Initial Value Problem of the Einstein-Maxwell Field

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On an initial hypersurface, $x^0=0$, in the presence of gravitation and source-free electromagnetism one can specify the metric tensor, $g_{\mu\nu}$, and its partial derivatives, $g_{\mu\nu,0}$, as well as the electromagnetic tensor, $f_{\mu\nu}$. These quantities must be specified so that on the initial hypersurface two of Maxwell's equations are satisfied and so that the components $R_\alpha{}^0$ of the Ricci tensor are proportional to the components $T_\alpha{}^0$ of the electromagnetic energy-momentum tensor. It is sometimes possible to specify a different electromagnetic tensor on the initial hypersurface which together with the old metric and Ricci tensors will describe a properly set initial value problem such that the geometry in advance of the initial hypersurface is different for the different electromagnetic fields. Thus, the Ricci tensor on the initial hypersurface does not always uniquely describe the geometry off the hypersurface in the Einstein-Maxwell theory. The conditions when this nonuniqueness exists are explicitly derived. An initial value problem could be set by specifying $g_{\mu\nu}$ and $g_{\mu\nu,0}$ on the initial hypersurface and deriving an appropriate $f_{\mu\nu}$; however, $g_{\mu\nu}$ and $g_{\mu\nu,0}$ cannot be arbitrarily specified but are subject to one rather complicated constraint condition on the hypersurface.

I. INTRODUCTION

THE Einstein-Maxwell theory describes a space-time which contains only gravitation and source-free electromagnetism by means of a set of nonlinear second-order partial differential equations in which the dependent variables are the components, $g_{\mu\nu}$, of the symmetric metric tensor and the components, $f_{\alpha\beta}$, of the antisymmetric electromagnetic tensor. It is appropriate for a theory of this kind to ask what information can and should be given on an initial space-like hypersurface in order to determine the fields uniquely in a space-time region in advance of the hypersurface.

This question has been discussed[1] and the present situation can be briefly outlined. In appropriate units the combined Einstein-Maxwell field equations can be written, in the absence of electromagnetic sources,[2]

$$R_\alpha{}^\beta-\tfrac{1}{2}\delta_\alpha{}^\beta R=f_{\alpha\gamma}f^{\beta\gamma}+d_{\alpha\gamma}d^{\beta\gamma}, \tag{1}$$

$$f_{[\alpha\beta,\gamma]}=0, \tag{2}$$

$$d_{[\alpha\beta,\gamma]}=0. \tag{3}$$

The bracket in $[\alpha\beta,\gamma]$ stands for alternation and since $f_{\alpha\beta}$ is antisymmetric $f_{[\alpha\beta,\gamma]}\equiv\frac{1}{3}(f_{\alpha\beta,\gamma}+f_{\beta\gamma,\alpha}+f_{\gamma\alpha,\beta})$. Choose a coordinate system in which the equation of the initial space-like hypersurface is $x^0=0$. Suppose one gives as Cauchy data the values of $g_{\mu\nu}$, the "time" derivatives $g_{\mu\nu,0}$, and $f_{\alpha\beta}$ everywhere on the initial hypersurface. In order that these 26 quantities be an appropriate set of variables to specify the Einstein-Maxwell fields at the initial time $x^0=0$, it is necessary that Eqs. (1), (2), and (3) be satisfied everywhere on the initial hypersurface. Evidently it is necessary that on this hypersurface, the following six equations be satisfied:

$$R_\alpha{}^0-\tfrac{1}{2}\delta_\alpha{}^0 R=f_{\alpha\gamma}f^{0\gamma}+d_{\alpha\gamma}d^{0\gamma}, \tag{4}$$

$$f_{[12,3]}=0, \tag{5}$$

$$d_{[12,3]}=0. \tag{6}$$

These equations involve only quantities that have already been assumed given on the initial hypersurface. If these equations are satisfied, one can solve Eqs. (1), (2), (3) to get the time evolution of the system. To repeat, the Cauchy problem for the Einstein-Maxwell problem requires that the 26 quantities $g_{\mu\nu}$, $g_{\mu\nu,0}$, and $f_{\alpha\beta}$ can be given only subject to the six constraints (4), (5), and (6). One is led to expect that on $x^0=0$ it should be possible to find twenty quantities that can be given independently of any conditions and to determine the other six quantities by solving a set of differential equations derived from the six constraint equations.

It is more usual to say that the given Cauchy data involves giving 18 quantities, not 26. Eight of the quantities are readily seen to be physically meaningless and can be easily eliminated. Equations (4), (5), and (6) do not involve the four quantities $g_{0\mu,0}$ at all. Moreover, by using the second fundamental form $K_{ij}\equiv-(g^{00})^{-\frac{1}{2}}\Gamma^0{}_{ij}$ to replace $g_{ij,0}$ and the quantities $\mathcal{E}^i=(-g)^{\frac{1}{2}}f^{0i}$, $\mathcal{H}^i=(-g)^{\frac{1}{2}}d^{0i}$ to replace $f_{\mu\nu}$, one finds that they also do not contain $g_{0\mu}$. Only the 18 quantities g_{ij}, K_{ij}, $\mathcal{E}^i$, and $\mathcal{H}^i$ need be specified at $x^0=0$ to determine the solution of Eqs. (1), (2), and (3) uniquely (to within coordinate transformations at "times" different from $x^0=0$). The $g_{0\mu}$ and $g_{0\mu,0}$ may be given arbitrary values by coordinate transformations which reduce to the identity on the initial surface $x^0=0$. Hence, they can be given independent of any condition, and must be found among the twenty independent variables referred to above. (See, for example, the review by Misner and Wheeler.[3])

Of the given initial data, $g_{\mu\nu}$, $g_{\mu\nu,0}$, and $f_{\alpha\beta}$, those

[1] A. Lichnerowitz, *Théories Relativistes de la Gravitation et de L'Electromagnétisme* (Maisson et Cie, Paris, 1955), Chap. 2.

