interaction Hamiltonian

$$H_i = \sum_{\alpha\beta\gamma\delta} v_{\alpha\beta\gamma\delta} c_\alpha^* c_\beta^* c_\delta c_\gamma, \tag{3}$$

where the following antisymmetry relations must hold

$$v_{\alpha\beta\gamma\delta} = -v_{\beta\alpha\gamma\delta} = -v_{\alpha\beta\delta\gamma} = v_{\beta\alpha\delta\gamma}. \tag{4}$$

Next, we must write H_i in a way which exhibits its invariance under rotations and reflexions. This is accomplished by coupling two of the particles, say α and β, to angular momentum JM, coupling the other two also to JM, and writing an invariant tensor product. Thus, we are led to write $v_{\alpha\beta\gamma\delta}$ in the form

$$v_{\alpha\beta\gamma\delta} = -\tfrac{1}{2} \sum_{JM} G(abcdJ) C(j_\alpha j_\beta J; m_\alpha m_\beta M) \times C(j_\gamma j_\delta J; m_\gamma m_\delta M), \tag{5}$$

where the C's are the usual vector coupling coefficients. The minus sign is introduced for convenience, since the interactions are mostly attractive. The parity of $l_a + l_b$ must be the same as that of $l_c + l_d$, otherwise G vanishes. G must also conserve charge, i.e., $q_a + q_b = q_c + q_d$. There follows from Hermiticity of H_i and time-reversal invariance that G is real and

$$G(abcdJ) = G(cdabJ). \tag{6}$$

The relations (4) give, together with the symmetry properties of the C coefficients,

$$\begin{aligned} G(abcdJ) &= -\theta(abJ) G(bacdJ) \\ &= -\theta(cdJ) G(abdcJ) = \theta(abcd) G(badcJ), \end{aligned} \tag{7}$$

with the notation

$$\theta(abJ) = \theta(j_a j_b J) = (-)^{j_a + j_b + J}. \tag{8}$$

The relation between our G and the usual two-body matrix element of shell model calculations is

$$\langle abJM | H_i | cdJM \rangle = -\sigma_{ab}\sigma_{cd} G(abcdJ), \tag{9}$$

with

$$\sigma_{ab} = \begin{cases} 1 \text{ if } a \equiv b, \\ \sqrt{2} \text{ otherwise.} \end{cases} \tag{10}$$

But there was no compelling reason for coupling α and β together, and γ and δ together. We could also have coupled α and γ to $J'M'$, β and δ similarly; or alternatively α and δ to $J''M''$, as well as β and γ. This leads us to define another function F by

$$\begin{aligned} v_{\alpha\beta\gamma\delta} &= -\tfrac{1}{2} \sum_{J'M'} F(acdbJ') s_\gamma C(j_\alpha j_\gamma J'; m_\alpha \bar{m}_\gamma M') \\ &\qquad \times s_\beta C(j_\delta j_\beta J'; m_\delta \bar{m}_\beta M') \\ &= +\tfrac{1}{2} \sum_{J''M''} F(adcbJ'') s_\delta C(j_\alpha j_\delta J''; m_\alpha \bar{m}_\delta M'') \\ &\qquad \times s_\beta C(j_\gamma j_\beta J''; m_\gamma \bar{m}_\beta M''). \end{aligned} \tag{11}$$

Here and in the following, $\bar{m}$ stands for $-m$. We had to introduce the symbol

$$s_\gamma = (-)^{j_\gamma - m_\gamma}, \tag{12}$$

because, in actuality, it is angular momentum j_α which is the sum of j_γ and J'. The same function F appears

[10] U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic Press, New York, 1959).

[11] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, New York, 1935).

with both coupling schemes by virtue of Eq. (4). It is related to the two-body matrix element for a particle and a hole. Its relationship to G involves a Racah coefficient:

$$F(acdbJ') = -\sum_J (2J+1) W(j_a j_b j_c j_d; JJ') G(bacdJ). \quad (13)$$

It is real and has the following symmetry properties:

$$F(acdbJ') = F(dbacJ') = \theta(abcd) F(cabdJ'), \quad (14)$$

but nothing simple happens if only a and c are interchanged.

The pairing force used by KS is (for charge-conserving matrix elements)

$$G(abcdJ) = \delta_{ab}\delta_{cd}\delta_{J0}(j_a+\tfrac{1}{2})^{\frac{1}{2}}(j_c+\tfrac{1}{2})^{\frac{1}{2}} g, \quad (15)$$

where g is a constant (which KS called G).

B. Treatment of the Pairing Correlations

In spherical nuclei with partially filled shells, the most important effect of the two-body force is to produce pairing correlations. Those must be treated quite accurately, even if other effects of the nuclear force are not. A suitable treatment was discovered by Bardeen, Cooper, and Schrieffer[1] for superconductors, and applied to nuclear physics by the Copenhagen school [2–4] and others.[5] In the BCS ground state of an even-even nucleus, the particles are distributed in pairs, all coupled to angular momentum 0; the method is a generalization of Racah's[12] seniority ideas. The simplest way to introduce these correlations in the wave function is to perform the Bogoliubov-Valatin[9] transformation. We define a new set of creation and absorption operators by

$$\begin{aligned} a_\alpha &= u_a c_\alpha - s_\alpha v_a c_{-\alpha}^*, \\ a_\alpha^* &= u_a c_\alpha^* - s_\alpha v_a c_{-\alpha}, \end{aligned} \quad (16)$$

where $-\alpha$ is obtained from α by changing the sign of the magnetic quantum number, and u_a and v_a are real and related by

$$u_a^2 + v_a^2 = 1. \quad (17)$$

The a's satisfy the same anticommutation relations as the c's. They create and absorb "quasi-particles." For a level far above the Fermi level, $u_a \approx 1$, $v_a \approx 0$, and the quasi-particle is the same as a particle. For a level far below the Fermi level, $u_a \approx 0$, $v_a \approx 1$, and the quasi-particle is a hole. But for levels in the neighborhood of the Fermi level, a quasi-particle is partly particle and partly hole. The vacuum of the new operators is the BCS ground state. The converse of relations (16) are

$$\begin{aligned} c_\alpha &= u_a a_\alpha + s_\alpha v_a a_{-\alpha}^*, \\ c_\alpha^* &= u_a a_\alpha^* + s_\alpha v_a a_{-\alpha}. \end{aligned} \quad (18)$$

We shall express H in terms of the operators a_α. A difficulty arises because the number of quasi-particles does not commute with the number of neutrons $\mathfrak{N}_n$ and that of protons $\mathfrak{N}_p$,

$$\mathfrak{N}_n = \sum_n c_\alpha^* c_\alpha, \quad \mathfrak{N}_p = \sum_p c_\alpha^* c_\alpha, \quad (19)$$

where $\sum_n$ runs over all neutron states and $\sum_p$ all proton states. This forces us to introduce two chemical potentials λ_n and λ_p, and instead of diagonalizing H, we try to diagonalize

$$\mathfrak{K} = H - \lambda_n \mathfrak{N}_n - \lambda_p \mathfrak{N}_p, \quad (20)$$

subject to the condition that the expectation values of $\mathfrak{N}_n$ and $\mathfrak{N}_p$ are the given numbers of neutrons and protons in the nucleus. This procedure has the unfortunate consequences that our equations really represent a mixture of neighboring even-even nuclei, and that some of their solutions are "spurious," i.e., do not correspond to any state of a single nucleus; the damage will be partially repaired in Sec. 3.

