# Gravitational-Electromagnetic Coupling and the Classical Self-Energy Problem*

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The gravitational effect on the classical Coulomb self-energy of a point charge is calculated rigorously. It is shown that the total mass then becomes finite (although still quite large), and that it depends only on the charge and not on the bare mechanical mass. Thus, a particle acquires mass only when it has non-gravitational interactions with fields of nonzero range. In order to treat this problem, it is necessary to extend the canonical formalism, previously obtained for the free gravitational field, to include coupling with the Maxwell field and the point charge system. It is shown that the canonical variables of the gravitational field are unaltered while those of the matter system are natural generalizations of their flat space forms. The determination of the total energy of a state can still be made from knowledge of the spatial metric at a given time. The self-mass of a particle is then the total energy of a pure one-particle state, i.e., a state containing no excitations of the canonical variables of the Maxwell or Einstein fields. Solutions corresponding to pure particle states of two like charges are also obtained, and their energy is shown consistent with the one-particle results.

## 1. INTRODUCTION

THAT gravitational coupling may be of relevance in the self-energy problem has recently been suggested by a number of authors.[1] We shall here investigate the effect of gravitation on classical self-energies of static point charges.

The aim of classical point electron theory, since its inception, has been to obtain a finite, model-independent electromagnetic self-energy, and if possible, to dispense with mechanical mass altogether. Thus the total mass of the particle would arise from its coupling to the field. Such a program, however, was not feasible, since the self-energy diverged linearly, with no realistic compensation possible. In terms of renormalization theory, this implied an infinite "bare" mechanical mass. Since gravitational interaction energy is negative on the Newtonian level, this field may be expected to provide compensation. Indeed, a simple argument yields a limit on the self-energy due to just such a compensation. Consider a *bare* mass $m_0$ distributed in a sphere of radius $\epsilon$. In the Newtonian limit, the total energy (i.e., the clothed mass) $m$ is given by $m = m_0 - \frac{1}{2}\gamma m_0^2/\epsilon$. For sufficiently small $\epsilon$, $m$ becomes zero and then negative. In general relativity, the principle of equivalence states that it is the total energy that interacts gravi-

tationally and not just the bare mass. Thus, as the interaction energy grows more negative, were a point reached where the total energy vanished, there could be no further interaction energy. Consequently, there can be no negative total energy, in contrast to the negative infinite self-energy of Newtonian theory. General relativity effectively replaces $m_0$ by $m$ in the interaction term: $m = m_0 - \frac{1}{2}\gamma m^2/\epsilon$. Solving for $m$ yields $m = \gamma^{-1}[-\epsilon + (\epsilon^2 + 2\gamma m_0\epsilon)^{\frac{1}{2}}]$. This relation is the rigorous one obtained[2] in IV for a neutral point particle and shows that $m \to 0$ as $\epsilon \to 0$ in this case.

More interesting is the fact that the gravitational interactions produce a natural cutoff for the Coulomb self-energy of a point charge. Here the self-mass resides in the Coulomb field, $\frac{1}{2}\int (e/4\pi r^2)^2 d^3r$. By the general argument above, on gravitational compensation, one expects that the Coulomb energy near the origin (which is, in fact, "denser" than the neutral particle's $\delta$-function distribution) will have a very strong gravitational self-interaction, resulting in a vanishing *total* contribution to the self-mass. Thus, the integral effectively extends down only to some radius $a$, yielding $m_{EM} = \frac{1}{2}\int_a^\infty (e/4\pi r^2)^2 d^3r = (e^2/4\pi)/2a$. We can determine this effective flat-space cutoff $a$ by the same equivalence principle argument. Without the gravitational contribution, the mass is $m_0 + \frac{1}{2}(e^2/4\pi\epsilon)$, so the total clothed mass is determined by the equation

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[1] L. D. Landau, in *Niels Bohr and the Development of Modern Physics* (Pergamon Press, London, 1955), p. 60; W. Pauli, Suppl. Helv. Phys. Acta 4, 69 (1956); O. Klein, Suppl. Helv. Phys. Acta 4, 61 (1956); S. Deser, Revs. Modern Phys. 29, 417 (1957).

[2] Other papers in this series are referred to as I, II, etc. They are: (I) R. Arnowitt and S. Deser, Phys. Rev. 113, 745 (1959); (II) R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 116, 1322 (1959); (III) 117, 1595 (1960); (IIIa) J. Math. Phys. 1, 434 (Sept.-Oct. 1960); (IV) 118, 1100 (1960); (IVa) *Heisenberg Representation in Classical General Relativity* (to be submitted to Nuovo cimento); (Va) following paper [Phys. Rev. 120, 321 (1960)]. The present paper is V.

$m=m_0+\frac{1}{2}e^2/4\pi\epsilon-\frac{1}{2}\gamma m^2/\epsilon$. This yields

$$m=\gamma^{-1}\{-\epsilon+[\epsilon^2+2m_0\epsilon\gamma+(e^2/4\pi)\gamma]^{\frac{1}{2}}\}. \quad (1.1)$$

We will see in Sec. 3 that this formula is a rigorous consequence of the field equations. In the limit $\epsilon\to0$ we have $m=(e^2/4\pi)^{\frac{1}{2}}\gamma^{-\frac{1}{2}}$. The bare mechanical mass $m_0$ again does not contribute to the clothed mass. Our result is then equivalent to a cutoff $a=\frac{1}{2}(e^2/4\pi)^{\frac{1}{2}}\gamma^{\frac{1}{2}}$ on the flat space Coulomb integral. The solution of the field equations for the case of two like charges will also be given in Sec. 3 and seen to be consistent with the conclusions of the one-particle case.

In order to derive the above, it is necessary first to have a canonical formalism for the coupled fields in question. This is required to ascertain that the various one- and two-particle solutions are truly pure particle states and involve no independent gravitational or electromagnetic excitations (waves). Only then is the total energy of the system the particle's mass. In previous work (III, IV), the general theory of relativity has been put into Hamiltonian form in terms of two independent canonical degrees of freedom. In Sec. 2, we begin with the extension of these results to include the situation in which the electromagnetic field and point charges are coupled to the gravitational field. We shall see that this generalization leaves unaltered many of the formal and physical characteristics of the system. Thus, the gravitational canonical variables and the coordinate conditions (choice of intrinsic coordinates) remain unchanged. The canonical variables for the Maxwell and matter systems are found to be simple generalizations of the familiar flat space ones. Also, the expression for total energy is formally identical to that given in III and IV for the free field and expressible entirely in terms of the metric at spatial infinity.