[2] Greek subscripts assume values 1, 2, 3, 0; Latin subscripts 1, 2, 3; a comma denotes a partial derivative; a semicolon a covariant derivative; $R_\alpha{}^\beta$ represents the Ricci tensor; $d^{\alpha\beta}\equiv\frac{1}{2}(-g)^{-\frac{1}{2}}\epsilon^{\alpha\beta\gamma\delta}f_{\gamma\delta}$ is the dual of $f_{\alpha\beta}$; $\epsilon^{\alpha\beta\gamma\delta}$ is the antisymmetric tensor density, equal to $+1$ or -1 depending on whether $\alpha\beta\gamma\delta$ is an even or odd permutation of 1, 2, 3, 0. g stands for the determinant of $g_{\mu\nu}$ and $\Gamma^\alpha{}_{\beta\sigma}$ is the Christoffel symbol of the second kind.

[3] C. W. Misner and J. A. Wheeler, Ann. Phys. 2, 525 (1957).

involving the $f_{\alpha\beta}$ do not immediately pertain to geometry while the remaining twenty are of course inherently geometrical. With the same geometric initial data, can one replace $f_{\alpha\beta}$ by a different electromagnetic tensor $f'_{\alpha\beta}$ such that the constraint equations will still be obeyed? If this can be done, it will be possible to have situations where the metric tensor and its first time derivatives on the hypersurface will not be sufficient uniquely to predict the future space-time. It turns out that this nonuniqueness sometimes does exist and in the next section we explicitly derive the condition on $f_{\alpha\beta}$ which must be obeyed in order to find a new suitable electromagnetic tensor $f'_{\alpha\beta}$. Suppose the given data satisfy the energy constraint, (4), but not Maxwell's equations, (5) and (6). In the next section is also shown that, if two different relations are obeyed by $f_{\alpha\beta}$, one can find a new $f'_{\alpha\beta}$ which together with the given $g_{\mu\nu}$ and $g_{\mu\nu,0}$ will satisfy all the appropriate initial constraint equations.

In the third section we take up the question whether $g_{\mu\nu}$ and $g_{\mu\nu,0}$ can be arbitrarily specified and the $f_{\alpha\beta}$ remain unspecified but determined by solving the appropriate constraint equations. It turns out to be, in general, an impossible task. The given twenty metric quantities must still satisfy one or, at most, two constraint equations which will then allow the appropriate electromagnetic tensor to be found. The resulting problem may not be unique; there may be a family of suitable electromagnetic tensors. All of the above results are derived in the usual Einstein-Maxwell theory. The situation is then discussed in Sec. 4 from the completely geometric point of view of Rainich and a proof outlined to show that the situation is the same from this point of view also (as of course it must be).

II. UNIQUENESS OF ELECTROMAGNETIC FIELD FOR GIVEN INITIAL GEOMETRY

Suppose the Cauchy data $g_{\mu\nu}$, $g_{\mu\nu,0}$, and $f_{\alpha\beta}$ are given so that (4), (5), and (6) are fulfilled and the Einstein-Maxwell equations can be solved. Keeping the same $g_{\mu\nu}$ and $g_{\mu\nu,0}$, can one find a new electromagnetic field $f'_{\alpha\beta}$ that will satisfy (4), (5), and (6) and that will properly define a different Einstein-Maxwell universe? The answer is that this can sometimes be done and we will now display under what circumstances this nonuniqueness exists.

If one defines the $f'_{\alpha\beta}$ by

$$f'_{\alpha\beta}=f_{\alpha\beta}\cos\theta+d_{\alpha\beta}\sin\theta, \tag{7}$$

$$d'_{\alpha\beta}=-f_{\alpha\beta}\sin\theta+d_{\alpha\beta}\cos\theta, \tag{8}$$

one sees that Eq. (4) remains satisfied,

$$R_\alpha{}^0-\tfrac{1}{2}\delta_\alpha{}^0R=f'_{\alpha\gamma}f'^{0\gamma}+d'_{\alpha\gamma}d'^{0\gamma}.$$

θ is an arbitrary function of the coordinates. We shall refer to the transformation from $f_{\alpha\beta}$, $d_{\alpha\beta}$ to $f'_{\alpha\beta}$, $d'_{\alpha\beta}$ described by Eqs. (7) and (8) as a phase transformation; it is sometimes referred to as a duality rotation. In a notation to be introduced in the Appendix, θ is a phase angle in a phase transformation.

In order that $f'_{\alpha\beta}$ be usable as an electromagnetic field in an initial value problem, it is necessary that it satisfy (5) and (6). Under a condition now to be derived, $f'_{\alpha\beta}$ can indeed be a suitable electromagnetic field. On the hypersurface, use the notation ${}^3g=\det g_{ij}$; $(-g)^{\frac{1}{2}}f^{0i}\equiv\mathcal{E}^i\equiv({}^3g)^{\frac{1}{2}}E^i$; $(-g)^{\frac{1}{2}}d^{0i}\equiv\mathcal{H}^i\equiv({}^3g)^{\frac{1}{2}}H^i$; $H_i=g_{ij}H^j$; $\boldsymbol{\mathcal{E}}\cdot\mathbf{H}=\mathcal{E}^iH_i$; $(\mathbf{E}\times\mathbf{H})_i=\epsilon_{ijk}\mathcal{E}^jH^k$; $(\nabla)_i\equiv\partial/\partial x^i$; $\nabla\cdot\boldsymbol{\mathcal{E}}\equiv\mathcal{E}^i{}_{,i}$; etc. With this notation, we can use the language of three-dimensional vector analysis.

