Proton-Proton Effective-Range Theory with Vacuum Polarization*

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The effect of vacuum polarization upon p-p scattering is considered by first solving the problem in the Coulomb plus vacuum polarization potentials (called "electric" potential) without the nuclear potential. The nuclear phase shifts are then defined with respect to the electric wave functions, and the scattering cross section is written in terms of these phase shifts. The connection with other nuclear phase shifts (in the presence of vacuum polarization) which appear in the literature is established.

The effective-range expansion for the nuclear s-wave phase shift is derived. An analysis of three low-energy p-p scattering experiments indicates that the omission of vacuum polarization from the analysis results in a value for the shape-dependent parameter which is 0.02 smaller than the value obtained when vacuum polarization is included in the analysis. A discussion of the accuracy required to usefully delimit this parameter is included.

I. INTRODUCTION

FOLDY and Eriksen[1] originally demonstrated the presence of the vacuum polarization (v.p.) potential in the interaction between two protons by showing that one obtains a better fit to the energy dependence of the s-wave proton-proton scattering phase shift (and slightly different values for the p-p scattering length and effective range) if the v.p. potential is included in the analysis than if it is omitted. They[1] did this by deriving a correction ΔK due to v.p. to the effective-range theory function

$$\frac{K}{R}=C^2k\cot\delta_0+\frac{h(\eta)}{R}=-\frac{1}{a}+\tfrac{1}{2}r_0k^2-Pr_0^3k^4+\cdots. \quad (1)$$

Durand[2] showed that the long range of the v.p. potential, of the order of the electron Compton wavelength, produces scattering in many orbital angular momentum states even at low proton energies. To deduce the correct value of the nuclear s-wave phase shift from experiment, one must include the complete v.p. scattering amplitude along with the Coulomb and nuclear amplitudes. Precision p-p angular distributions at Wisconsin[3] are better fitted with the inclusion of the v.p. scattering amplitude as calculated by Durand[2] than if v.p. is omitted.[3]

It should be possible before too long to assign a good value to the coefficient, P, of the third term in the expansion of K in powers of k^2 [Eq. (1)], the shape-dependent parameter. We shall return to this question later. This number is important in deciding whether or not there is a hard core in the p-p potential, since if P is positive, it appears to rule out[4] a hard core greater than 0.3×10^{-13} cm. Since the term involving P in Eq. (1) is of the order of 10^{-3} of the leading term $(-1/a)$ at 1 Mev, one must be sure to include all effects in the function K which could modify it by a part in a thousand, if the correct value of P is to be deduced from the experiments. This is really true only for energy-dependent corrections to K. Short-range interactions, e.g., finite size of the protons and the contact part of the magnetic moment interaction,[5] produce corrections to K which at low energies are energy independent and consequently they primarily affect the values of the scattering length, a, and the effective range, r_0. Long-range interactions such as v.p. produce energy-dependent corrections to K which do affect the value of P (as well as r_0 and a).

One difficulty with the Foldy-Eriksen[1] procedure for calculating ΔK due to v.p. is the fact that approximate p-p wave functions have to be known, since they treated the v.p. potential as a perturbation upon the combined Coulomb plus nuclear potentials. Because of the sensitivity of P to changes in K, one must ascertain whether or not the assumed shape of the nuclear potential used in calculating $\Delta K_{\text{v.p.}}$ affects the value of P thereby deduced from the experiments.

We take a different point of view and first solve the problem of the Coulomb plus v.p. potentials (called the "electric" potential) with no nuclear potential (Sec. II). Wave functions and the scattering amplitude (Durand's[2]) are obtained for this electric potential (treating v.p. as a perturbation). Then the nuclear potential is turned on (Sec. III): the nuclear phase shifts are defined with respect to the electric wave functions; and the nuclear scattering amplitude is written in terms of these phase shifts. The connection with the phase shifts used in references 1 and 2 is established. The effective range expansion for the s-wave nuclear phase shift is derived in Sec. IV. Section V contains a discussion of the accuracy needed in the phase shifts to obtain a given accuracy in P, as well as the effect which v.p. has upon the value of P which is deduced from the experiments.

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[1] L. L. Foldy and E. Eriksen, Phys. Rev. **98**, 775 (1955).

[2] L. Durand, III, Phys. Rev. **108**, 1597 (1957).

[3] D. J. Knecht, S. Messelt, E. D. Berners, and L. C. Northcliffe, Phys. Rev. **114**, 550 (1959).

[4] R. L. Preston and M. A. Preston, Can. J. Phys. **36**, 579 (1958).

[5] J. Schwinger, Phys. Rev. **78**, 135 (1950).

It should be pointed out that the procedure which is employed here for the v.p. potential can be used equally well, and additively, for other electromagnetic corrections to the Coulomb potential which can be treated in perturbation theory.

