$K\pi$ Interaction in the Double Dispersion Representation*

BENJAMIN W. LEE†
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania
(Received April 27, 1960; revised manuscript received June 22, 1960)

The low-energy $K\pi$ interaction is studied on the basis of the double dispersion representation. Exact dispersion relations for partial wave amplitudes are derived for $K\pi$ scattering and for the process $\pi+\pi\rightarrow K+\bar{K}$. These relations are reduced to manageable form and effective-range formulas are derived under the assumption that the $K\pi$ interaction proceeds principally through a contact "potential." For the process $\pi+\pi\rightarrow K+\bar{K}$, the initial-state interaction is taken into account; for the p wave of $K\pi$ scattering, the two-pion exchange mechanism is considered. Other types of solutions are briefly discussed.

I. INTRODUCTION

RECENTLY, Chew and Mandelstam[1] have formulated the theory of the low-energy pion-pion interaction based on the analytic properties of the transition amplitudes as proposed previously by Mandelstam.[2] The role of the pion-pion interaction in the pion-nucleon system has attracted a considerable attention in recent times.[3] It is clear that in elementary phenomena where K mesons participate significantly, the $K\pi$ interaction should play a role similar to that of the pion-pion interaction.[4–6] In particular the low-energy behavior of KN scattering is in a large measure determined by the two-pion exchange between the nucleon and the K meson, and the hyperon (Λ,Σ) exchange. The mechanism suggested by Barshay,[6] in which the two-pion exchange takes place through the boson-boson interaction term in the Hamiltonian of the form

$$H_{K\pi}=4\pi\lambda_{K\pi}(K^{\dagger}\cdot K)\phi_{\pi}^{2}, \tag{1}$$

and the $K-\Lambda$ $(K-\Sigma)$ relative parity relative to the nucleon is assumed to be odd (even), appears to be entirely adequate to explain the small charge exchange in $K^{+}n$ scattering at extremely low energies, and the rise in the charge-exchange to non-charge-exchange ratio at the same time as the p wave in the isotopic spin zero state makes its appearance. A quantitative analysis is being made of the KN scattering based on the double dispersion relation.[7] In this paper we address ourselves to the effective-range analysis of the $K\pi$ interaction. Our analysis will be based on the double dispersion representation. This work is motivated as preliminary to quantitative understanding of the K-meson nucleon interaction.

Dispersion relations for partial wave amplitudes will be deduced from the double dispersion representation. A procedure of calculating the scattering amplitudes by means of the partial wave dispersion equation has been proposed by Chew and Mandelstam.[1] The initial steps of the procedure, the effective-range approximation, applied to the pion-kaon scattering will be carried out in this paper. The dispersion equation for the s-wave amplitude will include an inhomogeneous term reflecting the fact that in the conventional (renormalizable) Lagrangian field theory, an independent coupling constant appears in the $K\pi$ interaction. Complete neglect of the unphysical branch cuts associated with the crossed processes then affords a simple approximation for the low-energy s-wave amplitude. We shall subsequently investigate the process $\pi+\pi\rightarrow K+\bar{K}$. Our main interest in this process is its relevance to the kaon-nucleon scattering. Hence, we shall mainly concentrate on the energy region below the physical threshold. The Pauli principle gives a rather stringent selection rule for this process which simplifies our consideration to some extent.

For the p-wave amplitude, we must take into account the unphysical branch cuts, since the "coupling constant" does not appear explicitly in the dispersion equation for the p wave. We shall make an approximation in which we only consider the singularities associated with the low-energy $\pi+\pi\rightarrow K+\bar{K}$ process. Here the phase shift of the pion-pion scattering in the $T=0$, $J=0$ state plays an important role, and at the moment even a qualitative estimate of the p-wave amplitude seems to have little meaning.

The solutions presented here will be an analog of the s-wave dominant solutions of Chew *et al.*[1,8] Other types of solutions[9,10] will be discussed and physical implications in the present approximate reduction of the dispersion relations are clarified in the last section.

* Supported in part by the U. S. Atomic Energy Commission.
† Harrison Special Fellow.