On the initial hypersurface $\nabla\cdot\boldsymbol{\mathcal{E}}=0$; $\nabla\cdot\boldsymbol{\mathcal{H}}=0$; and $\boldsymbol{\mathcal{H}}'=\boldsymbol{\mathcal{H}}\cos\theta+\boldsymbol{\mathcal{E}}\sin\theta$; $\boldsymbol{\mathcal{E}}'=-\boldsymbol{\mathcal{H}}\sin\theta+\boldsymbol{\mathcal{E}}\cos\theta$. The question is whether θ can be chosen so that

$$\nabla\cdot\boldsymbol{\mathcal{H}}'=0,\quad \nabla\cdot\boldsymbol{\mathcal{E}}'=0, \tag{9}$$

$$\nabla\cdot\boldsymbol{\mathcal{H}}'=\nabla\cdot\boldsymbol{\mathcal{H}}'\cos\theta-\sin\theta\boldsymbol{\mathcal{H}}'\cdot\nabla\theta+\nabla\cdot\boldsymbol{\mathcal{E}}\sin\theta+\cos\theta\boldsymbol{\mathcal{H}}\cdot\nabla\theta=0, \tag{10}$$

$$\nabla\cdot\boldsymbol{\mathcal{E}}'=-\nabla\cdot\boldsymbol{\mathcal{H}}\sin\theta-\cos\theta\boldsymbol{\mathcal{H}}\cdot\nabla\theta+\nabla\cdot\boldsymbol{\mathcal{E}}\cos\theta-\sin\theta\boldsymbol{\mathcal{E}}\cdot\nabla\theta. \tag{11}$$

Multiply (10) by $\cos\theta$, (11) by $\sin\theta$, and subtract; then multiply (10) by $\sin\theta$, (11) by $\cos\theta$, and add. The result will be that the two equations (10) and (11) are equivalent to

$$\nabla\cdot\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{E}}\cdot\nabla\theta=0, \tag{12}$$

$$\nabla\cdot\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{H}}\cdot\nabla\theta=0. \tag{13}$$

If E_i is not proportional to H_i, $\nabla\theta$ can be written

$$\nabla\theta=\alpha\mathbf{E}+\beta\mathbf{H}+\gamma\mathbf{E}\times\mathbf{H}. \tag{14}$$

The components α, β, γ are as yet undetermined. If E_i is proportional to H_i, one can write an expression analogous to (14) expressing $\nabla\theta$ in terms of its components in the direction of $\mathbf{E}$ and any two other independent directions. Using expression (14), one can rewrite (12) and (13):

$$\nabla\cdot\boldsymbol{\mathcal{H}}+\alpha\cdot\mathbf{E}+\beta\mathbf{E}\cdot\boldsymbol{\mathcal{H}}=0, \tag{15}$$

$$\nabla\cdot\boldsymbol{\mathcal{E}}-\alpha\mathbf{E}\cdot\boldsymbol{\mathcal{H}}-\beta\mathbf{H}\cdot\boldsymbol{\mathcal{H}}=0. \tag{16}$$

If the determinant of coefficients of α, β does not vanish, the equations can be solved to yield

$$\alpha=\frac{(\nabla\cdot\boldsymbol{\mathcal{H}})\mathbf{H}\cdot\boldsymbol{\mathcal{H}}+(\nabla\cdot\boldsymbol{\mathcal{E}})\mathbf{E}\cdot\boldsymbol{\mathcal{H}}}{(\mathbf{E}\cdot\boldsymbol{\mathcal{H}})^2-(\mathbf{E}\cdot\boldsymbol{\mathcal{E}})(\mathbf{H}\cdot\boldsymbol{\mathcal{H}})}, \tag{17}$$

$$\beta=\frac{(\nabla\cdot\boldsymbol{\mathcal{H}})\mathbf{E}\cdot\boldsymbol{\mathcal{H}}+(\nabla\cdot\boldsymbol{\mathcal{E}})\mathbf{E}\cdot\boldsymbol{\mathcal{E}}}{(\mathbf{E}\cdot\boldsymbol{\mathcal{H}})^2-(\mathbf{E}\cdot\boldsymbol{\mathcal{E}})(\mathbf{H}\cdot\boldsymbol{\mathcal{H}})}. \tag{18}$$

Using the knowledge that $\nabla\cdot\boldsymbol{\mathcal{H}}=0$ and $\nabla\cdot\boldsymbol{\mathcal{E}}=0$ shows that $\alpha=0$ and $\beta=0$ so that $\nabla\theta=\gamma\mathbf{E}\times\mathbf{H}$, γ, an arbitrary function, solves (12) and (13). We could have deduced this more readily by a prior use of the divergence-free nature of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$; however, we shall want to use (17) and (18) later to discuss the situation when $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ are not divergence-free. In order that $\gamma\mathbf{E}\times\mathbf{H}$ represent a gradient, it is necessary that

$$\nabla\times(\gamma\mathbf{E}\times\mathbf{H})=\nabla\gamma\times(\mathbf{E}\times\mathbf{H})+\gamma\nabla\times(\mathbf{E}\times\mathbf{H})=0. \quad (19)$$