II. THE ELECTRIC POTENTIAL

We define the electric potential, $V_E(r)$, to be the sum of the Coulomb and v.p. potentials,

$$V_E(r) \equiv \frac{e^2}{r} + \lambda I(r)\frac{e^2}{r}, \tag{2}$$

where $\lambda = 2\alpha/3\pi = 1.549\times 10^{-3}$ and the function $I(r)$, shown in Fig. 1, can be written[6]

$$I(r) = \int_1^\infty dx\, e^{-2\kappa r x}\left(\frac{1}{x^2}+\frac{1}{2x^4}\right)(x^2-1)^{\frac{1}{2}}, \tag{3}$$

with $(2\kappa)^{-1} = \hbar/2mc = 193.1\times 10^{-13}$ cm. The limiting forms for $I(r)$ are as follows ($z\equiv 2\kappa r$):

$$I(r) = -\gamma - \tfrac{5}{6} - \ln(\kappa r) + O(z), \qquad (z\ll 1)$$

$$I(r) = \frac{3(2\pi)^{\frac{1}{2}}}{4}\frac{e^{-2\kappa r}}{(2\kappa r)^{\frac{3}{2}}}\left[1+O\left(\frac{1}{z}\right)\right], \qquad (z\gg 1) \tag{4}$$

γ is Euler's constant $\gamma = 0.5772\cdots$.

The radial wave equation in the electric potential can be written

$$\frac{d^2u_L(r)}{dr^2} + \left[k^2 - \frac{1}{Rr} - \frac{L(L+1)}{r^2}\right]u_L(r) = \frac{\lambda I(r)}{Rr}u_L(r), \tag{5}$$

where $R = \hbar^2/Me^2 = 28.82\times 10^{-13}$ cm. There are two linearly independent solutions of this equation. One of these S_L vanishes at the origin, and we choose it to have

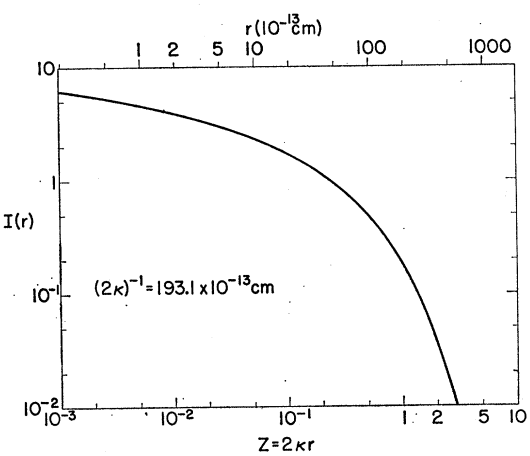

FIG. 1. The function $I(r)$ which appears in the vacuum polarization potential vs distance.

[6] See reference 2 for a bibliography on the history and derivation of the v.p. potential.

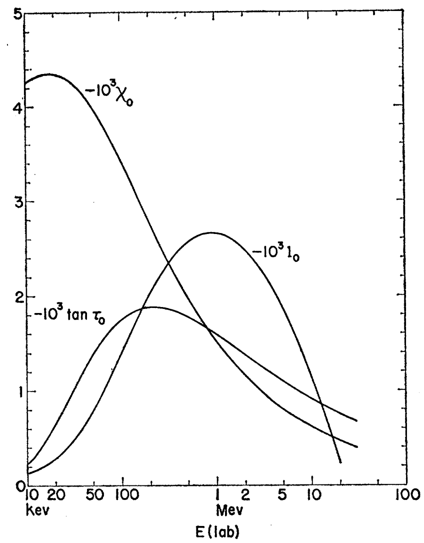

FIG. 2. The quantities which occur in the effective-range expansion for $\delta_0{}^E$ vs the laboratory energy of the incident proton. Note that all three quantities are negative.

the asymptotic form

$$S_L(r) \underset{(2\kappa r\gg 1)}{=} F_L(r) + \tan\tau_L G_L(r), \tag{6}$$

where F_L and G_L are the regular and irregular Coulomb functions, respectively. The integral equation satisfied by S_L is

$$S_L(r) = F_L(r)\left(1 - 2\eta\lambda\int_r^\infty dr'\frac{G_L(r')I(r')S_L(r')}{r'}\right) + G_L(r)\left(\tan\tau_L + 2\eta\lambda\int_r^\infty dr'\frac{F_L(r')I(r')S_L(r')}{r'}\right), \tag{7}$$

where $\eta = e^2/\hbar v$ is the Coulomb parameter[7] and

$$\tan\tau_L \equiv -2\eta\lambda\int_0^\infty dr\frac{F_L I S_L}{r} \tag{8}$$

is the tangent of the v.p. phase shift.

The other solution T_L is taken to have the asymptotic form

$$T_L(r) \underset{(2\kappa r\gg 1)}{=} G_L(r) - \tan\tau_L F_L(r). \tag{9}$$

The integral equation for T_L is

$$T_L(r) = G_L(r)\left(1 + 2\eta\lambda\int_r^\infty dr'\frac{F_L(r')I(r')T_L(r')}{r'}\right) - F_L(r)\left(\tan\tau_L + 2\eta\lambda\int_r^\infty dr'\frac{G_L(r')I(r')T_L(r')}{r'}\right). \tag{10}$$

For later reference, the behavior near the origin

[7] The relativistic η has been used in all numerical calculations.

of these functions and their derivatives for $L=0$ is presented.