The task of expressing $\mathfrak{K}$ in terms of the a's is straightforward and will only be sketched. We introduce the notation N for the normal product[13] of an operator, obtained by rewriting all quasi-particle creation operators to the left of the absorption operators, changing the sign whenever two Fermion operators are inverted, but ignoring commutators. For instance,

$$\begin{aligned} N(c_\alpha^* c_\gamma) = {} & u_a u_c a_\alpha^* a_\gamma - s_\alpha s_\gamma v_a v_c a_{-\gamma}^* a_{-\alpha} \\ & + s_\alpha v_a u_c a_{-\alpha} a_\gamma + u_a s_\gamma v_c a_\alpha^* a_{-\gamma}^*. \end{aligned} \quad (21)$$

A well-known theorem[14] enables one to write any operator as a sum of normal products, for instance

$$\begin{aligned} c_\alpha^* c_\beta^* c_\delta c_\gamma = {} & N(c_\alpha^* c_\beta^* c_\delta c_\gamma) \\ & + \delta_{\alpha,-\beta} s_\alpha u_a v_a N(c_\delta c_\gamma) + \delta_{\gamma,-\delta} s_\gamma u_c v_c N(c_\alpha^* c_\beta^*) \\ & + \delta_{\alpha\gamma} v_a^2 N(c_\beta^* c_\delta) + \delta_{\beta\delta} v_b^2 N(c_\alpha^* c_\gamma) \\ & - \delta_{\alpha\delta} v_a^2 N(c_\beta^* c_\gamma) - \delta_{\beta\gamma} v_b^2 N(c_\alpha^* c_\delta) \\ & + \delta_{\alpha,-\beta}\delta_{\gamma,-\delta} s_\alpha s_\gamma u_a v_a u_c v_c + (\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) v_a^2 v_b^2. \end{aligned} \quad (22)$$

This is substituted in H_i and use is made of well-known properties of the C coefficients. For instance, in view of the relation

$$\begin{aligned} \sum_{m_\alpha M} & C(j_\alpha j_\beta J; m_\alpha m_\beta M) C(j_\alpha j_\delta J; m_\alpha m_\delta M) \\ & = (2J+1)(2j_\beta+1)^{-1} \delta_{j_\beta j_\delta} \delta_{m_\beta m_\delta}, \end{aligned} \quad (23)$$

the first term of the third line of (22), when substituted in H_i, gives

$$\begin{aligned} -\tfrac{1}{2} \sum_{\alpha\beta\delta J} & G(abadJ)(2J+1) \\ & \times (2j_\beta+1)^{-1} \delta_{j_\beta j_\delta} \delta_{m_\beta m_\delta} v_a^2 N(c_\beta^* c_\delta). \end{aligned} \quad (24)$$

At this point we introduce an essential simplification. We assume that, among all our levels a, b, $\cdots$, a given

[12] G. Racah, Phys. Rev. **63**, 367 (1943).

[13] S. S. Schweber, H. A. Bethe, and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, 1955), Vol. I. p. 203.

[14] Reference 13, p. 210. The theorem is closely related to Wick's theorem.

combination of charge, parity, and j value occurs only once. This enables us to replace $\delta_{j\beta j\delta}$ in (24) by δ_{bd} and to sum immediately over δ. In practice, this restriction is not serious, as it is satisfied in most shell model calculations. The same assumption has to be invoked for all the terms in the middle three lines of (22).

Introduce the notations

$$\Delta_a=(2j_a+1)^{-\frac{1}{2}}\sum_c(2j_c+1)^{\frac{1}{2}}u_cv_cG(aacc0), \quad (25)$$

$$\mu_a=2(2j_a+1)^{-1}\sum_{bJ}(2J+1)v_b{}^2G(ababJ)$$
$$=2(2j_a+1)^{-\frac{1}{2}}\sum_b(2j_b+1)^{\frac{1}{2}}v_b{}^2F(aabb0), \quad (26)$$

$$\eta_a=\epsilon_a-\mu_a-\lambda_a. \quad (27)$$

After the Bogoliubov-Valatin transformation, $\mathcal{H}$ can be written as the sum of four parts,

$$\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+\mathcal{H}_4. \quad (28)$$

The first part is a pure number, the energy of the ground state,

$$\mathcal{H}_1=\sum_\alpha[v_a{}^2(\eta_a+\tfrac{1}{2}\mu_a)-\tfrac{1}{2}u_av_a\Delta_a]. \quad (29a)$$

The others are

$$\mathcal{H}_2=\sum_\alpha[(u_a{}^2-v_a{}^2)\eta_a+2u_av_a\Delta_a]a_\alpha{}^*a_\alpha, \quad (29b)$$

$$\mathcal{H}_3=\sum_\alpha s_\alpha[u_av_a\eta_a+\tfrac{1}{2}(v_a{}^2-u_a{}^2)\Delta_a]$$
$$\times(a_\alpha{}^*a_{-\alpha}{}^*+a_{-\alpha}a_\alpha), \quad (29c)$$

$$\mathcal{H}_4=\sum_{\alpha\beta\gamma\delta}\mathcal{V}_{\alpha\beta\gamma\delta}N(c_\alpha{}^*c_\beta{}^*c_\delta c_\gamma). \quad (29d)$$

One chooses the u's and v's in such a way that $\mathcal{H}_3$ vanishes. This is what Bogoliubov[9] calls "the elimination of the dangerous terms"; it is equivalent to the BCS procedure of minimizing $\mathcal{H}_1$. The result is

$$2u_av_a=\Delta_a/E_a, \quad u_a{}^2-v_a{}^2=\eta_a/E_a, \quad (30)$$

with

$$E_a=(\eta_a{}^2+\Delta_a{}^2)^{\frac{1}{2}}. \quad (31)$$

Then, $\mathcal{H}_2$ takes the form

$$\mathcal{H}_2=\sum_\alpha E_aa_\alpha{}^*a_\alpha, \quad (32)$$

which shows that E_a is the energy of a quasi-particle. Δ_a is half the energy gap. The quantity η_a is the single-particle energy, corrected for the self-energy μ_a, and counted from the Fermi level λ_a. The two λ's are determined by writing that the expectation value of $\mathfrak{N}$ for the ground state is a given number n, for both neutrons and protons, for instance

$$n_n=\sum_n v_a{}^2=\tfrac{1}{2}\sum_n(1-\eta_a/E_a). \quad (33)$$

As for $\mathcal{H}_4$, it is the residual interaction between quasi-particles. Its explicit expression in terms of the a operators involves sixteen terms.

In the case of a general nuclear force G, the values of Δ_a, μ_a, λ_a can only be obtained by solving the complicated set of coupled equations given above. The problem simplifies itself[7] in the case of the pairing force (15), because μ_a can be neglected and one sees that Δ_a depends only on the charge. For a more realistic force, the various Δ_a's for a given charge are still roughly equal; hence the notion of an energy gap, without reference to a single-particle state, is still a meaningful one.

3. THE APPROXIMATION PROCEDURE

A. Basic Equations

If the interaction were really a pairing force, $\mathcal{H}_2$ would be the most important part of the Hamiltonian and $\mathcal{H}_4$ could be treated by perturbation theory, at least for states with $J\neq0$, because F would be small and would have only diagonal elements. In this case, one obtains the picture of nearly independent quasi-particles, the creation of a pair of quasi-particles requiring an energy at least equal to the energy gap. KS have shown that this picture agrees well with many detailed properties of heavy nuclei, hence the pairing force is certainly an important element in the real force. But, to properly take into account other parts of the force, a better treatment of $\mathcal{H}_4$ must be given. The first idea that comes to mind is to diagonalize $\mathcal{H}_4$ exactly between all two-quasi-particle states. The neglect of states containing four or more quasi-particles is justified by saying that, since there is an energy gap, their excitation requires more energy. In the language of Feynman diagrams, the method consists in summing exactly all diagrams similar to Fig. 1(a). It is also known as the Tamm-Dancoff approximation. The calculation is very similar to a standard shell-model calculation for two particles, which is not hard. This procedure cannot yield collective states, which necessarily contain a large number of quasi-particles.