The renormalization treated in Sec. 3 (which relates the clothed mass to the bare parameters) did not involve any divergent manipulations, unlike the usual procedures of Lorentz covariant field theory. In Sec. 4, we shall apply the renormalization techniques used in divergent problems to the finite results obtained in Sec. 3 to see if a consistent reinstatement of mechanical mass can thereby be made. The gravitational coupling constant must then also be renormalized (with infinite renormalization constant) in order to obtain the correct Newtonian limit at large interparticle separation. However, the higher order terms still diverge, requiring an infinite number of counter terms, as in a non-renormalizable field theory. Thus, such a renormalization procedure is actually inconsistent and does not provide an acceptable alternative.

## 2. CANONICAL REDUCTION OF THE COUPLED SYSTEMS

In this section we analyze the coupled gravitational-electromagnetic point charge system in order to obtain the canonical form, and hence the independent degrees of freedom that characterize the different parts of the system. As in III and IV, we will make use of the first-order form of the Lagrangian throughout. In order to illustrate the methods involved, we first carry out the reduction for the Maxwell-point charge theory in flat space. The extension to the full theory then follows in a straightforward fashion.

The Lagrangian density in flat space may be taken to be[3]

$$\mathcal{L}_M=\mathcal{L}_{EM}+\mathcal{L}_P+\mathcal{L}_I=-A_\mu\mathfrak{F}^{\mu\nu}{}_{,\nu}$$

$$+\frac{1}{4}\mathfrak{F}^{\mu\nu}\mathfrak{F}^{\alpha\beta}\eta_{\mu\alpha}\eta_{\nu\beta}+\int ds\{\pi_\mu dx^\mu/ds$$

$$-\frac{1}{2}\lambda'(s)[\pi_\mu\pi_\nu\eta^{\mu\nu}+m_0^2]\}\delta^4(x-x(s))$$

$$+e\int ds(dx^\mu/ds)A_\mu(x)\delta^4(x-x(s)). \quad (2.1)$$

Here $A_\mu$ and $\mathfrak{F}^{\mu\nu}$ are the vector potential and the contravariant electromagnetic field strength density respectively, while $\eta_{\mu\nu}$ is the Lorentz metric $(-1, 1, 1, 1)$. The variable $x^\mu(s)$ is the trajectory of the particle in terms of an arbitrary parameter[4] $s$; $\pi^\mu(s)$ is essentially the four velocity and $\lambda'(s)$ is a Lagrange multiplier whose variation gives the energy law for the particle, $\pi_\mu\pi^\mu+m_0^2=0$. In the action, $A_\mu$ and $\mathfrak{F}^{\mu\nu}$ are to be varied independently, as are $\pi_\mu(s)$ and $x^\mu(s)$. Variation of $A_\mu$ and $\mathfrak{F}^{\mu\nu}$ gives rise to the usual Maxwell equations in first order form; variation of $\pi_\mu(s)$ and $x^\mu(s)$ similarly yields the particle's first-order equations of motion with the Lorentz force, provided one chooses $\lambda'ds=d\tau/m_0$ where $d\tau$ is the conventional proper time. One may next perform the integration over the parameter $s$ to obtain:

$$\mathcal{L}_P+\mathcal{L}_I=\{\pi_i(dx^i/dt)+\pi_0-\frac{1}{2}\lambda(\pi_\mu\pi_\nu\eta^{\mu\nu}+m_0^2)$$
$$+e(dx^\mu/dt)A_\mu\}\delta^3(\mathbf{r}-\mathbf{r}(t)), \quad (2.2)$$

where $\lambda\equiv\lambda'[ds/dx^0(s)]_{x^0(s)=t}$. The total Lagrangian, written in "3+1 dimensional" form, becomes

$$\mathcal{L}_M=A_{i,0}\mathcal{E}^i-A_{i}\mathfrak{F}^{ij}{}_{,j}$$
$$-A_0[\mathcal{E}^i{}_{,i}-e\delta^3(\mathbf{r}-\mathbf{r}(t))]+[(\pi_i+eA_i)(dx^i/dt)$$
$$+\pi_0-(\lambda/2)(\pi_i\pi^i+m_0^2-\pi_0^2)]\delta^3(\mathbf{r}-\mathbf{r}(t))$$
$$-\frac{1}{2}(\mathcal{E}^i\mathcal{E}_i+\frac{1}{2}\mathfrak{F}^{ij}\mathfrak{F}_{ij}), \quad (2.3)$$

where $\mathcal{E}^i\equiv\mathfrak{F}^{0i}$. In this structure, it is easy to pick out

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[3] Notation and units are as in III: $\kappa=16\pi\gamma c^{-4}=1$, $c=1$, where $\gamma$ is the Newtonian gravitational constant. Latin indices run from 1 to 3, Greek from 0 to 3, and $x^0=t$. All tensors and covariant operations are *three*-dimensional unless specified, $g^{ij}$ being the matrix inverse to $g_{ij}$ and "|" indicating covariant differentiation with respect to $g_{ij}$ (not $^4g_{\mu\nu}$). The sign convention on curvature tensors is determined by

$$R_{ab}=\Gamma_a{}^c{}_{b,c}-\Gamma_a{}^c{}_{c,b}+\Gamma_c{}^c{}_d\Gamma_a{}^d{}_b-\Gamma_a{}^c{}_d\Gamma_c{}^d{}_b.$$

The totally antisymmetric symbols $\epsilon_{ijk}=\epsilon^{ijk}=0$, $\pm1$ are three-tensor densities of weight $-1$ and $+1$, respectively.