$$S_0(r)=Ckr(1+\chi_0)+O(r^2\ln r),$$

$$T_0(r)=\frac{1}{C}(1-\phi_0)+O(r\ln^2 r),$$

$$S_0'(r)=Ck(1+\chi_0)+O(r\ln r),$$

$$T_0'(r)=(1/CR)[\ln(r/R)+2\gamma+h(\eta)](1-\phi_0) -Ck\left(\tan\tau_0+2\eta\lambda\int_r^\infty dr'\frac{G_0(r')I(r')T_0(r')}{r'}\right) +O(r\ln^2 r), \quad (11)$$

where

$$C=\left(\frac{2\pi\eta}{\exp(2\pi\eta)-1}\right)^{\frac{1}{2}},$$

$h(\eta)$ is the function defined in Jackson and Blatt,[8] and

$$\chi_0\equiv-2\eta\lambda\int_0^\infty dr\frac{G_0IS_0}{r},$$
$$\phi_0\equiv-2\eta\lambda\int_0^\infty dr\frac{F_0IT_0}{r}. \quad (12)$$

The Wronskian relation is

$$(\cos^2\tau_0)(1+\chi_0)(1-\phi_0)=1.$$

Due to the fact that λ is a very small number, a perturbation expansion can be made, and we shall only need the first-order correction terms which are obtained by replacing S_L and T_L wherever they occur in integrands by F_L and G_L, respectively. Note that the logarithmic singularity in $I(r)$, because of its integrability, is of no consequence. Actually the finite size of the protons eliminates this singularity, but it is simpler to work with the expressions for point protons bearing in mind the discussion of short-range corrections given in the Introduction. In perturbation theory, the expressions given above become

$$\tau_L=-2\eta\lambda\int^\infty dr\frac{F_L{}^2I}{r},$$
$$\chi_0=\phi_0=-2\eta\lambda\int^\infty dr\frac{F_0G_0I}{r}. \quad (13)$$

These quantities were calculated[9] with the aid of an electronic computer and are plotted in Fig. 2 vs the laboratory energy of the incident proton ($L=0$ only). Durand[2] has given an analytic expression for τ_0 (called δ_0 by him) which is good above 1 Mev and which we agree with in that domain. The same remarks apply to the formula for τ_1 derived by Eriksen[10] (called $\delta_1{}^{\text{v.p.}}$). Figures 3 and 4 show the v.p. phase shifts versus L at

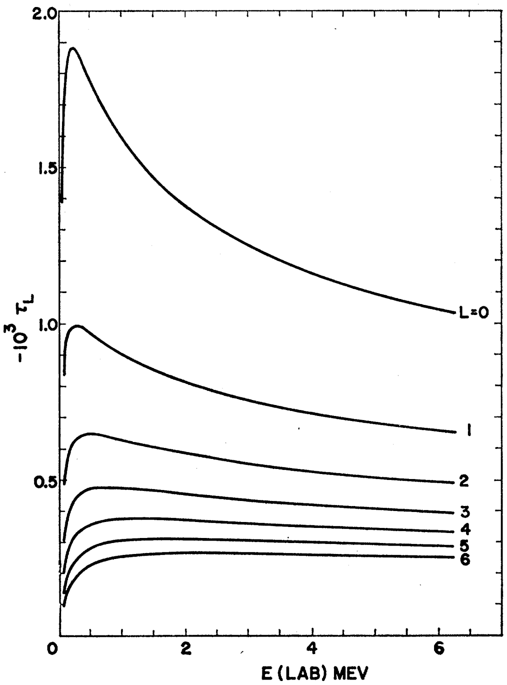

FIG. 4. The vacuum polarization phase shifts versus laboratory energy of the incident proton for the first few orbital angular momentum states.

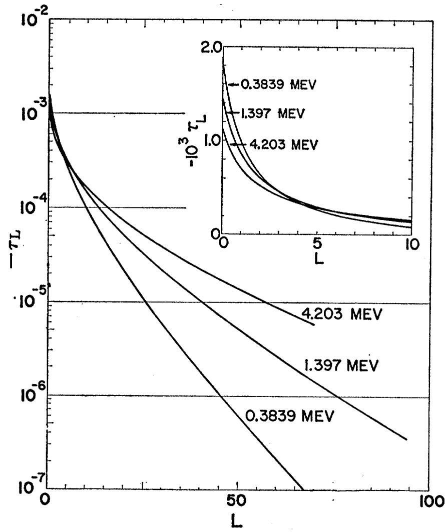

FIG. 3. The vacuum polarization phase shifts versus orbital angular momentum for three laboratory energies of the incident proton. The phase shifts are all negative, and only integral values of L are meaningful.

[8] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 96–97 (1950).

[9] I am indebted to Dr. M. L. Gursky for his procedure for calculating the Coulomb functions and for considerable help in obtaining the numerical answers.

[10] E. Eriksen, L. L. Foldy, and W. Rarita, Phys. Rev. **103**, 781 (1956).

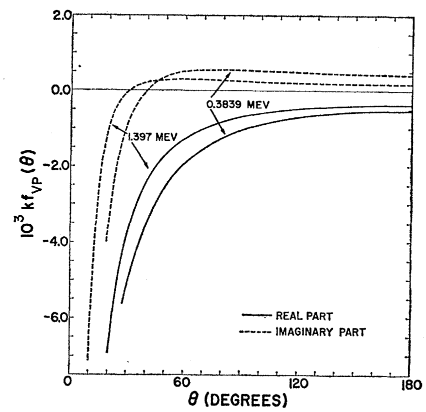

FIG. 5. The real and imaginary parts of the vacuum polarization scattering amplitude multiplied by the wave number versus center-of-mass angle for two laboratory energies of the incident proton.

three energies and as functions of energy for several L-values, respectively.