1 G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960), hereafter referred to as CM.
2 S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
3 G. Takeda, Phys. Rev. **100**, 440 (1955); L. S. Rodberg, Phys. Rev. Letters **3**, 58 (1959); P. Federbush, M. L. Goldburger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958). W. R. Frazer and J. Fulco, Phys. Rev. Letters **2**, 365 (1959); Phys. Rev. **117**, 1609 (1960).
4 J. Schwinger, Phys. Rev. **104**, 1164 (1956).
5 A. Pais, Phys. Rev. **112**, 624 (1958).
6 S. Barshay, Phys. Rev. **109**, 2160 (1958); Phys. Rev. **110**, 743 (1958); Phys. Rev. Letters **1**, 97 (1958). See also F. Gürsey, Phys. Rev. Letters **1**, 98 (1958).
7 B. W. Lee (to be published).
8 G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev. **119**, 478 (1960).
9 G. F. Chew and S. Mandelstam, University of California Radiation Laboratory Report, UCRL-9126, 1960 (unpublished).
10 B. W. Lee and M. T. Vaughn, Phys. Rev. Letters **4**, 578 (1960).

II. KINEMATICS

We commence with the kinematical characterization of the pion-kaon scattering and the crossed processes. Let (p_1,α) and $(-p_3,\beta)$ be the pion momenta and isotopic spin indices of the incoming and outgoing pions, respectively, and p_2, $-p_4$ be the incoming and outgoing kaon momenta. The convenient invariant variables for the double dispersion representation are the squares of the total center-of-mass energies for the three reactions

$$\begin{aligned}
&\text{I.} && \pi(p_1,\alpha)+K(p_2)\rightarrow\pi(-p_3,\beta)+K(-p_4),\\
&\text{II.} && \pi(p_3,\beta)+K(p_2)\rightarrow\pi(-p_1,\alpha)+K(-p_4),\qquad(2)\\
&\text{III.} && \pi(p_1,\alpha)+\pi(p_3,\beta)\rightarrow\bar{K}(-p_2)+K(-p_4).
\end{aligned}$$

We define

$$\begin{aligned}
s&=(p_1+p_2)^2=(p_3+p_4)^2,\\
u&=(p_3+p_2)^2=(p_1+p_4)^2,\qquad(3)\\
t&=(p_1+p_3)^2=(p_2+p_4)^2,
\end{aligned}$$

where s, u, and t are not all independent, but

$$s+u+t=2M^2+2\mu^2,$$

M, μ being the masses of the kaon and the pion, respectively.

In reaction I, s and t are related to k, the magnitude of the three-momentum, and θ, the scattering angle in the barycentric system, by[11]

$$s=M^2+\mu^2+2k^2+2[(k^2+M^2)(k^2+\mu^2)]^{\frac{1}{2}},\qquad(4,\text{I})$$

$$t=-2k^2(1-\cos\theta).\qquad(5,\text{I})$$

In reaction III, s and t are given by

$$s=-p^2+2pq\cos\varphi-q^2,\qquad(4,\text{III})$$

$$t=4(p^2+M^2)=4(q^2+\mu^2),\qquad(5,\text{III})$$

where p and q are the magnitudes of the three-momenta of the kaon and the pion and φ is the scattering angle in the barycentric system.

In the isotopic spin space of the kaon, the transition amplitude has the form[12]

$$A_{\beta\alpha}=A^{(+)}\delta_{\beta\alpha}+A^{(-)}\tfrac{1}{2}[\tau_\beta,\tau_\alpha].\qquad(6)$$

In reaction (I), there are two independent isotopic spin states, $T=\frac{1}{2}$ and $T=\frac{3}{2}$. The relations between the eigenamplitudes of the total isotopic spin $A^{(\frac{1}{2})}$, $A^{(\frac{3}{2})}$, and $A^{(\pm)}$ are

$$\begin{aligned}
A^{(\frac{1}{2})}&=A^{(+)}+2A^{(-)},\\
A^{(\frac{3}{2})}&=A^{(+)}-A^{(-)}.\qquad(7,\text{I})
\end{aligned}$$

For reaction (II)

$$\begin{aligned}
\bar{A}^{(\frac{1}{2})}&=A^{(+)}-2A^{(-)},\\
\bar{A}^{(\frac{3}{2})}&=A^{(+)}+A^{(-)}.\qquad(7,\text{II})
\end{aligned}$$

For reaction (III)[13]

$$\begin{aligned}
B^{(0)}&=6A^{(+)},\\
B^{(1)}&=2A^{(-)},\qquad(7,\text{III})
\end{aligned}$$

where $B^{(0)}$, $B^{(1)}$ are the eigenamplitudes of total isotopic spin $T=0$, and 1, respectively.