The method which we propose to use is more powerful and almost as easy. It is well known in the theory of

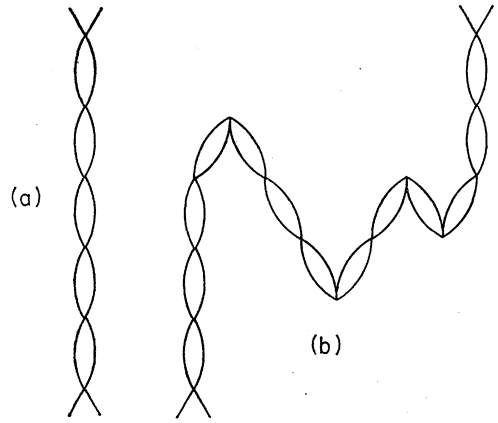

FIG. 1. (a) Feynman diagram included in the Tamm-Dancoff approximation. The lines are quasi-particles. No arrows are shown since a quasi-particle is part particle, part hole. There are two quasi-particles present at all times. The complete interaction between quasi-particles, $\mathcal{H}_4$, acts at each vertex. (b) Additional diagram included in the approximation of linearized equations of motion. The chain may double upon itself any number of times. There may be 2, 6, 10, $\cdots$ quasi-particles at once.

superconductors, under such varied names as random phase approximation,[15] method of linearized equations of motion,[15] method of approximate second quantization.[16] It is also allied to Dyson's new Tamm-Dancoff method.[17] It is equivalent to summing all diagrams[18] similar to Fig. 1(b). Our treatment will be closest to that of Anderson.[15] Let us define some creation and absorption operators for pairs of quasi-particles coupled to JM,

$$A^*(abJM)=\sum_{m_\alpha m_\beta} C(j_\alpha j_\beta J; m_\alpha m_\beta M) a_\alpha^* a_\beta^*, \quad (34a)$$

$$A(abJM)=\sum_{m_\alpha m_\beta} C(j_\alpha j_\beta J; m_\alpha m_\beta M) a_\beta a_\alpha. \quad (34b)$$

Upon interchange of a and b, we have for instance

$$A(baJM)=-\theta(abJ)A(abJM). \quad (35)$$

Let Ψ_0 be the real ground state (not the BCS approximate state) and Ψ_{BM} an excited state with angular momentum JM, the subscript B being used to distinguish it from other excited states. We define two amplitudes ψ and φ (independent of M) by

$$\psi_{abB}=\langle\Psi_0|A(abJM)|\Psi_{BM}\rangle, \quad (36a)$$

$$\varphi_{abB}=s_{JM}\langle\Psi_0|A^*(abJ\bar{M})|\Psi_{BM}\rangle, \quad (36b)$$

with

$$s_{JM}=(-)^{J-M}.$$

The symmetry relations for ψ and φ are

$$\psi_{baB}=-\theta(abJ)\psi_{abB}, \quad (37a)$$

$$\varphi_{baB}=-\theta(abJ)\varphi_{abB}. \quad (37b)$$

To obtain equations for these amplitudes, one takes matrix elements of the equations of motion for the A operators. If we call ω_B the excitation energy of Ψ_{BM} with respect to Ψ_0, we can write

$$\langle\Psi_0|[A,\mathcal{H}]|\Psi_{BM}\rangle=\omega_B\langle\Psi_0|A|\Psi_{BM}\rangle, \quad (38)$$

since both Ψ_0 and Ψ_{BM} are eigenstates of $\mathcal{H}$. If the commutator of $\mathcal{H}$ with one of the A operators of (34) is calculated, it is found to contain terms with two a operators and terms with four a operators. The approximation consists in dropping the latter terms, i.e., linearizing the equations of motion, on the grounds that they involve more energetic excitations. A serious discussion of the validity of this approximation is very difficult and we shall not attempt it. We are left with a set of linear, homogeneous equations for matrix elements of pairs of quasi-particle operators. Of course, we must include combinations of products of the type $a_\alpha^* a_\beta$ as well as the combinations of Eq. (34). But these other combinations are found to be completely uncoupled from each other and from the A's. Amplitudes of the type

$$\langle\Psi_0|a_\alpha^* a_\beta|\Psi_{BM}\rangle, \quad (39)$$

must therefore be set equal to 0, since they vanish in the limit of independent quasi-particles,[19] $\mathcal{H}_4=0$. Define two functions P and R through the equations

$$\begin{aligned}[A(abJM),\mathcal{H}]&=\sum_{cd}P(abcdJ)A(cdJM)\\&\quad+\sum_{cd}R(abcdJ)s_{JM}A^*(cdJ\bar{M})\\&\quad+\text{terms in } a^*a+\text{terms with four } a\text{'s.}\end{aligned} \quad (40a)$$

$$\begin{aligned}[s_{JM}A^*(abJ\bar{M}),\mathcal{H}]&=-\sum_{cd}P(abcdJ)s_{JM}A^*(cdJ\bar{M})\\&\quad-\sum_{cd}R(abcdJ)A(cdJM)\\&\quad+\text{terms in } a^*a+\text{terms with four } a\text{'s.}\end{aligned} \quad (40b)$$

They satisfy the symmetry relations

$$\begin{aligned}P(cdabJ)&=P(abcdJ),\\R(cdabJ)&=R(abcdJ).\end{aligned} \quad (41)$$

With our approximations, the quantities ψ, φ, and ω are solutions of the eigenvalue equation

$$\begin{aligned}\omega_B\psi_{abB}&=\sum_{cd}P(abcdJ)\psi_{cdB}+\sum_{cd}R(abcdJ)\varphi_{cdB},\\-\omega_B\varphi_{abB}&=\sum_{cd}P(abcdJ)\varphi_{cdB}+\sum_{cd}R(abcdJ)\psi_{cdB}.\end{aligned} \quad (42)$$

This simple system of equations resembles very much the equations for a two-particle shell-model calculation, but we shall see in the next subsection that it also includes some collective solutions. It is clear that the problem is equivalent to diagonalizing a nonsymmetric matrix whose number of lines and columns is twice the number of two-particle states. For every solution B with $\omega_B>0$, there exists another solution B' with $\omega_{B'}=-\omega_B$ and ψ and φ interchanged. The latter solution must be rejected; only solutions with $\omega_B>0$ have physical significance.

The task of expressing P and R in terms of the nuclear force G or F of Sec. 2 involves a fair amount of manipulating and only the results will be given. It is convenient

[15] P. W. Anderson, Phys. Rev. **112**, 1900 (1958). The method was applied to the electron gas by D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953); K. Sawada, Phys. Rev. **106**, 372 (1957); K. Sawada, K. A. Brueckner, N. Fukuda, and R. Brout, Phys. Rev. **108**, 507 (1957).

[16] N. N. Bogoliubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau, New York, 1959). See also V. M. Galitskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 1011 (1958) [translation: Soviet Phys.-JETP **34**, (7), 698 (1958)].

[17] F. J. Dyson, Phys. Rev. **91**, 1543 (1953). The present approximation keeps an infinite subset of new Tamm-Dancoff amplitudes.

[18] The importance of these backward-going diagrams in the treatment of nuclear collective oscillations has been emphasized repeatedly by S. Fallieros and R. Ferrell, who have also discussed the corresponding modification (69) of the normalization. See in particular S. Fallieros, Ph.D. thesis, University of Maryland, 1959 (Physics Department Technical Report No. 128), and R. A. Ferrell, Bull. Am. Phys. Soc. **4**, 59 (1959).

[19] The amplitude φ also vanishes in that limit, but it is coupled to ψ.

for the following to define two new amplitudes by

$$f_{abB}=\psi_{abB}+\varphi_{abB}, \tag{43a}$$

$$g_{abB}=\psi_{abB}-\varphi_{abB}. \tag{43b}$$

The equations for f and g are found to be

$$\begin{aligned}\omega_B f_{abB}&=(E_a+E_b)g_{abB}-\textstyle\sum_{cd}U(abcdJ)g_{cdB},\\ \omega_B g_{abB}&=(E_a+E_b)f_{abB}-\textstyle\sum_{cd}V(abcdJ)f_{cdB},\end{aligned} \tag{44}$$

with

$$\begin{aligned}U(abcdJ)=&(u_au_b+v_av_b)(u_cu_d+v_cv_d)G(abcdJ)\\&+(u_av_b-v_au_b)(u_cv_d-v_cu_d)H(abcdJ),\end{aligned} \tag{45a}$$

$$\begin{aligned}V(abcdJ)=&(u_au_b-v_av_b)(u_cu_d-v_cv_d)G(abcdJ)\\&+(u_av_b+v_au_b)(u_cv_d+v_cu_d)K(abcdJ),\end{aligned} \tag{45b}$$

$$H(abcdJ)=F(abcdJ)+\theta(cdJ)F(abdcJ), \tag{46a}$$

$$K(abcdJ)=F(abcdJ)-\theta(cdJ)F(abdcJ). \tag{46b}$$

It is important to note that, in all this work, the summations run over all values of the indices, which means that a given pair of distinct single-particle levels appears twice. Since, upon interchange of the indices, the amplitudes f and g transform in the same way as ψ and φ, i.e., by Eq. (37), the matrices U and V should have the following symmetry properties:

$$\begin{aligned}U(abcdJ)&=U(cdabJ)\\&=-\theta(abJ)U(bacdJ)=-\theta(cdJ)U(abdcJ)\\&=\theta(abcd)U(badcJ),\end{aligned} \tag{47}$$

and similarly for V. It can be checked that they do. One of the two functions, say g, can be eliminated from Eq. (44). Then f is seen to be the eigenvector of a non-symmetric matrix for eigenvalue ω_B^2.