The complete scattering amplitude in the electric potential $f_E(\theta)$ is the sum of the Coulomb and v.p. amplitudes

$$f_E(\theta)=f_C(\theta)+f_{\text{v.p.}}(\theta), \tag{14}$$

where

$$f_C(\theta)=-\frac{\eta}{2k\sin^2(\theta/2)}e^{-2i\eta\ln\sin(\theta/2)}, \tag{15}$$

and

$$f_{\text{v.p.}}(\theta)=\frac{1}{k}\sum_L(2L+1)\tau_L e^{2i(\sigma_L-\sigma_0)}P_L(\cos\theta). \tag{16}$$

Use has been made of the fact that $\tau_L\ll 1$, and σ_L is the Coulomb phase shift. This formula was given by Durand,[2] who derived the first three terms in an expansion of $f_{\text{v.p.}}(\theta)$ in powers of η. Figures 5 and 6 show the v.p. scattering amplitude versus angle at two energies and as a function of energy at three angles, respectively. The portion of Fig. 6 above 1.4 Mev was computed from Durand's expansion,[2] which is satisfactory at these energies.

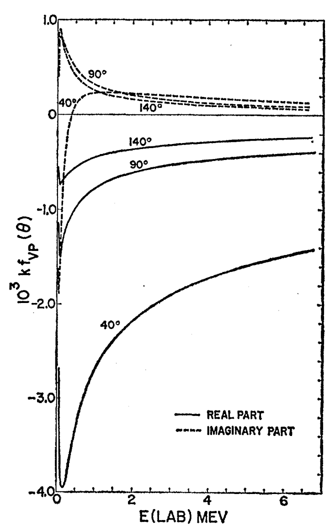

FIG. 6. The real and imaginary parts of the vacuum polarization scattering amplitude multiplied by the wave number versus the laboratory energy of the incident proton for three center-of-mass angles.

III. ELECTRIC PLUS NUCLEAR POTENTIALS

With the nuclear potential on (we consider only the singlet state at present) the radial wave function becomes, at distances large compared to the range of the nuclear force b,

$$R_L(r)\underset{(r\gg b)}{=}\cos\delta_L{}^E S_L(r)+\sin\delta_L{}^E T_L(r), \tag{17}$$

with $\delta_L{}^E$ the nuclear phase shift *in the electric potential*. Using Eqs. (6) and (9), this in turn becomes

$$R_L(r)\underset{(2\kappa r\gg 1)}{=}\cos K_L F_L(r)+\sin K_L G_L(r), \tag{18}$$

with

$$K_L=\delta_L{}^E+\tau_L. \tag{19}$$

K_L is the sum of the nuclear and v.p. phase shifts with the nuclear phase shift defined as above.

One can ask for the relation between $\delta_L{}^E$ and the nuclear phase shift, $\delta_L{}^C$, which *the same nuclear potential* would produce *if there were no* v.p. The superscript "C" indicates that this is a nuclear phase shift in the Coulomb potential; i.e., if $W_L(r)$ is the radial wave function in the same nuclear plus Coulomb potentials, but with no v.p., then $\delta_L{}^C$ is defined by

$$W_L(r)\underset{(r\gg b)}{=}\cos\delta_L{}^C F_L(r)+\sin\delta_L{}^C G_L(r). \tag{20}$$

Using perturbation theory, one can show that

$$K_L=\delta_L{}^E+\tau_L=\delta_L{}^C+\Delta_L, \tag{21}$$

with

$$\Delta_L\equiv -2\eta\lambda\int_0^\infty dr\frac{W_L{}^2 I}{r}. \tag{22}$$

The complete (unsymmetrized singlet) scattering amplitude $f(\theta)$ is the sum of the electric amplitude and the nuclear amplitude

$$f(\theta)=f_E(\theta)+f_N(\theta), \tag{23}$$

where $f_N(\theta)$ is calculated by the customary procedure to be

$$f_N(\theta)=\frac{1}{2ik}\sum_L(2L+1)e^{2i(\sigma_L-\sigma_0)}e^{2i\tau_L}\times[\exp(2i\delta_L{}^E)-1]P_L(\cos\theta). \tag{24}$$

Note that this expression differs from the customary (no v.p.) formula by the presence of the v.p. phase shift, along with the Coulomb phase shift. $f(\theta)$ can be decomposed in another way:

$$f(\theta)=f_C(\theta)+\frac{1}{2ik}\sum_L(2L+1)e^{2i(\sigma_L-\sigma_0)}\times(e^{2iK_L}-1)P_L(\cos\theta). \quad (25)$$