We define the partial wave amplitudes for reaction I by

$$\begin{aligned}
A^{(I)}(s)&=\tfrac{1}{2}\int_{-1}^{1}d\cos\theta\,P_l(\cos\theta)A^{(I)}[s,t(\cos\theta)]\\
&=\frac{\sqrt{s}}{k}\exp(i\delta_l^{(I)})\sin\delta_l^{(I)},\qquad(8,\text{I})
\end{aligned}$$

where I can be either $\frac{1}{2}$ or $\frac{3}{2}$. The last equality in fact defines the normalization of $A^{(I)}$, while the kinematical factor $s^{\frac{1}{2}}/k$ follows from the Lorentz invariance of the S matrix. For reaction III, we define

$$B_l^{(\pm)}(t)=\tfrac{1}{2}\frac{1}{(pq)^l}\int_{-1}^{1}d\cos\varphi\,P_l(\cos\varphi)A^{(\pm)}[s(\cos\varphi),t],\qquad(8,\text{III})$$

where $B_l^{(\pm)}(t)$ are related to $B_l^{(0,1)}(t)$ through Eq. (7,III).

III. ANALYTICITY OF THE AMPLITUDES

According to the Mandelstam representation, the amplitudes $A^{(\pm)}$ are analytic functions of s and t except for singularities associated with the possible intermediate states of the reactions (I) to (III). Therefore, we have branch lines for $s\geqslant(M+\mu)^2$, $t\geqslant(2\mu)^2$, and $u\geqslant(M+\mu)^2$. To give a valid meaning to the Mandelstam representation, some subtractions will be necessary. It is unfortunate, but true, that we must infer the nature of subtractions needed from perturbation theory.[14]

If one takes the view that the primary $K\pi$ interaction should be charge independent, and parity conserving, then the only choice is the Hamiltonian of the form of Eq. (1). From the viewpoint of the conventional renormalizable Lagrangian theory, the $(\bar{K}\cdot K)\pi^2$ interaction term is required as an infinite counterterm to cancel the infinities associated with the baryon loops in the $K\pi$ scattering. In the past, proposals have been made for the Yukawa coupling of two K mesons and a pion.[4,5] There seems, however, no compelling reason, within the present experimental knowledge, to assume the violation of charge independence, or the parity doubling of the K multiplet, or the odd relative parity between K^+ and K^0.

The postulation of the Hamiltonian, Eq. (1), then leads to the conclusion that one needs to add an over-all subtraction constant and single dispersion integrals in

[11] R. Oehme, Phys. Rev. Letters 4, 249 (1960); 4, 320 (E) (1960), hereafter referred to as Oe.

[12] G. F. Chew, M. L. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[13] W. R. Fraser and J. Fulco, Phys. Rev. **117**, 1603 (1960), hereafter referred to as FF.

[14] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

s, t, and u in the double dispersion representation[15] for $A^{(+)}$, while one need not for $A^{(-)}$.

The analytic properties of the partial wave amplitudes, Eq. (8,I), have been investigated by McDowell,[16] by Oehme,[11] and by Frazer and Fulco,[17] for meson-nucleon scattering. The partial wave amplitude $A_l^{(I)}$ is analytic in the complex s plane except for (a) the physical branch cut extending from $s=(M+\mu)^2$ to $+\infty$, (b) the branch cut along the circle $|s|=M^2-\mu^2$ associated with the intermediate states of reaction (III), and (c) the branch cut $s=-\infty$ to $(M-\mu)^2$ associated with the intermediate states of reactions (II) and (III).

The analytic properties of the amplitudes $B_l(t)$ have been discussed by Fraser and Fulco[13] for the process $\pi+\pi\to\bar{N}+N$. The quantity $B_l(t)$ is analytic in t except for the physical branch cut from $t=(2\mu)^2$ to $+\infty$, and the unphysical branch cut from $-\infty$ to 0 along the real axis.