Here $\nabla\times A\equiv\epsilon^{ijk}A_{k,j}$; ϵ^{ijk} is completely antisymmetric and equal to $+1$ or -1, depending on whether i, j, k is an even or odd permutation of 1, 2, 3. Equation (19) represents three equations whose components become explicit after taking the dot product with **E**, **H**, and $\mathbf{E}\times\mathbf{H}$, respectively. The last-mentioned dot product will show that Eq. (19) can only be satisfied if

$$(\mathbf{E}\times\mathbf{H})\cdot[\nabla\times(\mathbf{E}\times\mathbf{H})]=0. \quad (20)$$

The entire set of equations (9) through (20) has been written in a form which is covariant under all transformations which maintain the equation $x^0=0$ for the initial surface. The explicit use of covariant derivatives has been avoided by taking gradients only of 3-scalars, divergences only of 3-vector densities, and curls only of 3-vectors. We now show that Eq. (20) is not only necessary but also sufficient to ensure the existence of a γ which will satisfy (19). Define $\psi\equiv\ln\gamma$. Then Eq. (19) becomes

$$\nabla\psi\times(\mathbf{E}\times\mathbf{H})=-\nabla\times(\mathbf{E}\times\mathbf{H}). \quad (21)$$

These determine two components, say $\psi_{,1}$ and $\psi_{,2}$, of $\nabla\psi$ in directions perpendicular to $\mathbf{E}\times\mathbf{H}$. In order that these equations can be integrated to give ψ, it is necessary and sufficient that $(\psi_{,1})_{,2}=(\psi_{,2})_{,1}$. This can be written

$$(\nabla\times\nabla\psi)\cdot(\mathbf{E}\times\mathbf{H})=0.$$

We show that this is a consequence of (21) by forming the divergence of that equation:

$$\begin{aligned}\nabla\cdot[\nabla\psi\times(\mathbf{E}\times\mathbf{H})]&\\ \equiv(\nabla\times\nabla\psi)\cdot(\mathbf{E}\times\mathbf{H})&-(\nabla\psi)\cdot\nabla\times(\mathbf{E}\times\mathbf{H})\\ &=-\nabla\cdot[\nabla\times(\mathbf{E}\times\mathbf{H})]\equiv0.\end{aligned}$$

However, using (21) we have

$$\nabla\psi\cdot[\nabla\times(\mathbf{E}\times\mathbf{H})]=-\nabla\psi\cdot[\nabla\psi\times(\mathbf{E}\times\mathbf{H})]=0.$$

Consequently, we have the result that if (20) is satisfied everywhere on the hypersurface one can find γ (usually γ can be a family of functions) that satisfies (19). For example, if $\nabla\times(\mathbf{E}\times\mathbf{H})=0$, then $\nabla\gamma=\lambda(\mathbf{E}\times\mathbf{H})$, where λ varies arbitrarily along lines of tangency to $\mathbf{E}\times\mathbf{H}$ but does not vary along lines of tangency to **E** or **H** [i.e., $\nabla\lambda\cdot(\mathbf{E}\times\mathbf{H})\neq0$, $\nabla\lambda\cdot\mathbf{E}=0$, $\nabla\lambda\cdot\mathbf{H}=0$], will satisfy (19). Penrose[4] has given an example of such a situation involving a collision between two electromagnetic waves.

Thus, providing Eq. (20) holds, one can give a family of γ which will in turn give a family of $f_{\alpha\beta}$ which together with $g_{\mu\nu}$, $g_{\mu\nu,0}$ will satisfy the initial constraint equations. Although the geometry ($g_{\mu\nu}$ and $g_{\mu\nu,0}$) on the initial hypersurface is the same for the whole family of fields, the geometry is not the same for regions of space-time in advance of the hypersurface. This is true because γ, in general, is not a constant but may have different values as a function of points on the initial hypersurface. Two electromagnetic tensors $f_{\alpha\beta}$ and $f'_{\alpha\beta}$ can satisfy Maxwell's equations and produce the same electromagnetic energy-momentum tensor throughout a four-dimensional space-time region if, and only if, they are connected by relations (7) and (8) with θ being a constant independent of space-time. Any solution of the Einstein-Maxwell equations without electromagnetic sources can be used to generate a whole family of solutions by means of relations (7) and (8) with θ constant without changing the geometry. However, it is not this special case with which we are here dealing, and the geometry does change off the hypersurface as $f_{\alpha\beta}$ changes on the hypersurface with varying θ.

Suppose the initial $f_{\alpha\beta}$ were such that (1) was satisfied but not (2) and (3). Is it possible to find a new field, $f'_{\alpha\beta}$, so that (1) will remain unchanged, the given $g_{\mu\nu}$ and $g_{\mu\nu,0}$ will be unchanged, but (2) and (3) will be satisfied for $f'_{\alpha\beta}$? In other words, if $\nabla\cdot\mathcal{E}\neq0$ and $\nabla\cdot\mathcal{H}\neq0$, can one express an $\mathcal{H}'=\mathcal{H}\cos\theta+\mathcal{E}\sin\theta$ and an $\mathcal{E}'=-\mathcal{H}\sin\theta+\mathcal{E}\cos\theta$ so that $\nabla\cdot\mathcal{H}'=0$ and $\nabla\cdot\mathcal{E}'=0$? The above analysis shows that for this to be possible one must choose $\nabla\theta$ as given by Eq. (14) with α and β obtained from (17) and (18), respectively. This choice, however, can only be made if

$$\nabla\times\nabla\theta=\nabla\times(\alpha\mathbf{E}+\beta\mathbf{H}+\gamma\mathbf{E}\times\mathbf{H})=0. \quad (22)$$