The v.p. phase shift does not appear explicitly in this formula, but it must be remembered that it is contained in K_L according to Eq. (19). Indeed at low energies while only one or two δ_L^E are appreciably different from zero, very many K_L are nonzero. Hence, the advantage of Eqs. (23) and (24) over (25). It is nevertheless possible to use a hybrid notation which involves K phase shifts in the important angular momentum states and δ^E phase shifts in the unimportant ones. We shall illustrate this with the symmetrized singlet cross section in the case that only δ_0^E is large. First of all, entirely in terms of the δ^E phase shifts, from Eqs. (23) and (24)

$$[f^s(\theta)\equiv f(\theta)+f(\pi-\theta)],$$

$$\begin{aligned}|f^s(\theta)|^2=&|f_C{}^s(\theta)|^2+\frac{4}{k^2}\sin^2\delta_0^E\\&+\frac{4}{k}\mathrm{Re}\{f_C{}^s(\theta)[e^{2i\tau_0}\exp(i\delta_0^E)\sin\delta_0^E]^*\}\\&+2\,\mathrm{Re}\{f_{\mathrm{v.p.}}{}^s(\theta)[f_C{}^s(\theta)+(2/k)\exp(i\delta_0^E)\sin\delta_0^E]^*\}\\&+\frac{4}{k}\mathrm{Re}\{[\sum_{L=2,4,\cdots}(2L+1)e^{2i(\sigma_L-\sigma_0)}\delta_L^EP_L]\\&\times[f_C{}^s(\theta)+(2/k)\exp(i\delta_0^E)\sin\delta_0^E]^*\}. \quad (26)\end{aligned}$$

In Eq. (26), products of small terms have been dropped, namely, v.p. with v.p., v.p. with d and higher phase shifts, and d and higher phase shifts with themselves. Now the same quantity in the mixed notation,

$$\begin{aligned}|f^s(\theta)|^2=&|f_C{}^s(\theta)|^2+(4/k^2)\sin^2K_0\\&+(4/k)\,\mathrm{Re}[f_C{}^s(\theta)(e^{iK_0}\sin K_0)^*]\\&+2\,\mathrm{Re}\{[f_{\mathrm{v.p.}}{}^s(\theta)-2\tau_0/k][f_C{}^s(\theta)+(2/k)e^{iK_0}\sin K_0]^*\}\\&+\frac{4}{k}\mathrm{Re}\left\{[\sum_{L=2,4,\cdots}(2L+1)e^{2i(\sigma_L-\sigma_0)}\delta_L^EP_L]\right.\\&\left.\times\left[f_C{}^s(\theta)+\frac{2}{k}e^{iK_0}\sin K_0\right]^*\right\}. \quad (27)\end{aligned}$$

The same types of small terms have been dropped from Eq. (27) as from (26). The only differences in the structure of these two expressions, to the approximation considered, occur in the third and fourth terms. Using the K_0 form, Eq. (27), there is no factor of $e^{2i\tau_0}$ in the third term, and the s-wave part of the v.p. amplitude is subtracted off in the fourth term. The reason for both of these differences is just that K_0 includes both the nuclear and v.p. s-wave phase shifts.

A similar analysis can be performed for the triplet state, writing the cross section either in terms of $\delta_{1,j}^E$ phase shifts or in terms of $K_{1,j}$ phase shifts. If the latter choice is made, then there will be two differences just as in the singlet state: The nuclear-Coulomb interference term will not involve τ_1, and the v.p. term will occur with the combination

$$[f_{\mathrm{v.p.}}{}^T(\theta)-(6/k)e^{2i(\sigma_1-\sigma_0)}\tau_1\cos\theta].$$

Comparison of these formulas with those given by Durand[2] shows that any analysis of data based upon his $\Delta\sigma_{\mathrm{v.p.}}$ [Eq. (24.3) in that paper], e.g., the Wisconsin[3] analysis, necessarily yields for the s state a K_0 phase shift and for the p state, δ_1^E phase shifts and not K_1 phase shifts. We believe that it is simpler and more meaningful to express the cross section solely in terms of δ^E phase shifts as in Eq. (26), since these are the ones which tell about the nuclear potential.

IV. s-WAVE EFFECTIVE-RANGE EXPANSION

The first question in seeking an effective-range expansion[11] is which phase shift shall be used. From the discussion in Sec. III, it is clear that either δ_0^E (the nuclear phase shift in the electric potential) or δ_0^C (the nuclear phase shift that would be obtained from the same nuclear potential if there were no v.p.) is a satisfactory candidate, but that K_0 is not because it contains the v.p. phase shift in it explicitly. The experimental cross section yields either δ_0^E or K_0. If δ_0^C is to be used (this was the Foldy-Eriksen[1] choice), it must first be computed using Eqs. (21) and (22), and then the ordinary effective-range expansion in the Coulomb potential applies. We shall derive the expansion for δ_0^E.

One proceeds exactly as in the case of the Coulomb potential[12] except that the electric wave functions are used instead of Coulomb functions. One writes down the s-wave differential equation first with the electric potential [solution $\theta(r)$, the asymptotic function], then with the electric plus nuclear potentials [solution $v(r)$].

$$v(r)\underset{(r\gg b)}{=}\theta(r),\quad v(0)=0. \quad (28)$$

Normalize $\theta(r)$ so that $\theta(0)=1$, i.e., from Eq. (11)

$$\theta(r)=\frac{C}{1-\phi_0}[T_0(r)+\cot\delta_0^ES_0(r)]. \quad (29)$$

If the differential equations are written for two different energies, and one multiplies, subtracts, and integrates in the customary way,[12] the result for sufficiently small r (in the limit going to zero) is

$$\theta_b'(r)-\theta_a'(r)\underset{(r\to0)}{=}(k_b^2-k_a^2)\int_r^\infty(\theta_a\theta_b-v_av_b)dr. \quad (30)$$

[11] M. deWit and L. Durand, III, have also looked into v.p. effects in the effective-range expansion [Phys. Rev. 111, 1597 (1958)].