As we mentioned earlier, we need a subtraction for the amplitude $A^{(+)}(s,t)$. The subtraction point is in principle arbitrary, but for convenience we impose the following conditions to be fulfilled by the subtraction point: First, it must be symmetric in s and u; secondly, it should not overlap with the singularities of the partial wave amplitudes. A convenient point, satisfying the above criteria, is

$$s_0=u_0=M^2,\quad t_0=2\mu^2,$$
$$s_0+u_0+t_0=2M^2+2\mu^2. \tag{9}$$

We shall henceforth adhere to this convention.[18]

IV. EFFECTIVE-RANGE APPROXIMATION. ADIABATIC APPROACH

We can now derive a set of integral equations for the low-energy amplitudes, knowing their analytic properties as discussed in the last section. A procedure for solving the integral equations by iteration has been well set forth by Chew and Mandelstam. A "complete" solution, however, must depend on the gigantic brain of the modern electronic computer. Nothing so elaborate will be attempted here, but the initial stages of the iterative procedure, which yield what are commonly known as the effective-range approximations, will be carried out in this paper, with an estimate of the coupling constant.

A. *S* Wave in the $K\pi$ Scattering

Our point of view that an over-all subtraction constant is necessary to validate the double dispersion representation for $A^{(+)}$ indicates that one subtraction is necessary for the dispersion equation for $A_0^{(I)}$. Applying Cauchy's theorem to $A_0^{(I)}$ in the usual way (see Fig. 1),

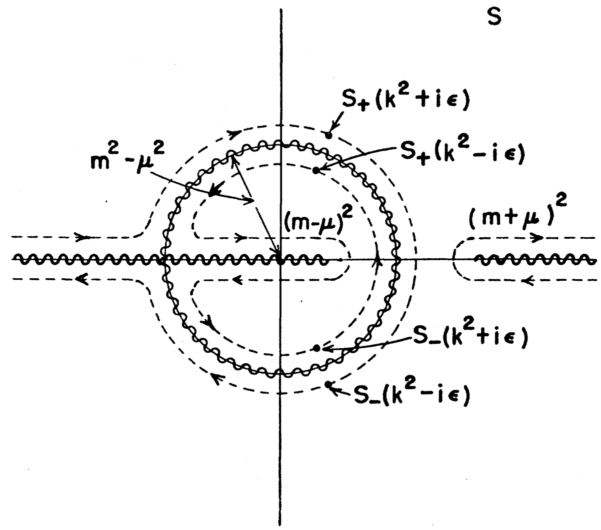

FIG. 1. The singularities of $A_l^{(\pm)}(s)$ and the contours of integration. The branch cuts are indicated by wavy lines: the dotted lines are the contours of integration for the partial wave dispersion relation.

$$A_0^{(I)}(s)=\alpha^{(I)}+\frac{s-M^2}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\mathrm{Im}A_0^{(I)}(s')}{(s'-s)(s'-M^2)}$$
$$+\frac{s-M^2}{\pi}\int_{-\infty}^{(M-\mu)^2}ds'\frac{\mathrm{Im}A_0^{(I)}(s')}{(s'-s)(s'-M^2)}$$
$$+F_0^{(I)}(s)+F_0^{(I)*}(s^*), \tag{10}$$

where $I=\frac{1}{2}$ or $\frac{3}{2}$, $\alpha^{(I)}=A_0^{(I)}(M^2)$; the subtraction is made in accordance with the renormalization convention discussed previously, and

$$F_0^{(I)}(s)=-\frac{s-M^2}{\pi}\int_{-M^2}^{-\mu^2}d\lambda\frac{2s_+(\lambda)}{s_+(\lambda)-M^2-\mu^2-2\lambda}$$
$$\times\frac{1}{[s_+(\lambda)-s][s_+(\lambda)-M^2]} \tag{11}$$
$$\times[M_0^{(+)}(s_+)+\xi^{(I)}M_0^{(-)}(s_+)];$$
$$s_+(\lambda)=2\lambda+M^2+\mu^2+2i[(M^2-\lambda)(-\lambda-\mu^2)]^{\frac{1}{2}},$$

with $\xi^{(I)}=2$, -1 for $I=\frac{1}{2}$, $\frac{3}{2}$, respectively, and $M_l^{(\pm)}$ defined as in Oe [Eq. (9)]. The imaginary part of $A_l^{(\pm)}$ is given by Eqs. (8) and (9) of Oe.