Equation (22) represents three independent equations. Taking the dot product of the above expression with $\mathbf{E}\times\mathbf{H}$ permits one to solve for γ,

$$\gamma=-\frac{(\mathbf{E}\times\mathbf{H})\cdot[\nabla\times(\alpha\mathbf{E}+\beta\mathbf{H})]}{(\mathbf{E}\times\mathbf{H})\cdot\nabla\times(\mathbf{E}\times\mathbf{H})}. \quad (23)$$

Taking the dot product of (22) with **E** and **H** yields

$$\alpha\mathbf{E}\cdot\nabla\times\mathbf{E}+\mathbf{E}\cdot[\nabla\times(\beta\mathbf{H}+\gamma\mathbf{E}\times\mathbf{H})]=0, \quad (24)$$

$$\beta\mathbf{H}\cdot\nabla\times\mathbf{H}+\mathbf{H}\cdot[\nabla\times(\alpha\mathbf{E}+\gamma\mathbf{E}\times\mathbf{H})]=0. \quad (25)$$

α, β, and γ are given by (17), (18), and (23), respectively. Thus, if $\mathcal{E}$ and $\mathcal{H}$ do not satisfy $\nabla\cdot\mathcal{E}=0$, $\nabla\cdot\mathcal{H}=0$ they can still serve to provide a suitable Einstein-Maxwell problem provided they satisfy the very much more complicated relations (24) and (25). In this case the initial electric and magnetic fields are given not by **E** and **H** but by $\mathbf{E}'$ and $\mathbf{H}'$, θ being obtained by integrating Eq. (14) with α, β, and γ given by the appropriate expressions. Of course, our previous analysis has shown that $\mathbf{E}'$ and $\mathbf{H}'$ will not define a unique geometric problem if $(\mathbf{E}'\times\mathbf{H}')\cdot[\nabla\times(\mathbf{E}'\times\mathbf{H}')]=0$.

III. ARBITRARINESS OF $g_{\mu\nu}$ AND $g_{\mu\nu,0}$ ON INITIAL HYPERSURFACE

Suppose that only $g_{\mu\nu}$ and $g_{\mu\nu,0}$ are arbitrarily given on the initial hypersurface. Can the six equations, (4), (5), and (6), be solved for $f_{\alpha\beta}$ to yield a properly set initial value problem? In this section we find that the answer is "no"; the equations cannot always be solved.

[4] Roger Penrose (private communication).

It turns out that the metric quantities $g_{\mu\nu}$ and $g_{\mu\nu,0}$ cannot be given arbitrarily but are subject to at least one and possibly to two constraint equations.

Equation (1) is the energy-momentum relation of the Einstein-Maxwell theory. It can be shown[5] that for this relation to be true, in the case that $\rho^2 \equiv R_{\mu\nu}R^{\mu\nu}/4$ is not equal to zero, the Ricci tensor must be expressible in the form

$$R_\alpha{}^\beta = 2\rho^2\delta_\alpha{}^\beta - 4(l_\alpha k^\beta + k_\alpha l^\beta). \quad (26)$$

k_α and l_α are the two null eigenvectors of $R_{\alpha\beta}$; $l_\alpha l^\alpha = k_\alpha k^\alpha = 0$; $R_\alpha{}^\beta k_\beta = -2\rho^2 k_\alpha$; $R_\alpha{}^\beta l_\beta = -2\rho^2 l_\alpha$; and $l_\alpha k^\alpha = \rho^2$.

$R_\alpha{}^0$ can be calculated from the assumed initial data. An easy calculation from (26) shows that $R_0{}^\beta R_\beta{}^0 = \rho^2$ which is consequently given by the initial data. Taking the $R_\alpha{}^0$ components of Eq. (26) will yield four algebraic equations from which four of the components of l_α and k_α can be determined. Since $\rho^2 = l_\alpha k^\alpha$ is given by the initial data and since l_α and k_α are null, there remains only one undetermined component of the two null vectors. The four equations in (27) involving $R_\alpha{}^0$ are equivalent to Eq. (4) which is satisfied because of the appropriate choice of components of l_α and k_α. The remaining equations in (26) can be solved for $g_{ij,00}$ and will determine all but one of these second derivatives. This one will remain dependent on the remaining undetermined component of the null vectors.

From the null eigenvectors one can construct an antisymmetric tensor, $F_{\mu\nu}$, and its dual, $D_{\mu\nu}$, which satisfies Eq. (1):

$$F_{\mu\nu} = 2(l_\mu k_\nu - k_\mu l_\nu)/\rho. \quad (27)$$

From $F_{\mu\nu}$ and $D_{\mu\nu}$ one can obtain other antisymmetric tensors, $f_{\mu\nu}$, with their duals, $d_{\mu\nu}$, by phase transformations:

$$f_{\mu\nu} = F_{\mu\nu}\cos\theta + D_{\mu\nu}\sin\theta, \quad (28)$$

$$d_{\mu\nu} = -F_{\mu\nu}\sin\theta + D_{\mu\nu}\cos\theta. \quad (29)$$

θ is a function of position. θ can be chosen so that $f_{[12,3]} = d_{[12,3]} = 0$ if, and only if, the expressions $F^{i0} \sim E_i$ and $D^{i0} \sim H_i$ obey the two constraints (24) and (25). As there is only one free parameter still available in k_α and l_α it can be used to satisfy at most one of the constraints, leaving at least one still unsatisfied. This still unsatisfied constraint can be stated in terms of the geometric quantities $g_{\mu\nu}$ and $g_{\mu\nu,0}$ (since k_α and l_α are so described). Hence, we have the result that $g_{\mu\nu}$ and $g_{\mu\nu,0}$ cannot be arbitrarily given on an initial hypersurface for the Einstein-Maxwell equation to be properly set. These initial data must satisfy a constraint equation on the hypersurface which has a very complicated form and which can be derived by the method outlined above.