[12] See reference 8, Appendix IV.

The derivative is calculated from Eq. (11) to be

$$\theta'(r) \underset{(r\to 0)}{=} \frac{1}{R}\left(\ln\frac{r}{R}+2\gamma+h(\eta)\right)+\frac{C^2k}{1-\phi_0}[(1+\chi_0)\cot\delta_0{}^E-\tan\tau_0]-\frac{C^2}{1-\phi_0}\frac{\lambda}{R}\int_r^\infty dr'\frac{G_0(r')I(r')T_0(r')}{r'}. \quad (31)$$

$$\theta_b'(r)-\theta_a'(r) \underset{(r\to 0)}{=} \frac{C_b{}^2k_b}{1-\phi_{0b}}[(1+\chi_{0b})\cot\delta_{0b}{}^E-\tan\tau_{0b}]+\frac{h(\eta_b)}{R}-\frac{C_a{}^2k_a}{1-\phi_{0a}}[(1+\chi_{0a})\cot\delta_{0a}{}^E-\tan\tau_{0a}]$$
$$-\frac{h(\eta_a)}{R}-\frac{\lambda}{R}\int_r^\infty dr'\frac{I(r')}{r'}\left[\left(\frac{C^2G_0T_0}{1-\phi_0}\right)_b-\left(\frac{C^2G_0T_0}{1-\phi_0}\right)_a\right]. \quad (32)$$

Again using Eq. (11), this integral converges at the origin and r can be put equal to zero. Doing this and also choosing $k_a=0$ [$h(\eta_a)=0$, $\tan\tau_{0a}=0$] gives

$$\frac{C^2k}{1-\phi_0}[(1+\chi_0)\cot\delta_0{}^E-\tan\tau_0]+\frac{h(\eta)}{R}+\frac{l_0(\eta)}{R}=-\frac{1}{a^E}+\tfrac{1}{2}k^2\rho^E(0,E), \quad (33)$$

where a^E is the scattering amplitude in the electric potential defined by

$$-\frac{1}{a^E}\equiv\lim_{k\to 0} C^2k\frac{1+\chi_0}{1-\phi_0}\cot\delta_0{}^E; \quad (34)$$

$l_0(\eta)$ is defined to be

$$l_0(\eta)\equiv-\lambda\int_0^\infty dr\frac{I}{r}\left[\left(\frac{C^2G_0T_0}{1-\phi_0}\right)-\left(\frac{C^2G_0T_0}{1-\phi_0}\right)_{E=0}\right]; \quad (35)$$

and

$$\rho^E(0,E)\equiv 2\int_0^\infty(\theta\theta_{E=0}-vv_{E=0})dr. \quad (36)$$

If perturbation theory is used, Eqs. (33)–(35) become

$$C^2k[(1+2\chi_0)\cot\delta_0{}^E-\tau_0]+\frac{h(\eta)}{R}+\frac{l_0(\eta)}{R}=-\frac{1}{a^E}+\tfrac{1}{2}k^2\rho^E(0,E) \quad (37)$$

$$-\frac{1}{a^E}=\lim_{k\to 0}C^2k(1+2\chi_0)\cot\delta_0{}^E, \quad (38)$$

and

$$l_0(\eta)=-\lambda\int^\infty dr\frac{I}{r}[(CG_0)^2-(CG_0)_{E=0}{}^2]. \quad (39)$$

$l_0(\eta)$ is plotted in Fig. 2.

The customary argument[13] can be made that $\rho^E(0,E)$ receives its contribution from inside the range of the (strong) nuclear force so that at low energies, it can be expanded in powers of k^2:

$$\rho^E(0,E)=r_0{}^E-2P^Er_0{}^3k^2+\cdots. \quad (40)$$

Bethe's argument[13] that for a *given nuclear potential* r_0 and P must have substantially the same values whether or not the Coulomb potential is present, because of the fact that the Coulomb potential is weak compared to the nuclear potential inside the range of the nuclear force where r_0 and P receive their contributions, holds even more strongly whether or not v.p. is present, so that it is not necessary to put electric superscripts on r_0 and P. This does *not* mean, however, that analyzing the experiments with or without v.p. will give rise to the same value of P (or r_0). (The effect upon r_0, for example, of analyzing p-p experiments without the Coulomb effect would be very serious.) The actual effect of v.p. will be considered quantitatively in the next section.