[4] Equation (2.1) is covariant against any reparameterization $\bar{s}=\bar{s}(s)$ with $\lambda'$ transforming as a "vector": $\bar{\lambda}'=\lambda'ds/d\bar{s}$.

the constraint variables by looking at the differential constraint equations. These equations arise from varying the Lagrange multipliers $A_0$ and $\lambda$. (The algebraic constraints of course arise from varying $\mathfrak{F}^{ij}$, and merely tell one to replace $\mathfrak{F}_{ij}$ by $A_{j,i}-A_{i,j}$ in $\mathcal{L}_M$.) In terms of the orthogonal decomposition of a vector $f_i$ into its transverse and longitudinal parts $(f_i = f_i{}^T + f_i{}^L,\ f_i{}^T{}_{,i} = 0,\ \mathrm{curl} \mathbf{f}^L = 0)$, the Maxwell constraint equation reads:

$$\mathcal{E}^{iL}{}_{,i} = e\delta^3(\mathbf{r}-\mathbf{r}(t)), \qquad (2.4)$$

allowing one to eliminate $\mathbf{\mathcal{E}}^L$ in terms of the particle variables $\mathbf{r}(t)$. Similarly, one has that $\pi_0 = -(\pi_i\pi^i + m_0^2)^{\frac{1}{2}}$. In terms of the notation $p_i \equiv \pi_i + eA_i{}^T(\mathbf{r}(t),t)$ and $\mathfrak{B}^i \equiv \epsilon^{ijk}A_k{}^T{}_{,j}$, the reduced Lagrangian, with all constraints eliminated, reads (to within divergences and total time derivatives):

$$\mathcal{L}_M = (-\mathcal{E}^{iT})\partial_t A_i{}^T + p_i(dx^i/dt)\delta^3(\mathbf{r}-\mathbf{r}(t))$$
$$- \{\tfrac{1}{2}(\mathcal{E}^i\mathcal{E}_i + \mathfrak{B}^i\mathfrak{B}_i) + [(p_i - eA_i{}^T)(p^i - eA^{iT}) + m_0^2]^{\frac{1}{2}}\delta^3(\mathbf{r}-\mathbf{r}(t))\}. \quad (2.5)$$

In Eq. (2.5), $\mathcal{E}^{iL}$ is understood to be the solution of Eq. (2.4) and thus $\mathcal{L}_M$ depends only on $A_i{}^T$, $\mathcal{E}^{iT}$, $x^i(t)$ and $p_i(t)$. Note that $A_i{}^L$ has disappeared from $\mathcal{L}_M$ due to the particle's equation of continuity (which is an identity here). The Lagrangian is now in the desired canonical $p\dot{q}-H(p,q)$ form with $A_i{}^T$, $-\mathcal{E}_i{}^T$ being the Maxwell canonical variables and $x^i(t)$, $p_i(t)$ being those for the particle. The total Hamiltonian density is, of course, the usual one for this system.

The inclusion of gravitation is achieved simply by adding the term[4] $\mathcal{L}_G = (-^4g)^{\frac{1}{2}}{}^4R$ to $\mathcal{L}_M$ in Eq. (2.1), replacing $\eta_{\mu\nu}$ there by the metric $g_{\mu\nu}$, and inserting a factor $(-^4g)^{-\frac{1}{2}}$ in the $\mathfrak{F}^2$ term. The canonical reduction of the free gravitational field was performed in detail in III. We merely record here the form of $\mathcal{L}_G$ after algebraic constraints have been eliminated. Using the notation

$$\pi^{ij} = (-^4g)^{\frac{1}{2}}(^4\Gamma_{pq}{}^0 - g_{pq}{}^4\Gamma_{lm}{}^0 g^{lm})g^{ip}g^{jq}, \quad (2.6a)$$

$$N = (-^4g^{00})^{-\frac{1}{2}}, \quad N_i = {}^4g_{0i}. \quad (2.6b)$$

$\mathcal{L}_G$ becomes

$$\mathcal{L}_G = \pi^{ij}\partial_t g_{ij} - NR^0 - N_i R^i, \qquad (2.7)$$

where

$$-R^0 \equiv g^{\frac{1}{2}\,3}R + g^{-\frac{1}{2}}(\tfrac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}), \qquad (2.8a)$$

$$R^i \equiv -2\pi^{ij}{}_{|j} \qquad (2.8b)$$

The quantities $N$ and $N_i$ are the four Lagrange multipliers of the gravitational field. The covariantly generalized Lagrangian $\mathcal{L}_M$ of Eq. (2.1) can also be expressed in terms of this notation. Thus, using $^4g^{ij} = g^{ij} - (N^iN^j/N^2)$ where $N^i \equiv g^{ij}N_j$, and the definitions $\mathcal{E}^i \equiv \mathfrak{F}^{0i} \equiv (-^4g)^{\frac{1}{2}}F^{0i}$ and $\mathfrak{B}^i \equiv \epsilon^{ijk}A_k{}^T{}_{,j}$, one finds after a straightforward calculation[5]

$$\mathcal{L}_M = (-\mathcal{E}^{iT})\partial_t A_i{}^T + p_i(dx^i/dt)\delta^3(\mathbf{r}-\mathbf{r}(t))$$
$$- N\{\tfrac{1}{2}g^{-\frac{1}{2}}[g_{ij}(\mathcal{E}^i\mathcal{E}^j + B^iB^j)]$$
$$+ [(p_i - eA_i{}^T)(p_j - eA_j{}^T)g^{ij} + m_0^2]^{\frac{1}{2}}\delta^3(\mathbf{r}-\mathbf{r}(t))\}$$
$$+ N^i\{\epsilon_{ijk}\mathcal{E}^j\mathfrak{B}^k + (p_i - eA_i{}^T)\delta^3(\mathbf{r}-\mathbf{r}(t))\}. \quad (2.9)$$

The equation determining $\mathcal{E}^{iL}$ is again Eq. (2.4) and the orthogonal decomposition of $\mathcal{E}^i$ and $A_i$ is formally defined as in flat space. One can see from this form of $\mathcal{L}_M$ that the same matter variables are canonical as in flat space. It should be stressed that this would not be the case had one introduced, for example, $\mathcal{E}_i \equiv g_{ij}\mathcal{E}^j$ as the primary variables. Under these circumstances, one would have lost the simple $p\dot{q}-H$ form of $\mathcal{L}_M$. As will be seen below, it is equally important in order to reach a canonical form, that the Lagrangian be linear in $N$ and $N_i$. The fact that $\mathcal{E}^i \equiv \mathfrak{F}^{0i}$, $A_i$, $p_i(t)$, and $x^i(t)$ are the appropriate variables can also be seen from geometrical considerations in that these variables can be defined on an initial $t=\mathrm{const}$ surface independently of how the coordinates continue off the surface. Thus, both the requirement of canonical form and that of having appropriate Cauchy data single out the same set.