[5] J. L. Synge, *Relativity, The Special Theory* (Interscience Publishers, Inc., New York, 1956), Chap. 9; L. Witten, Phys. Rev. **115**, 206 (1959), see p. 211.

IV. RECAPITULATION FROM GEOMETRIC POINT OF VIEW

It is known from the work of Rainich[6] that the Einstein-Maxwell theory [Eqs. (1), (2), (3)] can be restated using only geometric quantities. It is instructive to see how the discussion of the previous section sounds using the geometric equations of Rainich. Since the result of this section is merely a review of that of the previous section, we shall only outline how the proof goes without making any attempt at producing a rigorous derivation. The first of Rainich's equations say that Eq. (1) or its equivalent, Eq. (26), can be true if, and only if,

$$R = 0, \quad (30)$$

$$R_\alpha{}^\beta R_\beta{}^\gamma = \delta_\alpha{}^\gamma R_{\mu\nu}R^{\mu\nu}/4. \quad (31)$$

If (30) and (31) hold, one can determine an $F_{\mu\nu}$ [Eq. (27)] and a family of $f_{\mu\nu}$ [Eq. (28)] by a phase transformation from $F_{\mu\nu}$ that satisfies (1). Out of the family $f_{\mu\nu}$, it is possible under some circumstances to choose a particular θ so that the corresponding $f_{\mu\nu}$ will obey Maxwell's equations. This choice of θ can be made only if

$$\alpha_{[\mu,\nu]} \equiv \alpha_{\mu,\nu} - \alpha_{\nu,\mu} = 0, \quad (32)$$

where α_μ is defined by

$$\alpha_\mu \equiv \epsilon_{\mu\nu\rho\sigma} R^{\nu\tau;\rho} R_\tau{}^\sigma / (-g)^{\frac{1}{2}} R_{\mu\nu}R^{\mu\nu}. \quad (33)$$

In order to insure that the energy density is positive, one must also impose the requirement

$$R_{00} < 0. \quad (34)$$

Equations (30), (31), (32), and (34) are equivalent in content to Eqs. (1), (2), and (3). If the former four equations are satisfied, one can find an electromagnetic tensor which will enter properly into the latter three equations. Of concern to us now is whether on an initial space-like hypersurface, $x^0 = 0$, it is possible to give $g_{\mu\nu}$ and $g_{\mu\nu,0}$ arbitrarily and to solve Eqs. (30), (31), and (32) to give the future development of the $g_{\mu\nu}$.

The initial data, as has been remarked, assume a knowledge of $R_\alpha{}^0$ everywhere on the initial hypersurface. The inequality (34) can be immediately verified. One can then solve Eqs. (30) and (31) for the second derivatives $g_{ij,00}$. How many of the six second derivatives are determined by these equations? In considering the equivalent equation (26) there were also six parameters available in ρ^2, k_α, and l_α of which the initial conditions determined five. Hence, Eq. (26) or the equivalent set (30) and (31) determine only five of the derivatives $g_{ij,00}$. The undetermined derivatives can be considered as usable in satisfying any constraints that Eq. (29) (Maxwell's equation) may impose on the

[6] G. Y. Rainich, Trans. Am. Math. Soc. **27**, 106 (1925); C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957); L. Witten, Phys. Rev. **115**, 206 (1959).

initial conditions. The $g_{0\alpha,00}$ are determined by the Bianchi identities and hence unavailable for this purpose. It will turn out of course that Eq. (29) will impose two constraint conditions on the $g_{ij,00}$, and there is only one derivative left to satisfy them. Hence, as determined before, there will be one residual constraint equation that the initial data must satisfy. If we now show how Eq. (29) imposes two constraint conditions on the $g_{ij,00}$ we shall have completed our restatement.

Equation (29) can be rewritten in terms of six scalar equations; contracting $\alpha_{\mu,\nu}-\alpha_{\nu\mu}$ with six independent bivectors yields six scalars whose vanishing is equivalent to the vanishing of the curl of α_μ. We shall rewrite Eq. (29) in this form. One of the bivectors to be used will be $l_{[\mu}k_{\nu]}$ which determines $F_{\mu\nu}$ by Eq. (27). $l_{[\mu}k_{\nu]}$ describes a time-like plane or 2-surface which contains time-like lines as well as space-like. Absolutely perpendicular to this 2-surface is a space-like plane or 2-surface, determined by $D_{\mu\nu}$, which we can describe by means of two orthogonal unit vectors p_μ and q_μ, each perpendicular to l_ν and k_ν. Six scalar equations equivalent to (29) are accordingly

$$0=\alpha_{[\mu,\nu]}k^{[\mu}l^{\nu]}=\alpha_{[\mu,\nu]}F^{\mu\nu}, \tag{35}$$

$$0=\alpha_{[\mu,\nu]}p^{[\mu}q^{\nu]}=\alpha_{[\mu,\nu]}D^{\mu\nu}, \tag{36}$$

$$\alpha_{[\mu,\nu]}k^{[\mu}p^{\nu]}=\alpha_{[\mu,\nu]}k^{[\mu}q^{\nu]}=\alpha_{[\mu,\nu]}l^{[\mu}p^{\nu]}=\alpha_{[\mu\nu]}l^{[\mu}q^{\nu]}=0. \tag{37}$$

Equations (35) and (36) are identities following from the definitions and the Bianchi identity. This has been proved by Rosen[7]; in the Appendix we outline an independent proof.