V. SHAPE-DEPENDENT PARAMETER FROM EXPERIMENT

An estimate of the accuracy needed in order that the shape-dependent parameter be known to a given uncertainty can be obtained from Eqs. (37) and (40) by considering three "equally accurate", equally spaced (in energy) experiments. Then it can be shown that $|\delta P|$, the uncertainty in P, is given by

$$|\delta P|=\frac{48(1+\epsilon)}{\Delta^2}\left(\frac{2|d\delta_0|}{\sin 2\delta_0}\right), \quad (41)$$

where Δ is the laboratory energy separation (in Mev) between adjacent experiments; $|d\delta_0|$ is the uncertainty in the s-wave phase shift; and ϵ is an energy-dependent quantity which arises from all the terms in Eq. (37) except those involving $\delta_0{}^E$ and the scattering length. Below 6 Mev, $|\epsilon|<\frac{1}{4}$. By "equally accurate", we mean that the three experiments have the same value for the quantity in the brackets in Eq. (41). For the three Wisconsin[3] experiments, we assign the approximate value

$$2|d\delta_0|/\sin 2\delta_0\simeq 2|d\delta_0|\simeq 6\times 10^{-4}.$$

With experiments of this accuracy,

$$|\delta P|\simeq 0.03_0/\Delta^2.$$

The Wisconsin[3] experiments have $\Delta\sim 0.5$ Mev, and

[13] H. A. Bethe, Phys. Rev. 76, 38 (1949).

consequently they do not, by themselves, give any information about P. With experiments of the same accuracy, Δ has to be at least 1.5 Mev to sensibly delimit P.

As an example, if one considers in addition to the 2.425-Mev Wisconsin[3] experiment the one at the p-p minimum[14] (383.9±1.5 kev) and the older Wisconsin[15] angular distribution at 4.203 Mev, then $\Delta\sim 2$ Mev, but these experiments are less accurate than the one at 2.425 Mev.

a. 2.425 Mev

For this experiment, the Wisconsin analysis[3] gave

$$K_0=48.273°,$$

with the uncertainty as given above. From Eq. (21) and Fig. 2,

$$\delta_0{}^E=48.348°.$$

b. 4.203 Mev

According to the Hall-Powell[16] analysis

$$\delta_0=53.808°\pm0.081°,$$

and therefore

$$2|d\delta_0|/\sin 2\delta_0 \simeq 3\times10^{-3}.$$

This analysis[16] did not include v.p., so we write the phase shift with no superscript. (They[16] used the symbol K_0, but it is not the same as the K_0 phase shift defined in the present work.)

A more recent analysis of the same data[17] including split p-wave phase shifts as well as v.p. and relativistic kinematics indicates that the s-wave phase shift is much more uncertain than the value given above. If the p-wave phase shifts are put equal to zero, then the fit to the data is only slightly poorer, and the value

$$K_0=53.912°,$$

is obtained.[17]

From Eq. (21) and Fig. 2,

$$\delta_0{}^E=53.978°.$$

c. 0.3839 Mev

The purpose of this experiment[14] is to locate the minimum with respect to energy of the 90° p-p scattering cross section. We have reconsidered the relation between the energy of the minimum and the value of the s-wave phase shift at the minimum, including v.p. in the analysis. In agreement with the statement made in reference 14, we find a one-to-one relation between $E_{\min}$ and the value of $\delta_0{}^E$ at $E_{\min}$, which is, however, different from the relation if v.p. is omitted. (In reference 14, the relation without v.p. was stated not for the minimum in the cross section but rather for the energy at which the ratio $\sigma/\sigma_{\text{Mott}}$ has its minimum. We shall work with the minimum in σ itself.) Figure 7 shows the relations in question. The upper curve (straight line) includes v.p. and represents $\delta_0{}^E$. The lower curve has no v.p. These lines do have a finite width of approximately 0.1 kev due to the spread in possible values of a and r_0 which are capable of producing a given $E_{\min}$ and also due to the possible presence of split p-wave phase shifts which can displace $E_{\min}$. With energy resolution poorer than 0.1 kev, this spread is of no consequence. A given pair (a,r_0) gives rise to an $E_{\min}$ which is 0.7 kev lower if the v.p. scattering amplitude is omitted than if it is included.

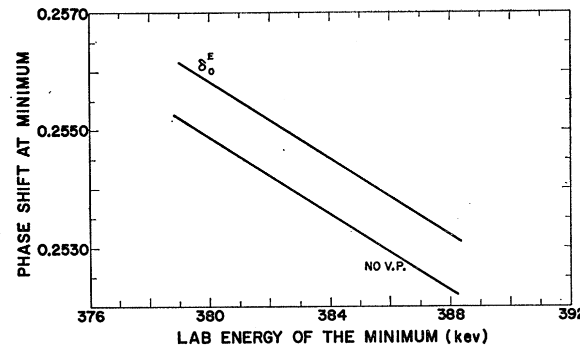

FIG. 7. The value of the s-wave phase shift at the minimum of the 90° p-p scattering cross section. The upper curve includes vacuum polarization and represents $\delta_0{}^E$. The lower curve omits vacuum polarization.