More will be said in the next subsection about the significance of Eqs. (42) or (44). But first, we shall show how the Tamm-Dancoff approximation referred to earlier is recovered in the limit of weak coupling, i.e., small $\mathcal{H}_4$. In that case, it is clear from Eq. (44) that f and g are about equal (since $\omega_B>0$), i.e., $\varphi\ll\psi$. If we neglect φ in Eq. (42) and realize that R is also small compared to P (because $\mathcal{H}_2$ does not contribute to R), we get an equation for ψ alone,

$$\omega_B\psi_{abB}=\textstyle\sum_{cd}P(abcdJ)\psi_{cdB}. \tag{48}$$

This same equation would come out of the Tamm-Dancoff method. Then, one would define ψ by

$$\psi_{abB}=\langle 0|A(abJM)|\Psi_{BM}\rangle, \tag{49}$$

where $\langle 0|$ is the BCS ground state, and neglect all other amplitudes. The Schrödinger equation would be

$$W_B\psi_{abB}=\tfrac{1}{2}\textstyle\sum_{cd}\langle 0|A(abJM)\mathcal{H}A^*(cdJM)|0\rangle\psi_{cdB}, \tag{50}$$

where W_B is the energy, and one would normalize ψ (which is real) by

$$\textstyle\sum_{ab}(\psi_{abB})^2=2. \tag{51}$$

The appearance of the factors $\frac{1}{2}$ and 2 is due to the fact, already mentioned, that our summations involve most pairs of levels twice. The matrix element in (50) can be written

$$\begin{aligned}&\langle 0|[A(abJM),\mathcal{H}]A^*(cdJM)|0\rangle\\&\qquad+\langle 0|\mathcal{H}A(abJM)A^*(cdJM)|0\rangle\\&=P(abcdJ)-\theta(cdJ)P(abdcJ)\\&\qquad+W_0[\delta_{ac}\delta_{bd}-\theta(cdJ)\delta_{ad}\delta_{bc}],\end{aligned} \tag{52}$$

where W_0 is the energy of the ground state. In view of the various symmetry relations, Eq. (48) would be obtained.

B. Collective Vibrations

To show how collective vibrations arise out of the formalism, we shall repeat the derivation of Eq. (42) somewhat differently. Let us look for some operators Q_{BM}, Q_{CM}, $\cdots$ with the following property

$$[Q_{BM},\mathcal{H}]=\omega_BQ_{BM}, \tag{53a}$$

$$[Q_{BM}{}^*,\mathcal{H}]=-\omega_BQ_{BM}{}^*, \tag{53b}$$

where ω_B is some number. Let us assume that the Q's can be expressed as linear combinations of the A's of a given J, with real coefficients,

$$\begin{aligned}Q_{BM}=&\textstyle\sum_{ab}X_{abB}A(abJM)\\&-\textstyle\sum_{ab}Y_{abB}s_{JM}A^*(abJ\bar{M}),\end{aligned} \tag{54a}$$

$$\begin{aligned}s_{JM}Q_{B\bar{M}}{}^*=&\textstyle\sum_{ab}X_{abB}s_{JM}A^*(abJ\bar{M})\\&-\textstyle\sum_{ab}Y_{abB}A(abJM).\end{aligned} \tag{54b}$$

If we substitute this expansion in Eqs. (53), and if we decide, as we did in subsection A, to ignore in the commutator terms of the form a^*a or terms containing four a's, we obtain the following set of equations for the X's and Y's:

$$\begin{aligned}\omega_BX_{abB}=&\textstyle\sum_{cd}P(cdabJ)X_{cdB}\\&+\textstyle\sum_{cd}R(cdabJ)Y_{cdB},\\-\omega_BY_{abB}=&\textstyle\sum_{cd}P(cdabJ)Y_{cdB}\\&+\textstyle\sum_{cd}R(cdabJ)X_{cdB},\end{aligned} \tag{55}$$

which is identical with the set (42). Thus, X and Y are proportional to ψ and φ. In the following, we consider only solutions of these equations for which ω_B is positive. One can easily deduce[20] from the equations a kind of orthogonality relation,

$$\textstyle\sum_{ab}(X_{abB}X_{abC}-Y_{abB}Y_{abC})=0, \tag{56a}$$

which holds for two different solutions B and C of the same J. Since one obtains another solution of Eq. (55) by exchanging X_B and Y_B, the following also holds for any two solutions with positive ω and the same J:

$$\textstyle\sum_{ab}(Y_{abB}X_{abC}-X_{abB}Y_{abC})=0. \tag{56b}$$

[20] Multiply Eqs. (55) by X_{abC} and Y_{abC}, respectively, and add. Repeat the procedure with B and C interchanged. Subtract the two and use Eq. (41).

Equations (53) show that Q_{BM}^* and Q_{BM} are the raising and lowering operators of a harmonic oscillator. Q_{BM}^* creates a vibrational quantum of energy ω_B and Q_{BM} destroys it. The ground state may be defined by the set of all equations

$$Q_{BM}|\Psi_0\rangle=0, \tag{57}$$

and a particular excited state would be, for instance

$$|\Psi_{BM}\rangle=Q_{BM}^*|\Psi_0\rangle, \tag{58}$$

whose energy with respect to $|\Psi_0\rangle$ is ω_B. The ground state so defined is more accurate than the BCS ground state, as it includes the zero-point motion of the oscillators. For most of the applications, it is not necessary to have an explicit form for $|\Psi_0\rangle$. But, if one were desired, one could assume the most general expansion

$$|\Psi_0\rangle=\chi_0|0\rangle+\sum_{\alpha\beta}\chi_2(\alpha\beta)a_\alpha^*a_\beta^*|0\rangle + \sum_{\alpha\beta\gamma\delta}\chi_4(\alpha\beta\gamma\delta)a_\alpha^*a_\beta^*a_\gamma^*a_\delta^*|0\rangle+\cdots, \tag{59}$$

and, after having solved Eqs. (55), determine the χ's from Eqs. (57). It will easily be seen that only terms with 0, 4, 8, $\cdots$ quasi-particles occur. On the other hand, the excited state (58) has only 2, 6, 10, $\cdots$ quasi-particles. This is consistent with our statement that the amplitude (39) vanishes. The χ's are determined by a set of recurrence relations which connect χ_n to χ_{n-4}. But since the sum in (59) actually has a finite number of terms, trouble will occur unless the χ's become small before the end of the sum is reached. A related cause of trouble is the Pauli principle, which requires that any χ with two equal arguments vanish and makes the number of equations larger than the number of χ's. We see therefore that it is essential for the validity of the approximation that the average number of quasi-particles in the ground state, ν_0, be small compared to the total number of states, which we call Ω. In other words, we wish to treat the Q operators as boson operators, i.e., creation and absorption operators for oscillator quanta; but since the Q's are actually made up of products of operators that obey the Pauli principle, this is possible only if the number of states available is much larger than the number of fermions present. Otherwise, the fermions start to get into each other's way and the whole picture breaks down. This would happen, presumably, if the interaction was made so strong that many vibrational frequencies were very low. It might be noted in this connection[4] that the BCS ground state itself is good only for large Ω.