The definition (33) of α_μ shows that $\alpha_{0;i}$ contains time derivatives of g_{ij} no higher than the second; $\alpha_{i;j}$ contains no higher than third time derivatives; and $\alpha_{i;0}$ contains no higher than fourth derivatives; each term contains the highest derivative specified. Hence, in the set of equations $\alpha_{[\mu,\nu]}=0$, three will involve third-order time equations and three will involve fourth-order equations. Barring the circumstance that k_μ or l_μ will lie along the perpendicular to the initial hypersurface, the identities (35) and (36) both involve fourth-order time derivatives. Hence, the set of equations (37) can be rewritten as a set of three third-order differential equations in time and a single fourth-order equation.

The three differential equations of the third order involving only the single undetermined one of the six functions $g_{ij,000}$ can only be satisfied if the functions $g_{ij,00}$ satisfy two conditions; these are the two constraints we have been looking for. Since only one $g_{ij,00}$ is still free to satisfy these two conditions, at least one of them will remain unspecified as a constraint on the initial $g_{\mu\nu}$ and $g_{\mu\nu,0}$ which therefore cannot be given arbitrarily.

If the appropriate choice of the initial data is made, $g_{ij,000}$ can be chosen so that three of the equations in (37) are obeyed and $g_{ij,0000}$ so that the fourth is satisfied. One can then be assured that the full set of Einstein-Maxwell equations are satisfied everywhere in the initial hypersurface.

In this paper we have not discussed the time evolution of the fields, we have merely discussed the possibility of choosing initial data that are consistent with the Einstein-Maxwell theory. The time evolution problem needs, however, careful consideration. In the case of flat space the problem is very simple. If $\mathbf{E}$ and $\mathbf{H}$ are given on a hypersurface so that $\nabla\cdot\mathbf{E}=0$ and $\nabla\cdot\mathbf{H}=0$, one can obtain the time evolution of the system from the remaining electromagnetic field equations: $\mathrm{curl}\mathbf{E}=-\partial\mathbf{H}/\partial t$, $\mathrm{curl}\mathbf{H}=\partial\mathbf{E}/\partial t$. These latter six equations determine the six quantities $\mathbf{E}$ and $\mathbf{H}$ and also assure that $\nabla\cdot\mathbf{E}=\nabla\cdot\mathbf{H}=0$ remains satisfied for all time. In the combined Einstein-Maxwell case the situation is somewhat different. The energy-momentum equation determines all but one of the free parameters in k_μ and l_μ. Left still are this one parameter and a phase function θ of somewhat limited usefulness. The Bianchi identity means that four of Maxwell's equations are automatically satisfied,[8] leaving only four independent equations. Two relations must be satisfied as identities on the hypersurface. Hence, two more equations are available whose solution must depend on an appropriate choice of the remaining parameter in k_μ and l_μ and on θ. It is not entirely clear that this can be done, and I am not aware of any existing proof that it can be.[9] The problem manifests itself in the geometric version by the realization that, after $g_{ij,000}$ is chosen to satisfy the three third-order equations in (37), all $g_{ij,000}$ are completely determined by this and the energy-momentum equations (26) and (27). Yet, the fourth order equation in (37) must still be satisfied. How to do this remains a problem for future investigation.

APPENDIX. IDENTICAL VANISHING OF TWO COMPONENTS OF $\alpha_{[\mu,\nu]}=0$

We now proceed to prove that Eqs. (35) and (36) vanish identically. First we outline the derivation of Eq. (32) that $\alpha_{[\mu,\nu]}=0$. If $R=0$ and $R_\alpha{}^\beta R_\beta{}^\gamma=\delta_\alpha{}^\gamma R_{\mu\nu}R^{\mu\nu}/4$, there are two null eigenvectors of $R_{\alpha\beta}$ from which $F_{\mu\nu}$ can be constructed by the procedure of Eq. (27). Define $\omega_{\mu\nu}\equiv F_{\mu\nu}+iD_{\mu\nu}$, so that

$$R_{\mu\nu}=\omega_\mu{}^\alpha\bar{\omega}_{\alpha\nu}. \tag{A-1}$$

If θ is an arbitrary function of space-time, $\omega'_{\mu\nu}=\omega_{\mu\nu}e^{i\theta}$ will also satisfy (A-1) whenever $\omega_{\mu\nu}$ itself does. This is the phase transformation entirely equivalent to that of Eqs. (7) and (8). Can θ be chosen so that $\omega'_{\mu\nu}$ obeys Maxwell's equations, $\omega'_\mu{}^\nu{}_{;\nu}=0$? ($\omega'_\mu{}^\nu$ being complex, the entire content of Maxwell's source-free equations are expressed in the preceding relation.) To satisfy this

[7] G. Rosen, Phys. Rev. **114**, 1179 (1959).

[8] This remark is proved in each reference cited in footnote 6.