The total $\mathcal{L}$, which is obtained by adding Eqs. (2.7) and (2.9), now has the form:

$$\mathcal{L} = \pi^{ij}\partial_t g_{ij} + (-\mathcal{E}^{iT})\partial_t A_i{}^T + p_i(t)\delta^3(\mathbf{r}-\mathbf{r}(t))dx^i(t)/dt$$
$$- N\bar{R}^0 - N_i\bar{R}^i, \qquad (2.10)$$

where $\bar{R}^\mu$ depend only on $\pi^{ij}$, $g_{ij}$, $\mathcal{E}^{iT}$, $A_i{}^T$, $p_i(t)$, and $x^i(t)$. The remaining step consists in reducing the gravitational variables to canonical form. This can be carried out in a manner identical to the free gravitational field procedure of III. Thus, one must eliminate the gravitational constraint variables by solving $\bar{R}^\mu = 0$, and impose coordinate conditions. Again we make the orthogonal decomposition of $g_{ij}$ and $\pi^{ij}$ according to

$$f_{ij} = f_{ij}{}^{TT} + f_{ij}{}^T + (f_{i,j} + f_{j,i}), \qquad (2.11)$$

where each of the quantities on the right-hand side can be uniquely expressed as a linear functional of $f_{ij}$. Here $f_{ij}{}^{TT}$ are the two transverse traceless components of $f_{ij}$ $(f_{ij}{}^{TT}{}_{,j} = 0,\ f_{ii}{}^{TT} = 0)$; $f_{ij}{}^T$ is transverse and is uniquely determined by its trace $f^T$,

$$f_{ij}{}^T = \tfrac{1}{2}[\delta_{ij}f^T - (1/\nabla^2)f^T{}_{,ij}],$$

which is also the trace of the transverse part of $f_{ij}$; and $f_{i,j} + f_{j,i}$ contain the longitudinal parts of $f_{ij}$. In the above, $1/\nabla^2$ is the inverse of the flat space Laplacian operator (with appropriate boundary conditions).

The constraint equations $\bar{R}^\mu = 0$ read

$$-R^0 = \tfrac{1}{2}g^{-\frac{1}{2}}[\mathcal{E}^i\mathcal{E}_i + \mathfrak{B}^i\mathfrak{B}_i]$$
$$+ [(p_i - eA_i{}^T)(p^i - eA^{iT}) + m_0^2]^{\frac{1}{2}}\delta^3(\mathbf{r}-\mathbf{r}(t)), \quad (2.12a)$$

$$R^i = [\epsilon_{klm}\mathcal{E}^l\mathfrak{B}^m + (p_k - eA_k{}^T)\delta^3(\mathbf{r}-\mathbf{r}(t))]g^{ki}. \quad (2.12b)$$

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[5] The four-dimensional $\delta$-function is defined according to $\int \delta^4(x-a)f(x)d^4x = f(a)$ for any scalar function $f(x)$. It thus transforms as a scalar density under coordinate transformations,

but is not a functional of the metric. The three-dimensional $\delta$-function is defined similarly with respect to three-space and in any coordinate frame one has the identity $\delta^4(x) = \delta^3(\mathbf{r})\delta(t)$.

These differ from the free field constraints only in having the matter's energy and momentum densities on the right-hand sides of Eqs. (2.12a) and (2.12b) respectively. Thus, Eqs. (2.12) can again be solved for $-\nabla^2 g^T$ and $-2\pi^{ij}{}_{,j}$ in an iteration series. The same coordinate conditions as used in the free field case,

$$-\tfrac{1}{2}(1/\nabla^2)\pi^T = x^0, \qquad (2.13a)$$

$$g_i = x^i, \qquad (2.13b)$$

put the full coupled theory into canonical form with the Lagrangian of Eq. (2.10) reduced to

$$\mathcal{L} = \pi^{ijTT}\partial_t g_{ij}{}^{TT} + (-\mathcal{E}^{iT})\partial_t A_i{}^T$$
$$+ p_i dx^i/dt \delta^3(\mathbf{r}-\mathbf{r}(t)) - \mathcal{H}, \qquad (2.14)$$

where $\mathcal{H} = -T^0{}_0[g_{ij}{}^{TT}, \pi^{ijTT}, A_i{}^T, \mathcal{E}^{iT}, x^i(t), p_i(t)]$ is the solution of the constraint equations for $-\nabla^2 g^T$ and is the Hamiltonian density of the *total* system. Again, $\mathcal{H}$ does not depend explicitly on the chosen coordinates $x^\mu$.

Associated with the Lagrangian of Eq. (2.10) is the generator arising from the endpoint variations of the action. The $\bar{R}^\mu$ terms do not contribute here either, since they vanish by virtue of the constraints. One has

$$G = \int d^3x \Big\{ \pi^{ijTT}\delta g_{ij}{}^{TT} + (-\mathcal{E}^{iT})\delta A_i{}^T + p_i \delta x^i \delta^3(\mathbf{r}-\mathbf{r}(t))$$
$$- (-\nabla^2 g^T)\delta[-\tfrac{1}{2}(1/\nabla^2)\pi^T] + (-2\pi^{ij}{}_{,j})\delta g_i \Big\}. \qquad (2.15)$$

Upon insertion of the coordinate condition (2.13) and the solutions of the constraint equations $\nabla^2 g^T = T^0{}_0$, $-2\pi^{ij}{}_{,j} = T^0{}_i$, the generator becomes:

$$G = \int d^3x [\pi^{ijTT}\delta g_{ij}{}^{TT} + (-\mathcal{E}^{iT})\delta A_i{}^T$$
$$+ p_i \delta x^i(t)\delta^3(\mathbf{r}-\mathbf{r}(t)) - (-T^0{}_0)\delta x^0 + T^0{}_i \delta x^i]. \qquad (2.16)$$

We see that $P^0 = \int d^3x(-T^0{}_0)$ generates time translations, while $P^i = \int d^3x T^0{}_i$ is to be interpreted as the total field momentum since it generates spatial translations. It is shown in IIIa that the $T^0{}_\mu$ derived directly from the canonical Lagrangian (2.14) agree with the $T^0{}_\mu$ of Eq. (2.16) to within at most a spatial divergence of the canonical variables.