If this interpretation is to be consistent, it must be verified that two different singly excited states, $|\Psi_{BM}\rangle$ and $|\Psi_{CM'}\rangle$, are orthogonal. We write

$$\begin{aligned}\langle\Psi_0|Q_{BM}Q_{CM'}^*|\Psi_0\rangle &=\langle\Psi_0|[Q_{BM},Q_{CM'}^*]|\Psi_0\rangle\\ &=\sum_{abcd}X_{abB}X_{cdC}\langle\Psi_0|[A(abJM),A^*(cdJ'M')]|\Psi_0\rangle\\ &\quad-\delta_{JM}\delta_{J'M'}\sum_{abcd}Y_{abB}Y_{cdC}\\ &\quad\times\langle\Psi_0|[A(cdJ'\bar{M}'),A^*(abJ\bar{M})]|\Psi_0\rangle,\end{aligned} \tag{60}$$

using Eq. (57) and the fact that two A^*'s or two A's commute. The commutator of an A with an A^* is found to be

$$\begin{aligned}&[A(abJM),A^*(cdJ'M')]\\ &=\sum_{m_\alpha m_\beta m_\gamma m_\delta}C(j_\alpha j_\beta J;m_\alpha m_\beta M)C(j_\gamma j_\delta J';m_\gamma m_\delta M')\\ &\quad\times[\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\gamma}a_\delta^*a_\beta\\ &\quad-\delta_{\beta\delta}a_\gamma^*a_\alpha+\delta_{\beta\gamma}a_\delta^*a_\alpha+\delta_{\alpha\delta}a_\gamma^*a_\beta],\end{aligned} \tag{61}$$

hence we need to know vacuum expectation values of products such as $a_\gamma^*a_\alpha$. This certainly vanishes unless the charge, the parity, and the angular momentum of α are the same as those of γ. If we invoke again the simplifying assumption made in Sec. 2B, this means that α must be the same as γ. Calling ν_a the average number of quasi-particles of type α in the ground state, we write

$$\langle\Psi_0|a_\gamma^*a_\alpha|\Psi_0\rangle=\delta_{\alpha\gamma}\nu_a. \tag{62}$$

After performing the summation over magnetic quantum numbers, one finds

$$\begin{aligned}&\langle\Psi_0|[Q_{BM},Q_{CM'}^*]|\Psi_0\rangle\\ &=2\delta_{JJ'}\delta_{MM'}\textstyle\sum_{ab}(X_{abB}X_{abC}-Y_{abB}Y_{abC})\\ &\quad\times(1-2\nu_a).\end{aligned} \tag{63}$$

The quantity ν_a is of order ν_0/Ω, which we assumed to be small and can be neglected. Then, the orthogonality relation (56a) shows that the expression vanishes for $B\neq C$. The case $B=C$ gives us the normalization, which we want to be

$$\langle\Psi_0|Q_{BM}Q_{BM}^*|\Psi_0\rangle=1, \tag{64}$$

and therefore, if $J_B=J_C$,

$$\textstyle\sum_{ab}(X_{abB}X_{abC}-Y_{abB}Y_{abC})=\frac{1}{2}\delta_{BC}. \tag{65}$$

The argument can be extended to show that the Q's satisfy approximately the commutation relations

$$[Q_{BM},Q_{CM'}]=[Q_{BM}^*,Q_{CM'}^*]=0, \tag{66a}$$

$$[Q_{BM},Q_{CM'}^*]=\delta_{BC}\delta_{MM'}, \tag{66b}$$

as far as all matrix elements between low excited states are concerned, relations (66a) being a consequence of Eq. (56b). Thus, as long as ν_0/Ω is small, we have truly a system of independent bosons and we can con-

sider higher excited states such as $Q_{BM}{}^{*2}|\Psi_0\rangle$ or $Q_{BM}{}^*Q_{CM'}{}^*|\Psi_0\rangle$, with energies $2\omega_B$ and $\omega_B+\omega_C$. Of course, the higher the excitation, the worse the approximations become. And, in any case, it is not allowed to carry these ideas to excitations larger than twice the gap, where we would get mixing with configurations that we have not properly taken into account. Since most ω's, by far, are larger than the gap, the corresponding Q^*'s can be applied only once and do not give rise to a vibrational spectrum, properly speaking, since there is only one excited state. Those are the excitations that can be described as of the single-particle type. We know that these excitations are also given correctly by our method, since it is an improvement upon the Tamm-Dancoff method that would normally be used for them. It is the virtue of the present approximation that the same equations describe single-particle and collective states, as well as all shades in between. Consequently, the various states are automatically orthogonal to each other (approximately) and the total number of states is correct. The main result of this section is therefore that, due to the special treatment of the pairing correlations and the importance of the energy gap, it has become possible to perform a certain type of shell-model calculation that will give the low excited states of spherical, nonclosed-shell nuclei, including the vibrational states, and whose difficulty is not much greater than that of a two-body shell-model calculation, except for the choice of parameters.

There remains to establish the connection between X and Y on the one hand and ψ and φ on the other. We defined ψ in Eq. (36a) by

$$\psi_{abB}=\langle\Psi_0|A(abJM)Q_{BM}{}^*|\Psi_0\rangle = \langle\Psi_0|[A(abJM),Q_{BM}{}^*]|\Psi_0\rangle. \quad (67)$$

All we have to do is express $Q_{BM}{}^*$ in terms of the A's by Eq. (54b) and insert the previously given value of the commutator of A and A^*, neglecting terms of order ν_0/Ω as earlier. The result is

$$\psi_{abB}=2X_{abB}, \quad \varphi_{abB}=2Y_{abB}, \quad (68)$$

and the orthonormalization conditions become[18]

$$\sum_{ab}(\psi_{abB}\psi_{abC}-\varphi_{abB}\varphi_{abC})=2\delta_{BC}, \quad (69a)$$

$$\sum_{ab}(\varphi_{abB}\psi_{abC}-\psi_{abB}\varphi_{abC})=0, \quad (69b)$$

provided $J_B=J_C$. We note that this agrees with Eq. (51) in the limit where φ is small. In terms of f and g, these equations become simpler, namely

$$\sum_{ab} f_{abB}g_{abC}=2\delta_{BC}, \quad (J_B=J_C). \quad (70)$$

One can also derive a simple sum rule. For this, it is more convenient to sum over a given pair of levels only once. We designate the pair (ab) or (ba), coupled to J, by the single Greek index Γ, and furthermore we define

$$f_{\Gamma B}{}'=\begin{cases} f_{abB} & \text{if } a\neq b \\ f_{abB}/\sqrt{2} & \text{if } a\equiv b \end{cases} \quad (71)$$

and $g_{\Gamma B}{}'$ similarly. Then Eq. (70) becomes

$$\sum_\Gamma f_{\Gamma B}{}'g_{\Gamma C}{}'=\delta_{BC}, \quad (J_B=J_C). \quad (72)$$

The number of possible values taken by Γ is also the number of possible values for B or C. If one assumes that the vectors $f_{\Gamma B}{}'$, for instance, form a complete set, then the following sum rule holds:

$$\sum_B f_{\Gamma B}{}'g_{\Delta B}{}'=\delta_{\Gamma\Delta}, \quad (J_\Gamma=J_\Delta), \quad (73)$$

since it is true when applied to every vector of the set. This sum rule is not accurate in the case $J=0$, because then the physical solutions of our equations do not form a complete set, due to the existence of spurious solutions to be discussed next.