With the minimum given as[14] $E_{\min}=383.9\pm1.5$ kev, the phase shift obtained from Fig. 7 is[18]

$$\delta_0{}^E=0.25452\pm0.00050.$$

The uncertainty in the phase shift agrees with the estimate made in the text of reference 14, but the uncertainty which is quoted by those authors is double this value. Using our uncertainty, we have

$$2|d\delta_0|/\sin 2\delta_0=2\times10^{-3}.$$

Treating the Hall-Powell[16] uncertainty at 4.203 Mev as a lower bound, these three experiments (0.3839, 2.425, and 4.203) of unequal accuracy imply

$$|\delta P|>0.02_5.$$

To see by how much the omission of v.p. affects the value of P, it is not useful to make a best fit to the data both with and without v.p., because this procedure will not distinguish between the experimental uncertainty and the effect in question. Instead, the following method shall be employed. At the low energy point, we regard

[14] D. I. Cooper, D. H. Frisch, and R. L. Zimmerman, Phys. Rev. **94**, 1209 (1954).

[15] H. R. Worthington, J. N. McGruer, and D. E. Findley, Phys. Rev. **90**, 899 (1953).

[16] H. H. Hall and J. L. Powell, Phys. Rev. **90**, 912 (1953).

[17] M. H. MacGregor, Phys. Rev. **113**, 1559 (1959).

[18] H. P. Noyes (private communication) reanalyzed the minimum of $\sigma/\sigma_{\text{Mott}}$, using the v.p. scattering amplitudes of this paper, and obtained a phase shift which is consistent with the value stated above. I would like to thank Dr. Noyes for keeping me informed of this work.

TABLE I. The effective-range parameters obtained from the three experiments discussed in the text, analyzed in three different ways.

Type of phase shift used in analysis	Scattering length (10^{-13} cm)	Effective range (10^{-13} cm)	Shape-dependent parameter
δ_0^E	-7.77	2.77	0.04_7
δ_0 (no v.p.)	-7.71	2.68	0.02_6
δ_0^C	-7.79	2.78	0.04_8

0.3839 Mev as the *exact* location of the minimum. Then corresponding to the value $\delta_0^E=0.25452$, there is a unique value for δ_0 (the no-v.p. phase shift) of

$$\delta_0=0.25359,$$

obtained from Fig. 7. At 2.425 and 4.203 Mev, we regard the phase shifts δ_0^E given above (using the MacGregor[17] analysis at 4.203 Mev) as *exact* numbers; calculate the 90° cross section which these phase shifts imply, assuming no p or higher waves contribute at this angle (which is consistent with the analysis that gave rise to the values of δ_0^E); and then ask what value must be assigned to δ_0 if this same 90° cross section is to be explained without v.p. In this way, we find

$$\delta_0=48.320° \quad (2.425 \text{ Mev}),$$

$$\delta_0=53.953° \quad (4.203 \text{ Mev}).$$

Using the ordinary effective-range expansion, Eq. (1), for the three δ_0 phase shifts, and the new expansion, Eq. (37), for the δ_0^E phase shifts, the effective-range parameters given in Table I are obtained. In the third row of this table, the parameters for the δ_0^C phase shifts calculated from Eqs. (21), (22) and the quantity ΔK tabulated in reference 1, are presented. The relation

$$\Delta_0=-\frac{\sin^2\delta_0^C}{C^2kR}(\Delta K), \qquad (42)$$

has been used. It is seen that the omission of v.p. leads to a value for P which is considerably smaller than the value obtained when v.p. is included in the analysis. (The uncertainty in P has been estimated above; the central value appears in the table.) On the other hand, there is essentially no difference between the results obtained from the δ_0^E and δ_0^C phase shifts. This was to be expected with $P\simeq 0.05$ because the Yukawa-like wave functions which were used[1] to calculate ΔK are consistent with this value of P. If P is only slightly different from 0.05, there will be very little difference between these two methods; a discrepancy might appear if P should be negative (e.g., hard core or square well potentials). In any event, the s-wave phase shift must be obtained correctly from experiment, i.e., the complete v.p. scattering amplitude should be included. In the example considered above, if v.p. is taken into account only in the s state, the error in P is about half of the error produced by completely neglecting v.p. We feel that it is simpler and safer to employ the effective-range expansion for δ_0^E.

VI. CONCLUSION

Using the effective-range expansion derived for the s-wave nuclear phase shift in the electric potential, a procedure which has the advantage that it makes no assumption about the nuclear potential, it has been shown that the omission of v.p. from the analysis of p-p scattering results in a value for the shape-dependent parameter which is considerably smaller than the value obtained when v.p. is included in the analysis. To obtain (from the effective range expansion) a value for P which is sufficiently accurate to decide whether or not there is an appreciable hard core in the singlet state will require more accurate values for the s-wave phase shifts at very low energy[19] and also near 4 Mev where the p-wave difficulty is present.

Note added in Proof.—It has been brought to my attention by Dr. M. H. MacGregor and Dr. D. J. Knecht that there were some systematic errors present in the 4.203 Mev data, and that improved analyses of the low energy experiments will soon be forthcoming. Consequently, the numbers in Table I should be regarded as illustrative of the relative effect of analyzing data with and without vacuum polarization, and not as the best values of the effective range parameters. If the 1.397 Mev data[3] are used instead of those at 4.203 Mev (along with 0.3839 Mev and 2.425 Mev), then one obtains

$$P=-0.03\pm0.06,$$

a result which is consistent with a hard core but does not confirm it.

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[19] J. E. Brolley and J. D. Seagrave at this Laboratory are considering relocating the minimum in the 90° cross section with considerably more accuracy than the Cooper, Frisch, and Zimmerman experiment.[14]