The $T_\mu{}^0$ are, of course, functionals of the metric and matter variables. In the flat space limit, one can immediately recover the Lorentz covariant expressions for the matter $T^0{}_\mu$. This follows from the fact that in this limit, the left-hand sides of Eqs. (2.12) are just $-\nabla^2 g^T$ and $-2\pi^{ij}{}_{,j}$, respectively, and the right-hand sides the correct energy-momentum densities. The $T^0{}_\mu$ obtained by solving Eqs. (2.12) rigorously for $-\nabla^2 g^T$ and $-2\pi^{ij}{}_{,j}$ are the energy-momentum densities of the coupled gravitational and matter systems (and are highly nonlinear in the two parts of the system). If one is interested in the numerical values of $P^\mu$ for a particular solution of the coupled equations, however, one need not express $T^0{}_\mu$ in terms of the canonical

variables. Here $P^\mu$ can be obtained directly from the metric quantities $g^T$ and $\pi^i$. This is no different from the fact that the electromagnetic variable $\mathcal{E}^L$ determines the total charge.

One need not have used the coordinate conditions (2.13) to reduce the theory to canonical form. Indeed, any coordinate conditions consistent with asymptotically flat boundary conditions could be employed to obtain a canonical form (in terms, of course, of different canonical variables). That the numerical values of $P^\mu$, as well as other physical quantities, are unchanged by such a procedure will be shown in a forthcoming paper (IVa). For the applications of this paper, we shall find it useful to consider the coordinate conditions

$$x^0 = -\tfrac{1}{2}(1/\nabla^2)(\pi^T + \nabla^2 \pi^L), \qquad (2.17a)$$

$$x^i = g_i - \tfrac{1}{4}(1/\nabla^2)g^T{}_{,i}. \qquad (2.17b)$$

In Eq. (2.17a) we have introduced the notation $\pi^i = \pi^{iT} + \tfrac{1}{2}\pi^L{}_{,i}$, which is the decomposition of the vector $\pi^i$ [arising in the orthogonal breakup (2.11)] into its transverse and longitudinal parts. By a simple rearrangement of the generator of Eq. (2.15) we obtain

$$G = \int d^3x \Big\{ \pi^{ijTT}\delta g_{ij}{}^{TT} + (-\mathcal{E}^{iT})\delta A_i{}^T + p_i \delta^3(\mathbf{r}-\mathbf{r}(t))\delta x^i(t)$$
$$- (-\nabla^2 g^T)\delta[-\tfrac{1}{2}(1/\nabla^2)(\pi^T + \nabla^2 \pi^L)]$$
$$+ [-2\nabla^2(\pi^{iT} + \pi^L{}_{,i})]\delta[g_i - \tfrac{1}{4}(1/\nabla^2)g^T{}_{,i}] \Big\}. \qquad (2.18)$$

Since $-\nabla^2 g^T$ and $-2\nabla^2(\pi^{iT} + \pi^L{}_{,i}) \equiv -2\pi^{ij}{}_{,j}$ can be chosen as the four quantities to be solved for in the constraint equations (2.12) and are the coefficients of $\delta x^\mu$ in $G$, they may be taken to be $-T^0{}_0$ and $T^0{}_i$ respectively. Note that, though the gravitational canonical variables are still $g_{ij}{}^{TT}$ and $\pi^{ijTT}$, these are different quantities from the ones previously considered since they are transverse-traceless with respect to a different coordinate frame. Similarly, $T^0{}_\mu$ are different functionals of the new canonical variables.

This coordinate system is of interest since the metric $g_{ij}$ takes on the form

$$g_{ij} = g_{ij}{}^{TT} + (1 + \tfrac{1}{2}g^T)\delta_{ij}, \qquad (2.19)$$

with the boundary conditions $g^T \to 0$ as $r \to \infty$. When the canonical variables $g_{ij}{}^{TT}$ vanish, the metric is therefore isotropic.

### 3. COULOMB SELF-ENERGIES OF POINT CHARGES

The formalism of the preceding section allows us to treat the problem of static self-energies of charged particles. As we have seen, the energy, $E$, of a system is given by the numerical value of the Hamiltonian:

$$E = -\int d^3x\, T^0{}_0 = -\int d^3x\, \nabla^2 g^T$$

$$= -\oint dS_i\, g^T{}_{,i}, \qquad (3.1)$$

where $dS_i$ is the two-dimensional surface element at spatial infinity. Thus, the energy is given by the coefficient of $1/(4\pi r)$ in the asymptotic expansion of $g^T$, since this term is the monopole part of $(1/\nabla^2)T^0{}_0$.

A one-particle state is one that contains no independent excitations of the gravitational or electromagnetic fields in the rest frame. This requires that, for this state, $g_{ij}{}^{TT}=\pi^{ijTT}=A_i{}^T=\mathcal{E}^{iT}=p_i=0$ on whatever $t=$ const surface the energy is being computed. With the coordinate conditions of Eq. (2.17), we are therefore dealing with the time symmetric situation, $\pi^{ij}=0$. According to Eq. (2.19), then, the metric is isotropic; it is convenient to write it as $g_{ij}=\chi^4(r)\delta_{ij}$. From the discussion of Eq. (3.1), the energy is the coefficient of $1/(32\pi r)$ in the asymptotic form of $\chi(r)$. The field equation determining $g^T$ and hence $\chi$ is Eq. (2.12a). One has[6]

$$g^{\frac{1}{2}}{}^3R = -8\chi\nabla^2\chi = m_0\delta^3(\mathbf{r})+\tfrac{1}{2}\chi^{-2}\mathcal{E}^{iL}\mathcal{E}^{iL}, \qquad (3.2a)$$

with the electric field, $\mathcal{E}^L$, determined by Eq. (2.4) to be

$$\mathcal{E}^{iL} = (-e/4\pi r)_{,i}. \qquad (3.3)$$

A formal solution of Eq. (3.2) may be found by setting $\chi^2=\psi^2-\varphi^2$. One finds

$$\begin{aligned} -8\chi\nabla^2\chi \equiv & 8(\varphi\nabla^2\varphi-\psi\nabla^2\psi) \\ & +8(\psi^2-\varphi^2)^{-1}(\varphi\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\varphi)^2 \quad (3.2b) \\ = & m_0\delta^3(\mathbf{r})+\tfrac{1}{2}(\psi^2-\varphi^2)^{-1}(\mathcal{E}^L)^2. \end{aligned}$$