C. The Spurious States

It is well known[4,7] that the independent quasi-particle picture yields too many states. There is one extra state of $J=0$ among all states containing a pair of neutron quasi-particles, and another one for proton pairs. Their existence is due to the fact that, since the BCS ground state is not eigenstate of $\mathfrak{N}_n$ or $\mathfrak{N}_p$, the states $\mathfrak{N}_n|0\rangle$ and $\mathfrak{N}_p|0\rangle$ are different from $|0\rangle$ and their components on two-quasi-particle states are spurious; only states orthogonal to them have equivalents in a physical nucleus. But the two-quasi-particle states resulting from an approximate diagonalization of the Hamiltonian are usually not orthogonal to the spurious states, with the result that the spurious states are mixed with various percentages among all the states that one calculates. It is a major advantage of the present method that this difficulty does not arise, as was pointed out by Anderson.[15] Here, two of the solutions of our equations are entirely spurious and the others not at all. The spurious solutions have positive parity, $J=0$, $\omega=0$, $f=0$, and

$$g_{aa0}=(2j_a+1)^{\frac{1}{2}}u_av_a, \quad (74)$$

a being a neutron state in one case, a proton in the other. It can easily be verified that these are solutions of Eqs. (44), provided one appeals once again to the assumption that the charge, the parity, and the angular momentum determine the state; as a consequence, for states of two neutrons and positive parity for instance, g_{ab0} exists only if $a\equiv b$.

The existence of these simple solutions is due to the fact that $\mathfrak{N}_n$ and $\mathfrak{N}_p$ satisfy Eq. (53) for Q_{BM} with $\omega_B=0$, since they commute exactly with $\mathfrak{IC}$. They can be written

$$\mathfrak{N}_n=\sum_n\{v_a{}^2+(u_a{}^2-v_a{}^2)a_\alpha{}^*a_\alpha + (2j_a+1)^{-\frac{1}{2}}u_av_a[A(aa00)+A^*(aa00)]\}, \quad (75)$$

and $\mathfrak{N}_p$ similarly. Therefore, two of our Q's are immediately known,

$$Q_n=\sum_n(2j_a+1)^{-\frac{1}{2}}u_av_a[A(aa00)+A^*(aa00)], \quad (76)$$

and Q_p similarly. We cannot apply to them all the

arguments of Sec. 2B because they are Hermitian. But the orthogonality relations (56) are still satisfied, and therefore we can repeat the argument leading to Eq. (63) to show that

$$\langle\Psi_0|\mathfrak{N}_n Q_{B0}^*|\Psi_0\rangle\approx 0, \tag{77}$$

for instance, where B is any other $J=0$ excited state. This shows that the other solutions of our equations are orthogonal to $\mathfrak{N}_n|\Psi_0\rangle$ and are nonspurious in the sense used above. They are not, however, eigenstates of $\mathfrak{N}_n$ or $\mathfrak{N}_p$. The states still describe a mixture of neighboring even-even nuclei, and in fact the expectation value of $\mathfrak{N}_n$ or $\mathfrak{N}_p$ varies from state to state, depending on the relative amounts of hole or particle character in the excitations. But, at least, the number of states is correct.

4. APPLICATIONS

We have seen that Eqs. (44), when used with a realistic effective two-body force, should be capable of accounting approximately for the properties of all low-lying levels of even-even spherical nuclei. A calculation of this type is necessarily lengthy and none have yet been performed. Here, we propose only to demonstrate the power of the method in dealing with collective effects. We shall use very simple forces and the calculations will be quite rough. We have never given explicit expressions for the various wave functions, but this is not so serious because we shall see that, for many applications, the amplitudes f and g are just what one needs.

A. Quadrupole Vibrations

Here, the aim is to understand the relatively small vibrational frequency and the large electric quadrupole matrix element between the first 2^+ excited state and the ground state.[21] We follow KS and take an interaction composed of a pairing force, Eq. (15), and a quadrupole force of the type introduced by Elliott[22] (the discussion is limited to particles of a single charge),

$$H_Q=-\tfrac{1}{2}\chi\sum_{\alpha\beta\gamma\delta\mu}(-)^\mu q_\mu(\alpha\gamma)q_{\bar\mu}(\beta\delta)c_\alpha^*c_\beta^*c_\delta c_\gamma, \tag{78}$$

$$q_\mu(\alpha\gamma)=\langle\alpha|r^2Y_{2\mu}(\theta,\varphi)|\gamma\rangle. \tag{79}$$

Essentially, the pairing force contributes only to $\mathfrak{IC}_2$ and the quadrupole force only to $\mathfrak{IC}_4$. The method used by KS to get the collective state consists in picking a collective coordinate, in this case the quadrupole moment Q; calculating the energy for a fixed value of Q; invoking the adiabatic approximation and using this energy as potential energy for the motion of Q; deriving a kinetic energy from Inglis' cranking formula; and finally solving the Schrödinger equation for Q, in this case a harmonic oscillator equation. We shall see on an example that Eqs. (44) give results that are nearly the same, without invoking the adiabatic approximation.

It is convenient to change the phases of the states of two particles (abJ). We multiply them all by

$$S_{abJ}=(-)^{j_b+\frac{1}{2}}\times\text{sgn } C(j_aj_bJ;\tfrac{1}{2}-\tfrac{1}{2}0). \tag{80}$$

We prime the matrices expressed with these new phases. For instance,

$$V'(abcdJ)=S_{abJ}S_{cdJ}V(abcdJ). \tag{81}$$

Their symmetry properties are simpler, namely

$$\begin{aligned}V'(abcdJ)&=V'(cdabJ)=V'(badcJ)\\&=(-)^JV'(bacdJ)=(-)^JV'(abdcJ).\end{aligned} \tag{82}$$

We need the matrices G', H', and K' for $J=2$. It turns out that G' and H' are small because, through recoupling of the angular momenta, the strength of the quadrupole force is distributed between many J values. But K' contains a large term for 2^+ states,

$$K'(abcd2^+)\approx(\chi/20\pi)L_{ab}^{\frac{1}{2}}L_{cd}^{\frac{1}{2}}, \tag{83}$$

$$L_{ab}=\mathcal{R}_{ab}^2(2j_a+1)(2j_b+1)\times[C(j_aj_b2;\tfrac{1}{2}-\tfrac{1}{2}0)]^2, \tag{84}$$

$$\mathcal{R}_{ab}=\int_0^\infty R_a(r)R_b(r)r^4dr. \tag{85}$$

The fact that all matrix elements of K' are positive leads us to expect a collective solution. Equations (44) become (for $J=2^+$)

$$\begin{aligned}&\omega f_{ab}=(E_a+E_b)g_{ab},\\&\omega g_{ab}=(E_a+E_b)f_{ab}-(\chi/20\pi)\sin(x_a+x_b)L_{ab}^{\frac{1}{2}}\\&\qquad\times\sum_{(cd)2^+}\sin(x_c+x_d)L_{cd}^{\frac{1}{2}}f_{cd},\end{aligned} \tag{86}$$

with

$$u_a=\cos x_a,\quad v_a=\sin x_a. \tag{87}$$

Solution is easy and the secular equation is[23]

$$\frac{20\pi}{\chi}=\sum_{(ab)2^+}\frac{\sin^2(x_a+x_b)L_{ab}(E_a+E_b)}{(E_a+E_b)^2-\omega^2}. \tag{88}$$

The collective solution is that value of ω which is below all (E_a+E_b). If it does not exist because χ is too large, then the nucleus is not spherical and this method cannot be applied.

As an example, we pick the following parameters which are very close to those used by KS for Sn^{118}: $\eta(h_{11/2})=0.80$, $\eta(d_{3/2})=0.24$, $\eta(s_{1/2})=-0.04$, $\eta(g_{7/2})=-1.70$, $\eta(d_{5/2})=-1.90$, $\Delta=1.02$, all in Mev; $\chi=0.090$ Mev α^4; $\alpha=(m\omega_0/\hbar)^{\frac{1}{2}}$, where ω_0 is the frequency of the oscillator well used; $\alpha^{-4}=2.46\times10^{-51}$ cm^4. We find the vibrational frequency to be $\omega=1.17$ Mev. The same

[21] G. S. Goldhaber and J. Weneser, Phys. Rev. **98**, 212 (1955).
[22] J. P. Elliott, Proc. Roy. Soc. (London) **A245**, 128, 562 (1958).
[23] Dr. Sorensen points out that this equation reduces to Eq. (32) of KS in the limit of small ω.

value is obtained with the method of KS if one takes $\chi=0.091$ Mev α^4. The deviations between the two methods would be larger for larger ω, but for states that are truly collective they are equivalent.