It is *a priori* expected, and can be verified *a posteriori* as in CM for the "adiabatic" solution, that

$$\alpha^{(\frac{3}{2})}\simeq\alpha^{(\frac{1}{2})}\simeq-\lambda_{K\pi}=A^{(+)}(s_0,u_0). \tag{12}$$

[Note that $A^{(-)}(s_0,u_0)=0$, since $A^{(-)}(s,u)=-A^{(-)}(u,s)$;

[15] We have in mind Eq. (2.13) of reference 14.

[16] S. W. McDowell, Phys. Rev. **116**, 774 (1960).

[17] W. R. Frazer and J. Fulco, Institute for Advanced Study, 1960 (to be published).

[18] It may be remarked that if one writes down the double dispersion relations for $A^{(I)}$; $I=\frac{1}{2}$, $\frac{3}{2}$, then one need make one subtraction for both $A^{(\frac{1}{2})}$ and $A^{(\frac{3}{2})}$. One finds, since $A^{(-)}(s_0,u_0)=0$ for $s_0=u_0$ because of the antisymmetry of $A^{(-)}$, $A^{(\frac{3}{2})}(s_0,u_0)=A^{(\frac{1}{2})}(s_0,u_0)$. The author is indebted to Dr. W. Frazer for this remark.

$u_0=s_0$.[18]] The effective-range approach in this case consists of neglecting the unphysical cut completely, assuming that the effects of the unphysical branch cuts are approximately represented by the subtraction constants $\alpha^{(I)}$. The resulting dispersion equation can be solved easily, for example by the N/D technique of CM, with the unitarity condition:

$$\operatorname{Im}\frac{1}{A_0^{(I)}(s)}=-\frac{k}{\sqrt{s}}=-\frac{[(s-M^2-\mu^2)^2-4M^2\mu^2]^{\frac{1}{2}}}{2s},$$

along the physical branch cut. In this approximation, $A_0^{(\frac{3}{2})}=A_0^{(\frac{1}{2})}$ and we may suppress the isotopic spin superscripts. There results

$$\frac{k}{\sqrt{s}}\cot\delta_0=\frac{-1}{l_0}+\frac{1}{2\pi}\left\{\frac{(\omega^2-4M^2\mu^2)^{\frac{1}{2}}}{\omega+M^2+\mu^2}\ln\frac{\omega+(\omega^2-4M^2\mu^2)^{\frac{1}{2}}}{2M\mu}-\frac{M-\mu}{M+\mu}\left(\frac{\omega-2M\mu}{\omega+M^2+\mu^2}\right)\ln\frac{M}{\mu}\right\},\quad(13)$$

with $\omega=s-M^2-\mu^2$, and

$$l_0^{-1}=\frac{1}{2\pi}\left\{\left[\frac{M^2-\mu^2}{M^2}-\frac{M^2-\mu^2}{M^2+\mu^2}\right]\ln\frac{M}{\mu}+\frac{(4M^2\mu^2-\mu^4)^{\frac{1}{2}}}{M^2}\tan^{-1}\frac{(4M^2\mu^2-\mu^4)^{\frac{1}{2}}}{\mu^2}\right\}-\frac{1}{\alpha}\simeq 0.19-\frac{1}{\alpha}.\quad(14)$$

The requirement that there be no "ghost" pole along the negative real axis within $|s|\leq L$ in our solution, with $L\simeq 5(M+\mu)^2$, say, puts a lower bound (for detailed discussions, see CM), $-1.02\lesssim\alpha<0$. For α positive the physical requirement that there be no bound state of the $K\pi$ system places the upper limit

$$\alpha<-\pi\left\{\left[\frac{M^2-\mu^2}{2(M+\mu)^2}-\frac{M^2-\mu^2}{2M^2}\right]\ln\frac{M}{\mu}-\frac{(4M^2\mu^2-\mu^4)^{\frac{1}{2}}}{2M^2}\tan^{-1}\frac{(4M^2\mu^2-\mu^4)^{\frac{1}{2}}}{\mu^2}\right\}^{-1}\simeq 5.08.$$

Therefore, the above considerations, together with Eq. (20), gives crude limits on the coupling constant:

$$-5.08<\lambda_{K\pi}\lesssim 1.02,$$
$$-1.02\lesssim\alpha<5.08.\quad(15)$$

The value $\lambda_{K\pi}\simeq(4\pi)^{-\frac{1}{2}}$ used by Barshay[6] is within these limits.

Equation (13) has been derived by Okubo[19] from perturbation theory. We note that the condition, Eq. (15), implies that there is no resonance in the region $(M+\mu)^2\leq s\lesssim L$.