If one then makes the assumption

$$\mathcal{E}^L = 4(\varphi\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\varphi), \qquad (3.4)$$

with $\varphi=e/(16\pi r)$ and $\psi=1+m/(32\pi r)$, Eq. (3.4) correctly reproduces Eq. (3.3). Equation (3.2b) then determines the total mass $m$ to be

$$m = \lim_{\epsilon\to 0} 16\pi\{-\epsilon+[\epsilon^2+(e/8\pi)^2+m_0\epsilon/8\pi]^{\frac{1}{2}}\}. \qquad (3.5)$$

In Eq. (3.5) the parameter $\epsilon$ has been introduced by setting $\delta^3(\mathbf{r})/r$ to $\delta^3(\mathbf{r})/\epsilon$, and is thus essentially the "radius" of the $\delta$-function. [This interpretation of the $\delta$-function corresponds to viewing $\delta^3(\mathbf{r})$ as the limit of a shell distribution, $\delta(r-\epsilon)/4\pi r^2$, of radius $\epsilon$. In the Appendix, it is shown that the results of this section are independent of the model chosen for $\delta^3(\mathbf{r})$ in the point limit.] The result (3.5) is that $m=2e$, or in conventional units, $m=(e^2/4\pi)^{\frac{1}{2}}\gamma^{-\frac{1}{2}}$. Hence, the total mass is finite

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[6] Professor L. N. Cooper has pointed out to us that Eq. (3.2a) with $e=0$ $[-8\chi\nabla^2\chi=\rho_0(r)$, where $\rho_0$ is any bare mass distribution] can be obtained by simple equivalence principle arguments starting from Newtonian theory. The Poisson equation $\nabla^2\phi=4\pi\gamma\rho_0=\frac{1}{4}\rho_0$, for the gravitational potential $\phi$, must be corrected to include the particle's gravitational self-energy, $\frac{1}{2}\rho\phi$, as part of the source, i.e., $\nabla^2\phi=\frac{1}{4}\rho\equiv\frac{1}{4}(\rho_0+\frac{1}{2}\rho\phi)$. Eliminating $\rho$, one obtains $\nabla^2\phi=\frac{1}{4}\rho_0(1-\frac{1}{2}\phi)^{-1}$. In terms of $\chi\equiv 1-\frac{1}{2}\phi$, this is just Eq. (3.2a) in the neutral case. For the point electric charge, the same argument with $\rho_0$ replaced by $\rho_0+\frac{1}{2}\rho_e\phi_e$ leads to an equation for $\phi$ which yields the correct total energy.

and independent of the bare mechanical mass.[7] The gravitationally renormalized electrostatic self-energy is now finite. The analysis also points out that mass only arises if a particle has nongravitational interaction with a field of finite range. For example, an electrically neutral particle coupled to a Yukawa field would still acquire a mass by virtue of this coupling.

A solution may also be obtained for the case of two particles of like charge. For simplicity, we consider a system of equal charges and equal bare masses. The field equations (3.2) and (3.3) are now

$$-8\chi\nabla^2\chi = m_0[\delta^3(\mathbf{r}_1)+\delta^3(\mathbf{r}_2)]+\tfrac{1}{2}\chi^{-2}\mathcal{E}^{iL}\mathcal{E}^{iL}, \qquad (3.6)$$

and

$$\mathcal{E}^i = \mathcal{E}^{iL} = [-e/(4\pi r_1)-e/(4\pi r_2)]_{,i}, \qquad (3.7)$$

where $\mathbf{r}_{1,2}\equiv\mathbf{r}-\mathbf{a}_{1,2}$ with $\mathbf{a}_{1,2}$, the positions of the charges. Again, $\chi^2=\psi^2-\varphi^2$, where $\varphi=(e/16\pi)(1/r_1+1/r_2)$ and $\psi=1+(E/64\pi)(1/r_1+1/r_2)$. As in the one-particle case, Eq. (3.4) holds. The parameter $E$ is the total energy of the two-particle state since again the canonical variables of the fields and the momenta of the particles are zero. One finds

$$\begin{aligned} E=\lim_{\epsilon\to 0} 32\pi\{&-\epsilon+[\epsilon^2+(e/8\pi)^2(1+x)^2 \\ &+(m_0\epsilon/8\pi)(1+x)]^{\frac{1}{2}}\}(1+x)^{-1}, \quad (3.8) \end{aligned}$$

where $x\equiv\epsilon/r_{12}\equiv\epsilon/|\mathbf{a}_1-\mathbf{a}_2|$. In the limit $\epsilon=0$, one obtains $E=4|e|=2m$. No interaction energy ($r_{12}$-dependent term) survives. This may be understood by examining the Newtonian limit of large $r_{12}$. One would expect to find there that $E-2m=(e^2/4\pi-\gamma m^2)/r_{12}$. However, as was seen, $m=(e^2/4\pi\gamma)^{\frac{1}{2}}$, and so the right-hand side vanishes. The fact that $E=2m$ generally, indicates that the electric and gravitational forces cancel in all orders of $1/r_{12}$ which further implies that the initial metric $g_{ij}=\chi^4\delta_{ij}$, $\pi^{ij}=0$ actually determines a static solution of the Einstein equations. This is indeed the case.[8]

The solution (3.8) correctly reduces to twice the $m$ of Eq. (3.5) (even for finite $\epsilon$) as $r_{12}\to\infty$. One can recover all the usual Newtonian results (including interaction energies) by taking $m_0$ and $e$ small and $r_{12}$ large before passing to the limit $\epsilon\to 0$. This may be viewed as the "dilute" limit since the extension of the particle is large compared to its bare mass and charge. Alternately, it corresponds to a perturbation expansion in powers of $\gamma$. One easily sees that

$$\begin{aligned} E\sim 2m_0+2(e^2/4\pi-\gamma m_0^2)/2\epsilon+(e^2/4\pi-\gamma m_0^2)/r_{12} \\ +O(1/\epsilon^2,1/(r_{12})^2,1/\epsilon r_{12}). \quad (3.9) \end{aligned}$$

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[7] It is interesting to note that, even though $m=0$ for the neutral particle, this does not imply that space is everywhere flat. The metric is indeed flat for $r>\epsilon$, but rises steeply in the interior. Using, for example, the model leading to Eq. (A.3), one finds that $\int_0^\epsilon (^3g)^{\frac{1}{2}}d^3x\equiv\int_0^\epsilon\chi^6 d^3x=(\pi/6)(32\pi)^{-6}m_0^3$ in the limit $\epsilon=0$. In general, $\chi\sim(m_0/\epsilon)^{\frac{1}{2}}$ for small $\epsilon$ in the interior and so this integral is always finite when one limits to the point particle. Also, $\int_0^\infty {}^3R(^3g)^{\frac{1}{2}}d^3x=m_0$, which shows, in a model independent way, that space is curved at the origin.