The strength of the electric quadrupole transition between the first excited state and the ground state is measured by[24]

$$B(E2)=\sum_{M\mu}|\langle\Psi_0|\mathfrak{M}(E2,\mu)|\Psi_{2^+M}\rangle|^2, \tag{89}$$

with

$$\mathfrak{M}(E2,\mu)=\int r^2Y_{2\mu}(\theta,\varphi)\rho(\mathbf{x})d^3x, \tag{90}$$

$$\rho(\mathbf{x})=e\sum_{m_s}\phi^*(\mathbf{x}m_s)\phi(\mathbf{x}m_s), \tag{91}$$

where $\phi(\mathbf{x}m_s)$ destroys a particle at position $\mathbf{x}$ with spin m_s, and e is the effective charge. To calculate $B(E2)$, one expresses the ϕ operators in terms of the c's,

$$\phi(\mathbf{x}m_s)=\sum_\alpha\langle\mathbf{x}m_s|\alpha\rangle c_\alpha, \tag{92}$$

then the c's in terms of the a's by Eqs. (18), then one introduces the amplitudes ψ and φ by Eqs. (36), and one finds

$$B(E2)=(5e^2/80\pi)\Big[\sum_{(ab)2^+}L_{ab}^{\frac{1}{2}}\sin(x_a+x_b)f_{ab}\Big]^2. \tag{93}$$

It is essential here to have the correct normalization. This is given by Eq. (70) and the final result is

$$B(E2)=\frac{50\pi e^2}{\chi^2\omega}\left[\sum_{(ab)2^+}\frac{\sin^2(x_a+x_b)L_{ab}(E_a+E_b)}{[(E_a+E_b)^2-\omega^2]^2}\right]^{-1}. \tag{94}$$

For the example above, setting e equal to unity, one finds $B(E2)=2.8\times10^{-49}$ cm^4, while the method of KS gives 3.0 and the experimental value[25] is 2.3. One can also use Eq. (73) to derive a sum rule for $B(E2)$. One multiplies Eq. (93) by ω to be able to replace ωf_{ab} by $(E_a+E_b)g_{ab}$, and one finds easily

$$\sum_B\omega_B[B(E2)]_B=(5e^2/40\pi)\sum_{(ab)2^+}\sin^2(x_a+x_b)L_{ab}(E_a+E_b), \tag{95}$$

where the sum on the left runs over all solutions of Eqs. (86). For our numerical example, the collective solution contributes 79% of the total sum.

Finally, we note that collective effects can be much stronger when protons are present as well as neutrons. The next section is an example.

B. Collective Oscillations of Closed Shells

Some collective phenomena vary smoothly through the periodic table without change at the magic numbers. They do not depend on the composition of the unfilled shells and must therefore involve transitions between whole shells. The giant photoresonance is evidence of 1^- oscillations with this property. There are also 3^- oscillations[26] which manifest themselves through large radiative matrix elements and anomalous inelastic scattering. Pairing correlations are not so important here, because the large spacing between shells already plays the role of an energy gap. Then, one can use the equations of Sec. 3, but set u equal to 1 for a particle and to 0 for a hole. Since we are interested in states composed of a particle and a hole, the G term does not appear in Eqs. (45).

We assume δ-function forces,

$$[V_S(1-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)/4+V_T(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)/4]\delta(\mathbf{x}_1-\mathbf{x}_2). \tag{96}$$

The triplet part acts only between a neutron and a proton. All standard force mixtures have[27]

$$V_S=0.6V_T. \tag{97}$$

The matrix elements involve radial integrals such as

$$F_S=(V_S/4\pi)\int_0^\infty R_a(r)R_b(r)R_c(r)R_d(r)r^2dr, \tag{98}$$

and F_T which is similarly defined. We follow Brown and Bolsterli[8] in keeping only those integrals which are obviously large and setting them all equal. We need matrix elements of H and K, both for (a,b,c,d) all of one charge and for (a,b) of one charge, (c,d) of the other. For odd J and parity, one finds that the largest elements by far (i.e., by a factor 3) are those of H between nonidentical particles. We keep only those. Let a and c be particles, b and d be holes. We keep only terms where a and b belong to adjacent major shells, and for which $l_b=l_a+1$; this gives R_a and R_b the same number of nodes and makes the radial integral large.[8] We do the same for c and d. Adopting again the phases (80), we find

$$H'(abcdJ)=\tfrac{1}{2}Fh_{ab}^{\frac{1}{2}}h_{cd}^{\frac{1}{2}}, \tag{99}$$

$$h_{ab}=\tfrac{1}{2}(2J+1)^{-1}(2j_a+1)(2j_b+1)\times[C(j_aj_bJ;\tfrac{1}{2}-\tfrac{1}{2}0)]^2, \tag{100}$$

$$F=F_S+3F_T. \tag{101}$$

We keep only terms with $j_b=j_a+1$, as they give bigger C coefficients in (100). The hole b associated with a given particle a is now completely determined and we shall use a single subscript, Roman for proton particle-hole pairs, Greek for neutrons.

With these rather drastic approximations, Eq. (44)

[24] K. Alder *et al.*, Revs. Modern Phys. **28**, 432 (1956).
[25] P. H. Stelson and F. K. McGowan, Phys. Rev. **110**, 489 (1958).
[26] A. M. Lane and E. D. Pendlebury, Nuclear Phys. **15**, 39 (1960).
[27] J. P. Elliott and A. M. Lane, *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1957), Vol. XXXIX, p. 337.

takes the form

$$\begin{aligned}\omega f_a &= E_0 g_a - F h_a^{\frac{1}{2}} \textstyle\sum_\alpha h_\alpha^{\frac{1}{2}} g_\alpha,\\ \omega g_a &= E_0 f_a,\\ \omega f_\alpha &= E_0 g_\alpha - F h_\alpha^{\frac{1}{2}} \textstyle\sum_a h_a^{\frac{1}{2}} g_a,\\ \omega g_\alpha &= E_0 f_\alpha,\end{aligned} \tag{102}$$

where E_0 is the distance between major shells. The secular equation is

$$(E_0^2-\omega^2)^2 = F^2 E_0^2 (\textstyle\sum_a h_a)(\sum_\alpha h_\alpha). \tag{103}$$

There are two solutions: one for which $E_0^2-\omega^2$ is positive and in which the protons and the neutrons oscillate in phase (i.e., f_a and f_α have the same sign); and the other for which $E_0^2-\omega^2$ is negative, f_a and f_α have opposite signs, the protons and the neutrons are in opposite phase. For light nuclei, those would be called $T=0$ and $T=1$, respectively. Oscillation in phase gives an attraction, out of phase a repulsion. For a numerical estimate, we took $\sum_a h_a$ and $\sum_\alpha h_\alpha$ equal, and calculated it using the 50–82 major shell for the particle and the 82–126 major shell for the hole. We found it equal to 9 for $J=1$ and 2.9 for $J=3$. We took E_0 equal to 7 Mev.

For the 3^- case and oscillation in phase, the experimental value of ω seems to be[26] 2.6 Mev. This is achieved by taking $F=2.1$ Mev. The 1^- in-phase solution corresponds to motion of the center of mass of the whole nucleus and is spurious.[28] The 1^- out-of-phase case is the giant dipole resonance which has been discussed by Brown and Bolsterli.[8] Our treatment differs from theirs in that we include the diagrams of Fig. 1(b), while they have only those of Fig. 1(a). Including diagrams 1(b) actually enhances the effect for the in-phase case, but decreases it for out-of-phase. Of course, one may wonder about the validity of either method when applied to such large excitation as that of the giant resonance. Be that as it may, one can get the experimental value, $\omega=15$ Mev (for heavy nuclei), by taking $F=2.8$ Mev.