B. Process $\pi+\pi\rightarrow K+\bar{K}$

The Pauli principle admits only the symmetric states for this process. For $T=0$ (1), the two pions in the initial channel must be in an even (odd) angular momentum state, and the G parity of the kaon system in the final channel must be even. Therefore, for $T=0$, the s state is the most important one, while for $T=1$, the main contributions to the amplitude $A^{(-)}$ come from the p state.

We write down the dispersion relations for the s wave with one subtraction and for the p wave with no subtraction:

$$B_0^{(+)}(t)=\beta+\frac{(t-2\mu^2)}{\pi}\int_{(2\mu)^2}^{\infty}dt'\frac{\operatorname{Im}B_0^{(+)}(t')}{(t'-t)(t'-2\mu^2)}+\frac{(t-2\mu^2)}{\pi}\int_{-\infty}^{0}dt'\frac{\operatorname{Im}B_0^{(+)}(t')}{(t'-t)(t'-2\mu^2)},\quad(16)$$

$$B_1^{(-)}(t)=\frac{1}{\pi}\int_{(2\mu)^2}^{\infty}dt'\frac{\operatorname{Im}B_1^{(-)}(t')}{t'-t}+\frac{1}{\pi}\int_{-\infty}^{0}dt'\frac{\operatorname{Im}B_1^{(-)}(t')}{t'-t},\quad(17)$$

where $\beta=B_0^{(+)}(2\mu^2)$, and $\operatorname{Im}B_l^{(\pm)}(t)$ for $t<0$ is given by

$$\operatorname{Im}B_l^{(\pm)}(t)=\int_{(M+\mu)^2}^{M^2+\mu^2+2p_-q_--t/2}\frac{ds'}{2(p_-q_-)^{l+1}}\times P_l\left(\frac{2s'-2M^2-2\mu^2+t}{4p_-q_-}\right)\times\tfrac{1}{3}\left[N^{(\frac{1}{2})}(s',t)+\binom{2}{-1}N^{(\frac{3}{2})}(s',t)\right],\quad(18)$$

where $p_-=(M^2-t/4)^{\frac{1}{2}}$, $q_-=(\mu^2-t/4)^{\frac{1}{2}}$, and $N^{(I)}$ are the absorptive parts of the amplitudes $A^{(I)}$.

From the unitarity, it follows[20] that the phases of the amplitudes $B_l^{(\pm)}(t)$ for $(2\mu)^2\leq t\leq(4\mu)^2$ are given by the phase shifts of the pion-pion scattering in the corresponding states. Equations (16), (17) can be solved as in FF once $N^{(I)}$ are known. Since $N^{(\frac{1}{2})}=N^{(\frac{3}{2})}$ in the present approximation as will be shown *a posteriori*, $B_1^{(-)}(t)=0$ for $(2\mu)^2\leq t\leq 16\mu^2$. A crude estimate of $B_0^{(+)}(t)$ may be obtained by neglecting the left-hand branch cut, assuming the subtraction constant $\beta\approx\alpha$

[19] S. Okubo, Phys. Rev. **118**, 357 (1960).

[20] S. Mandelstam, Phys. Rev. Letters **4**, 84 (1960).

represents approximately its effects:

$$B_0^{(+)}(t)\simeq\alpha\exp\left\{\frac{t-2\mu^2}{\pi}\int_{(2\mu)^2}^{\infty}\frac{dt'\delta(t')}{(t'-t)(t'-2\mu^2)}\right\},\quad\text{for}\quad 4\mu^2\lesssim t\lesssim 16^2,\quad(19)$$

where $\delta(t)$ is the phase shift of pion-pion scattering in the $T=0$, $J=0$ state, and in actual evaluation, the scattering length approximation:

$$\left(\frac{t-4\mu^2}{t}\right)^{\frac{1}{2}}\cot\delta(t)=\frac{1}{\eta_0}$$

may be used. To expedite the integration in Eq. (19), we further make the nonrelativistic approximation:

$$\delta(t)=\tan^{-1}[\eta_0(t-4\mu^2)^{\frac{1}{2}}/2\mu],$$

and obtain

$$B_0^{(+)}(t)\simeq\alpha\frac{2\mu/\eta_0-\sqrt{2}\mu}{(2\mu/\eta_0)-(4\mu^2-t)^{\frac{1}{2}}},\quad\text{for}\quad 4\mu^2\leq t\ll 16\mu^2.\quad(20)$$