[8] A. Papapetrou, Proc. Roy. Irish Acad. 51A, 191 (1947).

The higher terms include the gravitational interactions among the lower order structures. Since $m_0$ and $e$ are unrelated, there is no longer any cancellation in the second and third terms. Equation (3.9) contains the usual Coulomb and Newtonian infinite results as well as a whole series of more divergent terms. From the rigorous formula (3.8), it is clear that these actually sum to a finite answer, showing the inapplicability of the perturbation approach to the self-energy problem.

The cancellation of the interaction term in Eq. (3.8) can be traced to the fact that the two particles have the same sign of charge. In the case of two opposite charges, one would expect for large $r_{12}$ that $E - 2m = -(\gamma m^2 + e^2/4\pi)/r_{12}$, and hence a nonvanishing interaction energy. We have not been able to obtain a solution in this case for a pure two-particle state. While changing the relative sign in $\varphi$, i.e., writing $\varphi = b(1/r_1 - 1/r_2)$ does lead to a solution of the Einstein equations, the electric field calculated according to the right-hand side of Eq. (3.4) is no longer purely longitudinal. Thus $\mathcal{E}^T$ is nonzero and $E$ is no longer the energy of the two-particle system, but includes extraneous electromagnetic waves. To obtain a longitudinal $\mathcal{E}$, one must also change the relative sign in $\psi$. However, such a solution makes the metric singular on a two-dimensional surface and is therefore unacceptable. We have not been able to solve Eq. (3.2) with $\mathcal{E}^L = (-e/4\pi)\nabla(1/r_1 - 1/r_2)$.

The solutions derived here for the neutral and charged one particle states differ from the conventional Schwarzschild and Reissner-Nordstrom metrics. The matter source in the latter cases is not expressed in terms of the bare mass parameter. Thus no relation between $m$, $m_0$, and $e$ is obtained. A detailed discussion of this point is given in Va. It is also shown there that the complete solution (using our initial conditions and coordinate frame) is *nonsingular* and, in the point limit, static. Knowledge of the complete solution then allows one to calculate the total spatial self-stresses, which vanish everywhere in the point limit.

## 4. RENORMALIZATION METHODS

The results obtained in Sec. 3 show that the total renormalized mass is finite in general relativity, with *finite* bare parameters in all the cases considered. For the neutral point particle, the clothed mass is zero, independent of the bare mass, while the charged particle's mass arises entirely from its electromagnetic coupling. Although it is no longer logically necessary to do so, one may attempt to reinstate mechanical masses for point particles by utilizing the "divergent" renormalization techniques employed in Lorentz covariant field theory. In order that this renormalization be consistent, one must obtain, in the two neutral particle system, the Newtonian interaction energy between the two renormalized masses for large interparticle separation, $r_{12}$. As will be seen below, this

requirement will involve a further infinite renormalization, that of the coupling constant $\gamma$ of the gravitational field. The further requirement on the two-charge system, that for large separation one have the Coulomb interaction energy as well, requires an infinite renormalization of the charge $e$. Under these circumstances, the interaction energy is correctly given to order $1/r_{12}$ since this is precisely the renormalization imposed. However, the $(1/r_{12})^2$ and higher terms still remain divergent, showing that further renormalization of an infinite number of observables must be performed.

The neutral one-particle mass can be obtained from Eq. (3.5) by setting $e$ to zero. We will call this the renormalized mechanical mass $m_N$:

$$m_N = \gamma^{-1}\{-\epsilon + [\epsilon^2 + 2m_0\epsilon\gamma]^{\frac{1}{2}}\}. \tag{4.1}$$

In Eq. (4.1), $\gamma$ is the unrenormalized gravitational constant. The two-particle neutral energy [obtained from Eq. (3.8) with $e=0$] may be expressed now in terms of this renormalized mass. One finds here

$$E(1/r_{12}) = (2/\gamma)(1+x)^{-1}\{-\epsilon + [\epsilon^2 + (1+x) \times (\gamma^2 m_N^2 + 2\gamma m_N\epsilon)]^{\frac{1}{2}}\}, \tag{4.2}$$

where $x \equiv \epsilon/r_{12}$. For infinite $r_{12}$, $E(0)$ correctly equals $2m_N$. One determines the $\gamma$ renormalization by requiring that for large $r_{12}$, the $1/r_{12}$ term have just the Newtonian form. This is analogous to the definition of charge renormalization in electrodynamics by means of the asymptotic Coulomb energy. Thus,

$$(1/r_{12})[dE/d(1/r_{12})]_{r_{12}=\infty} = -\gamma_r m_N^2/r_{12}, \tag{4.3}$$

where $\gamma_r$ is the renormalized gravitational constant. One finds

$$\gamma_r = \gamma/(1 + \gamma m_N\epsilon^{-1}), \tag{4.4a}$$

or

$$\gamma = \gamma_r/(1 - \gamma_r m_N\epsilon^{-1}). \tag{4.4b}$$

This renormalization exhibits the characteristic "ghost" structure, since as $\epsilon \to 0$, $\gamma$ changes sign. Expressing $E$ in terms of $\gamma_r$, we obtain

$$E = (2/\gamma_r)(1+x)^{-1}\{(\gamma_r m_N - \epsilon) + [\epsilon^2 + x(2m_N\gamma_r\epsilon - m_N^2\gamma_r^2)]^{\frac{1}{2}}\}. \tag{4.5}$$

One must now investigate whether the higher terms in $1/r_{12}$ have been rendered finite by the above renormalizations. The next term in the expansion gives

$$\frac{1}{2r_{12}^2}\frac{d^2E}{d(1/r_{12})^2} = -\gamma_r\frac{m_N(-\gamma_r m_N^2/r_{12})}{r_{12}} -\frac{1}{2}\gamma_r\frac{(-\gamma_r m_N^2/2\epsilon)(-\gamma_r m_N^2/r_{12})}{r_{12}}. \tag{4.6}$$

This (and each higher term) is still divergent and a new renormalization is required in each term. The inadequacy of the renormalizations of Eqs. (4.1) and (4.4) is further indicated by the fact that the rigorous

formula (4.5) for $E$ in terms of the renormalized parameters approaches $2m_N$ as $\epsilon$ goes to zero. Hence, the Newtonian interaction is rigorously absent in spite of the fact that the $\gamma$ renormalization was carried out explicitly to obtain it.