The two values of F thus obtained are in rough agreement. Another estimate of F can be gotten from the work of KS, since the $J=0$ part of the G matrix for the pairing force is almost the same as for a δ-function force. Their value of F_S (their G) is (25 Mev)/A. By Eqs. (97) and (101), F should be six times larger. For heavy nuclei, this gives $F=0.8$ Mev. One possible reason why this estimate is smaller, is that the true force has a finite range, whose effect is to increase hole-particle matrix elements compared to particle elements. However, it should also be noted that, in his theory of the lead isotopes with δ-function forces, Pryce[29] uses values of F_S around 0.3 or 0.4 Mev, corresponding to $F\approx 2$ Mev. Finally, a meaningful comparison can also be made with the numbers of Sec. 4A because, for a δ-function force and identical particles, one has

$$\begin{aligned}&K'(abcd2^+)\\ &\quad=(F_S/10)[(2j_a+1)(2j_b+1)(2j_c+1)(2j_d+1)]^{\frac{1}{2}}\\ &\qquad\times|C(j_a j_b 2;\tfrac{1}{2}-\tfrac{1}{2}0)C(j_c j_d 2;\tfrac{1}{2}-\tfrac{1}{2}0)|.\end{aligned} \tag{104}$$

This is very close to Eq. (83). Since $\mathfrak{R}_{aa}^2$ is $(11/2)^2\alpha^{-4}$ for most values of a, one can identify F_S with $(\chi/2\pi)(11/2)^2\alpha^{-4}$. The value of F that one obtains this way is 2.6 Mev.

In conclusion, one can say that the order-of-magnitude agreement between these various estimates of the nuclear force makes it appear likely that, in the future, it will be possible to obtain a good fit of all these phenomena with the same effective two-body interaction.

C. Anomalous Inelastic Scattering

Anomalously large inelastic scattering has been observed with protons,[30] deuterons,[31] and α particles.[32] This happens in particular for scattering into the first quadrupole vibrational state, but there are other large anomalies at other energies. The main one is around 2.5 Mev in all heavy spherical nuclei. Cohen[33] has suggested that collective effects are responsible in all cases, and strong evidence[34] has been offered that the 2.5-Mev anomaly is due to 3^- vibrations. It might be that, in nuclei with proton and neutron shells both partially filled, there is also a contribution from the one-phonon 4^+ vibrational state[35] which must lie near the top of the gap.

A proper theory of this effect must take into account absorption, refraction, and diffraction of the projectile.[36] Here, we do not wish to attempt to calculate the absolute cross section or the angular distribution, but only try to make a very rough estimate of the enhancement compared to the single-particle value. For this, it might be enough to use a model where the projectile is spinless and interacts with the nucleons in Born approximation through a zero-range, spin-independent, charge-independent potential. Then, the scattering amplitude with

[28] J. P. Elliott and T. H. R. Skyrme, Proc. Roy. Soc. (London) **A232**, 561 (1955).

[29] M. H. L. Pryce, Nuclear Phys. **2**, 226 (1956/57); D. E. Alburger and M. H. L. Pryce, Phys. Rev. **95**, 1482 (1954).

[30] B. L. Cohen, Phys. Rev. **105**, 1549 (1957); B. L. Cohen and A. G. Rubin, Phys. Rev. **111**, 1568 (1958).

[31] J. L. Yntema and B. Zeidman, Phys. Rev. **114**, 815 (1959); B. L. Cohen and R. E. Price (to be published).

[32] D. R. Sweetman and N. S. Wall, *Comptes Rendus du Congrès International de Physique Nucléaire, Paris, 1958* (Dunod, Paris, 1959); H. W. Fulbright, N. O. Lassen, and N. O. Roy Poulsen, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31**, No. 10 (1959); J. L. Yntema, B. Zeidman, and B. J. Raz (to be published); D. K. McDaniels, J. S. Blair, S. W. Chen, and G. W. Farwell (to be published).

[33] B. L. Cohen, Phys. Rev. **116**, 426 (1959).

[34] See reference 26 and D. K. McDaniels *et al.*, reference 32.

[35] Not to be confused with the 4^+ component of the two-phonon 2^+ vibration.

[36] N. Austern and E. Rost (to be published); J. S. Blair, Phys. Rev. **115**, 928 (1959); S. I. Drozdov, J. Exptl. Theoret. Phys. (U.S.S.R.) **28**, 734, 736 (1955) [Translation: Soviet Phys.-JETP **1**, 588, 591 (1955)].

momentum transfer $\mathbf{k}$ is proportional to

$$\int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}}\langle\Psi_0|\sum_{m_sq}\phi^*(\mathbf{x}m_sq)\phi(\mathbf{x}m_sq)|\Psi_{BM}\rangle, \quad (105)$$

where $\phi(\mathbf{x}m_sq)$ destroys a particle with position $\mathbf{x}$, spin m_s, charge q. We shall calculate the quantity

$$\int d^3x|\langle\Psi_0|\sum_{m_sq}\phi^*(\mathbf{x}m_sq)\phi(\mathbf{x}m_sq)|\Psi_{BM}\rangle|^2, \quad (106)$$

which is a measure of the total strength of the transition. This can be expressed in terms of f and g by methods used earlier. With the phases (80), it is found equal to

$$\tfrac{1}{2}\sum_{abcd} Z(abcd)h_{ab}{}^{\frac{1}{2}}h_{cd}{}^{\frac{1}{2}}(u_av_b+v_au_b) \times(u_cv_d+v_cu_d)f_{ab}f_{cd}, \quad (107)$$

for even J and even parity, and to

$$\tfrac{1}{2}\sum_{abcd} Z(abcd)h_{ab}{}^{\frac{1}{2}}h_{cd}{}^{\frac{1}{2}}(u_av_b-v_au_b) \times(u_cv_d-v_cu_d)g_{ab}g_{cd}, \quad (108)$$

for odd J and odd parity, with

$$Z(abcd)=(4\pi)^{-1}\int_0^\infty R_a(r)R_b(r)R_c(r)R_d(r)r^2dr, \quad (109)$$

and h_{ab} as defined by Eq. (100). Other combinations of J and parity give a vanishing result. The charges of a and b must be the same, and those of c and d likewise. Once again, we shall set all large radial integrals equal and neglect the others.

For a single-particle transition, only two states are important, say a particle state a and a hole b. The amplitudes f_{ab} and g_{ab} are equal to each other, and to unity in view of the normalization (70). We can average h_{ab} over J, which gives $\frac{1}{2}$. Then, formulas (107) and (108) become just Z. Therefore, the enhancement over the single-particle transition is given by (107) or (108), but with Z omitted, provided that we include in the sum only terms whose Z is large.

It is a simple matter to apply this to the case of 3^- oscillations, using the approximations and parameters of Sec. 4B. The enhancement is given by $(4E_0/\omega)\sum_a h_a$, which is equal to 30. This is of the same order of magnitude as the enhancement of $B(E3)$ for heavy nuclei, quoted in reference 26. A more direct comparison with experiment is difficult.

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After completion of this work, we learned from Drs. Bohr and Sorensen that investigations of a similar nature are going on elsewhere. Equation (88) has been derived by A. Bohr (unpublished). A simplified version of it which assumes degeneracy of the single-particle levels was derived two years ago by B. Mottelson and developed by D. R. Bès and B. Mottelson (unpublished). The same equation has recently been given by Kobayasi and Marumori.[37] The work of Arvieu and Vénéroni[38] comes closest to the spirit of ours. These authors also propose the use of the Sawada method for determining collective oscillations in spherical nuclei. Their equations differ from ours by the absence of the G term in (45a) and (45b). This has little effect on the collective states, but these terms are essential if the equations are to be used also for other two-quasiparticle states, with a realistic nuclear force. They are also necessary for the elimination of the spurious states.[39]

[37] M. Kobayasi and T. Marumori, Progr. Theoret. Phys. (Kyoto) **23**, 387 (1960).

[38] R. Arvieu and M. Vénéroni, Compt. rend. **250**, 992, 2155 (1960).

[39] *Note added in proof.* Related papers recently appeared include T. Marumori, Progr. Theoret. Phys. (Kyoto) **24**, 331 (1960) and G. E. Brown, J. A. Evans, and D. J. Thouless, preprint.