C. *P* Wave in the $K\pi$ Scattering

In writing down the dispersion relations for higher angular momentum amplitudes, we take advantage of the fact that, as $s\to(M+\mu)^2$, $A_l(s)$ approaches zero as $[s-(M+\mu)^2]^l$, to suppress very high energies under the dispersion integrals. We write

$$A_l^{(I)}(s)=\frac{[s-(M+\mu)^2]^l}{\pi}\times\int_{(M+\mu)^2}^{\infty}ds'\frac{\mathrm{Im}A_l^{(I)}(s')}{[s'-(M+\mu)^2]^l(s'-s)}+\frac{[s-(M+\mu)^2]^l}{\pi}\times\int_{-\infty}^{(M-\mu)^2}ds'\frac{\mathrm{Im}A_l^{(I)}(s')}{[s'-(M+\mu)^2]^l(s'-s)}+F_l^{(I)}(s)+F_l^{(I)*}(s^*),\quad(21)$$

where

$$F_l^{(I)}=-\frac{[s-(M+\mu)^2]^l}{\pi}\times\int_{-M^2}^{-\mu^2}d\lambda\frac{2s_+(\lambda)\{M_l^{(+)}[s_+(\lambda)]+\xi^{(I)}M_l^{(-)}[s_+(\lambda)]\}}{[s_+(\lambda)-M^2-\mu^2-2\lambda^2][s_+(\lambda)-(M+\mu)^2]^l}.\quad(22)$$

Again in the spirit of effective-range approach, we shall only consider the nearest singularities: the branch cut along the circle $|s|=M^2-\mu^2$. For the p wave we write $[s-(M+\mu)^2]^{-1}A_1^{(I)}(s)=n_1^{(I)}(s)/d_1^{(I)}(s)$, where

$$d_1^{(I)}(s)=1-\frac{[s-(M+\mu)^2]}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{ds'\ k'n_1^{(I)}(s')}{2s'(s'-s)},\quad(23)$$

and

$$n_1^{(I)}(s)=f_1^{(I)}(s)+f_1^{(I)*}(s^*);$$

$$f_1^{(I)}(s)=\frac{-1}{\pi}\int_{-M^2}^{-\mu^2}d\lambda\frac{2s_+(\lambda)}{[s_+(\lambda)-M^2-\mu^2-2\lambda]}\times\frac{d_1(s_+)}{[s_+(\lambda)-s]}\frac{M_1^{(+)}+\xi^{(I)}M_1^{(-)}}{[s_+(\lambda)-(M+\mu)^2]}.\quad(24)$$

Equations (23), (24) can be solved in principle by the method outlined in CM. We shall be content here with the scattering length approximation:

$$\frac{k^3}{\sqrt{s}}\cot\delta_1^{(I)}=\lim_{s\to(M+\mu)^2}k^2[A_1^{(I)}(s)]^{-1}=\frac{M\mu}{(M+\mu)^2}\{n_1^{(I)}[(M+\mu)^2]\}^{-1}.\quad(25)$$

In the present approximation, $M_1^{(-)}=0$, and there will be no isotopic splitting of the phase shift. $M_1^{(+)}$ can be calculated from Eq. (20) by Eqs. (8) and (9) of Oe. It is

$$M_1^{(+)}[s_+(\lambda)]=\alpha\left(\sqrt{2}\mu-\frac{2\mu}{\eta_0}\right)\frac{1}{(4\lambda)^2}\left\{\tfrac{2}{3}(-4\lambda-4\mu^2)^{\frac{3}{2}}-\left(-2\lambda-4\mu^2+\frac{4\mu^2}{\eta_0}\right)\left[2(-4\lambda-4\mu^2)^{\frac{1}{2}}-\frac{4\mu}{\eta_0}\tan^{-1}\left(\frac{(-4\lambda-4\mu^2)}{(2\mu/|\eta_0|)}\right)\right]\right\}.\quad(26)$$

The facts that we have used the nonrelativistic scattering-length approximation for the phase shift of the pion-pion scattering and, as pointed out by Frazer and Fulco,[17] that the partial wave decomposition of $M^{(+)}[s_+(\lambda)]$ is valid only in a certain region (in our case, $\lambda\gtrsim-6\mu^2$) do not affect our estimate significantly, since it turns out that the main contribution to the integral in Eq. (24) comes from $\lambda\approx-\mu^2$.