[9] See, however, the remark in Lichnerowitz's book (reference 1, p. 51).

equation, θ must be chosen so that

$$\omega_{\mu}{}^{\nu}{}_{;\nu}+i\omega_{\mu}{}^{\nu}\theta_{,\nu}=0.$$

Multiplying by $\omega^{\mu}{}_{\rho}$ and using the identity

$$\omega_{\mu}{}^{\nu}\omega^{\mu}{}_{\rho}\equiv\tfrac{1}{2}\delta^{\nu}{}_{\rho}\omega_{\alpha\beta}\omega^{\alpha\beta}, \qquad \text{(A-2)}$$

which is a consequence of the duality of $F_{\mu\nu}$ and $D_{\mu\nu}$, one obtains

$$\theta_{,\rho}=\frac{2i\omega_{\mu}{}^{\nu}{}_{;\nu}\omega^{\mu}{}_{\rho}}{\omega_{\alpha\beta}\omega^{\alpha\beta}}\equiv\beta_{\rho}. \qquad \text{(A-3)}$$

The reality of the above expression follows from the Bianchi identity. It can now be shown that

$$\alpha_{[\rho,\sigma]}=\beta_{[\rho,\sigma]}. \qquad \text{(A-4)}$$

The reality of β_{ρ} and the validity of (A-4) are the essential steps in the proof that is omitted here, the details having been given in a previous paper.[10] From (A-3) and (A-4), with the notation $\omega^2=\omega_{\alpha\beta}\omega^{\alpha\beta}$, one has

$$\alpha_{[\rho,\sigma]}\omega^{\rho\sigma}=\frac{-4i\omega_{\mu}{}^{\nu}{}_{;\nu}\omega^{\mu}{}_{\rho;\sigma}\omega^{\rho\sigma}}{\omega^2}-\frac{4i\omega_{\mu}{}^{\nu}{}_{;\nu}\omega^{\mu}{}_{\rho}\omega^{\rho\sigma}}{\omega^2}+\frac{4i\omega_{\mu}{}^{\nu}{}_{;\nu}\omega^{\mu}{}_{\rho}\omega^{\rho\sigma}(\omega^2)_{,\sigma}}{\omega^4}. \qquad \text{(A-5)}$$

Using the identity (A-2) and the antisymmetry of $\omega^{\mu\rho}$, one obtains

$$\alpha_{[\rho,\sigma]}\omega^{\rho\sigma}=-4i\omega_{\mu}{}^{\nu}{}_{;\nu}\omega^{\mu}{}_{\rho;\sigma}\omega^{\rho\sigma}+\frac{2i\omega^{\sigma\nu}{}_{;\nu}(\omega^2)_{,\sigma}}{\omega^2}. \qquad \text{(A-6)}$$

Differentiating (A-2) yields

$$\omega^{\mu}{}_{\rho;\sigma}\omega^{\rho\sigma}+\omega^{\mu}{}_{\rho}\omega^{\rho\sigma}{}_{;\sigma}=\tfrac{1}{2}(\omega^2)_{,\rho}. \qquad \text{(A-7)}$$

Using this in (A-6) gives

$$\alpha_{[\rho,\sigma]}\omega^{\rho\sigma}=4i\omega_{\mu}{}^{\nu}{}_{;\nu}\omega^{\mu\rho}\omega_{\rho}{}^{\sigma}{}_{;\sigma}/\omega^2.$$

This vanishes because $\omega^{\mu\rho}$ is antisymmetric but $\omega_{\mu}{}^{\nu}{}_{;\nu}\omega_{\rho}{}^{\sigma}{}_{;\sigma}$ is symmetric in μ and ρ. So $\alpha_{[\rho,\sigma]}\omega^{\rho\sigma}=0$ identically; the real and imaginary parts correspond to Eqs. (35) and (36) which are identities.

[10] L. Witten, reference 6, p. 210.

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Invariance Under Antiunitary Operators*

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It is shown that for transitions between "weakly interacting" states, the transition matrix $\mathcal{T}$ can be expressed in terms of a Hermitean operator $\mathcal{T}+\mathcal{T}^{\dagger}$, and so invariance of the Hamiltonian under antiunitary operators such as T or TCP implies invariance of transition rates under kinematic transformations, without changing the direction of time.

An application is made to π^0 decay into 2 photons, where it is shown that invariance under TCP alone implies equality in the number of left and right circularly polarized photons, to 1 part in 10^4.

THE invariance of a Hamiltonian under a unitary transformation leads to the invariance of the transition rates under certain "kinematic" transformations of the quantum numbers in the initial and final states. For example, invariance of H under space reflection implies invariance of the transition rates under the change of sign of all momenta in the initial and final states. On the other hand, invariance of H under an anitunitary operator such as T or TCP does not in general lead to such an invariance of the transition rate, but rather to a relation between the transition rate from an initial state to final state, and the transition rate from the "kinematically reversed" final state to the kinematically reversed initial state. This is a physically distinguishable process, unless the initial and final states contain the same particles. It is, however, known that under some circumstances, invariance under an antiunitary operator nevertheless does imply a relation between transition rates for the same process. This will be the case, for example, when the following two conditions are satisfied[1]:

1. The transition matrix $\mathcal{T}$ can be taken equal to a Hamiltonian, i.e., when first-order perturbation theory is used.
2. The initial and final states $|a\rangle$, $|b\rangle$ are weakly interacting states.

In this note we shall show that the second condition alone is sufficient. Specifically, we show that if the initial and final states are such that all products of the form $\langle a|\mathcal{T}|n\rangle\langle n|\mathcal{T}^{\dagger}|b\rangle(|n\rangle\neq|a\rangle)$ can be neglected compared to $\langle a|\mathcal{T}|b\rangle$, then invariance under the antiunitary operator θ implies equality between the transition rates for $|a\rangle\rightarrow|b\rangle$ and for $|a_R\rangle\rightarrow|b_R\rangle$, where

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[1] T. D. Lee, R. Oehme, and C. N. Yang, Phys. Rev. 106, 340 (1957).