Approximating $d_1[s_+(\lambda)]$ by 1, we obtain

$$\frac{k^3}{\sqrt{s}}\cot\delta_1=\frac{M}{(M+\mu)^2}\left\{-\frac{2}{\pi}\mathrm{Re}\int_{\mu^2}^{m^2}d\nu\frac{2s_+(-\nu)}{[s_+(-\nu)-M^2-\mu^2+2\nu]}\times\frac{M_1^{(+)}[s_+(-\nu)]}{[s_+(-\nu)-(M+\mu)^2]^2}\right\}=\frac{\pi}{2}\frac{M\mu}{\alpha}g(\eta_0).\quad(27)$$

The quantity $g(\eta_0)$ is a rather sensitive function of η_0. An estimate shows that $g(\eta_0)=0.28$, 5.16 for $\eta_0=+1$ and -1, respectively.

V. CONCLUDING REMARKS

Implicit in our treatment is the assumption that $\alpha^{(I)}$ in the s-wave dispersion relation, Eq. (10), is large compared to the contributions from the left-hand branch cuts for not too high energies. This corresponds, physically, to the assumption that the $K\pi$ interaction can be approximately described by a contact interaction between two bosons. Such an interaction produces an s-wave dominant scattering. In the effective-range formula for the process $\pi+\pi\rightarrow K+\bar{K}$, Eq. (20), the initial-state interaction was taken into consideration. The p-wave scattering parameter was obtained by considering two-pion exchange between the pion and the K meson. On the other hand, if we assumed $\alpha^{(I)}$ to be small, the contributions from the unphysical branch lines should have been taken into account seriously. If, moreover, it turns out that $\mathrm{Im}A_l^{(\pm)}(s)$ for $s\leqslant(M-\mu)^2$ is large, then the possibility of the p-wave resonance appears in the kaon-pion scattering also.[1] It is unlikely, however, in view of the corresponding situation in the pion-pion problem,[8] that the "adiabatic" solution whose starting point we outlined here will actually develop a resonance in the p wave, even if the left-hand branch cuts are considered seriously.

The adiabatic solution we described here corresponds to the Hamiltonian of Eq. (1) in the conventional Lagrangian theory.[19] The equivalence of the phase shifts for the $T=\frac{1}{2}$ or $T=\frac{3}{2}$ states is true independent of the approximation as long as the Hamiltonian of Eq. (1) alone is considered. The adiabatic solution starting with $\alpha^{(\frac{3}{2})}=\alpha^{(\frac{1}{2})}$ will preserve this equivalence, and, what is equivalent, $B_l^{(-)}=0$, even if the left-hand branch cuts are taken into account. It must be emphasized that $A^{(-)}(s,t)=0$ is a *formal* solution to the present problem, satisfying the crossing relations on the unphysical branch cuts, although physically, mechanisms other than that of Eq. (1), such as the baryon-antibaryon intermediate states, will likely destroy the isotopic-spin degeneracy of the $K\pi$ system.

If, in the future, experimental developments confirm the isotopic-spin splitting of the phase shifts, we may introduce two phenomenological, independent parameters $\alpha^{(\frac{1}{2})}$ and $\alpha^{(\frac{3}{2})}$, both of which satisfy Eq. (15). Or, alternatively, self-consistent solutions sustained by a bootstrap mechanism[21] may be looked for.

An interesting possibility is to assume the coupling of an unstable vector boson[10] of isotopic spin one to the conserved isotopic vector current of the strongly interacting particles, as has been suggested by Sakurai.[22] In this case, the amplitude $B_1^{(-)}(t)$ may be computed by the resonance approximation[10] and inserted into the left-hand branch cuts to generate $A_l^{(I)}(s)$. Such a solution, with the resonance parameters to fit the electromagnetic structure of the nucleon, will give rise to a large isotopic splitting in the low-energy KN scattering.

ACKNOWLEDGMENT

The author is indebted to Professor A. Klein for encouragement throughout this work, and to Dr. W. R. Frazer for some comments.

[21] G. F. Chew and S. Mandelstam, University of California Radiation Laboratory Report, UCRL-9126, 1960 (unpublished).
[22] J. J. Sakurai, Ann. Phys. (to be published).