Similar difficulties occur in the charged particle case. One finds from the $1/r_{12}$ part of $E$ that $\gamma$ is to be renormalized as before. In order to obtain the Coulomb energy, $e^2$ must be renormalized in such a fashion that $e^2/\gamma$ is unchanged. Again, however, the $(1/r_{12})^2$ and higher terms are still divergent.

## 5. CONCLUSIONS

In this paper, we have analyzed the effects of general relativity on the classical self-energy problem. That gravitational interaction can effect a realistic compensation of positive self-energies follows from its intrinsically attractive nature and the fact that it is the total energy that interacts gravitationally. The gravitational coupling is then of such a type as to damp out large positive energies arising from the other interactions which the particle may have. This was seen rigorously in the case of the static Coulomb self-energy. In general, the total static energy remains positive definite if the matter energy density $-T^0{}_0$ is positive.[9] This follows from the general equation

$$-8\chi\nabla^2\chi = g^{\frac{1}{2}}(-T^0{}_0) = \chi^6(-T^0{}_0(\chi)), \qquad (5.1)$$

which can be integrated to yield

$$E = -8 \oint \chi_{,i}\, dS_i = \int d^3x\, \chi^5(-T^0{}_0(\chi)) > 0. \quad (5.2)$$

Equation (5.1) also gives an upper limit on $E$. If one integrates by parts,[10] on the left side, one obtains

$$E = \int d^3x\, g^{\frac{1}{2}}(-T^0{}_0(\chi)) - 8\int d^3x(\nabla\chi)^2. \quad (5.3)$$

Thus $E$ is bounded by the matter energy computed in the gravitational field $\chi$. For the coupling discussed in this paper, the further limitation that $E$ is less than its flat space value can be established, showing that compensation indeed takes place. Here, Eq. (5.2) takes the form

$$E(\chi) = m_0/\chi(0) + \tfrac{1}{2}\int d^3x\, \mathcal{E}^i\mathcal{E}^i/\chi^3. \quad (5.4)$$

Since, as we will see shortly, $\chi(\mathbf{r}) \geqslant 1$, one has $E(\chi) < E(\chi=1)$. This formula makes clear that $E(\chi)$ is

finite. As was derived in Sec. 3, $\chi(r) \sim r^{-\frac{1}{2}}$ near the origin. Thus, the first term vanishes (showing again that the bare mass does not contribute to the total mass), and the behavior of $\chi$ is also sufficient to converge the integral in Eq. (5.4).[11] The compensation of the Coulomb infinity is further brought out by the relation (with $m_0$ set to zero)

$$E = \tfrac{1}{2}\int \mathcal{E}^i\mathcal{E}^i d^3x - 24\int \chi^2(\nabla\chi)^2 d^3x, \qquad (5.5)$$

obtained by multiplying Eq. (5.1) by $\chi^2$. The first term on the right is infinite, its divergence being compensated by the last term.

That the function $\chi(r)$ is everywhere greater than unity follows from the condition

$$-\nabla^2\chi = g^{\frac{1}{2}}(-T^0{}_0)/8\chi \geqslant 0. \qquad (5.6)$$

The minimum principle for the Laplacian then guarantees that $\chi$ takes on its least value (unity) at infinity.

The numerical value obtained above for the mass of a classical point particle with electronic charge, $m = e/(4\pi\gamma)^{\frac{1}{2}} \simeq 10^{18}m_e$, is much too large. Of course, one would not expect classical theory to give correct numerical values for masses. Any realistic discussion of self-masses must be based on quantum theory. However, if the effective flat-space cutoff $a \sim [e/(4\pi)^{\frac{1}{2}}]\gamma^{\frac{1}{2}}$ obtained in this paper were to hold also in quantum theory,[12] the numerical values for the mass and charge would be quite different. For example, using such a cutoff in Landau's estimate[1] for the renormalized charge, one finds $e_r^2/4\pi \approx 10^{-2}$ (independent of the bare charge) as has been previously noticed by Landau. Thus, Landau obtains

$$e_r^2 = e^2[1 + (2/3\pi)\nu(e^2/4\pi)\ln\{(\hbar/mc)/a\}]^{-1}$$
$$\approx [(\nu/12\pi^2)\ln\{(\hbar/mc)/a\}]^{-1}, \quad (5.7)$$

where $\nu \simeq 10$ is essentially the number of charged fields. Equation (5.7) was obtained by summing the dominant terms in each self-energy diagram. With an effective cutoff of physical origin, the usual objections to making an estimate of what otherwise is a divergent series need no longer hold. The methods of Landau also yield a formula for the renormalized mass in terms of the bare mass and the charge.[13] In the classical theory, we saw that the bare mass did not enter. To what extent this is maintained in the quantum theory (i.e., to what extent

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[9] More precisely, $-T^0{}_0$ in Eq. (5.1) stands for the invariant expression $n_\mu n^\nu T^\mu{}_\nu$ where $n_\mu$ is the unit normal vector to the space-like surface. This corresponds to the fact that the left-hand side is $2g^{\frac{1}{2}}n_\mu n^\nu({}^4R^\mu{}_\nu - \delta^\mu{}_\nu{}^4R)$. For the case of a $t=$const surface with $g_{0i}=0$, $n_\mu n^\nu T^\mu{}_\nu = -T^0{}_0$.

[10] Here we have made use of the fact that $\oint \chi^n\chi_{,i}\, dS_i = \oint \chi_{,i}\, dS_i$ for the surface at infinity since $\chi^n$ may rigorously be replaced by its flat space value unity.

[11] It might be mentioned that even if the flat space theory were quadratically divergent, corresponding to $\mathcal{E}^i\mathcal{E}^i \sim r^{-5}$ near the origin, one still expects the general relativistic result to be finite. From Eq. (3.2a), one would then expect, on dimensional grounds, that $\chi \sim r^{-\frac{3}{4}}$. Thus the integral in Eq. (5.4) would still converge.

[12] An effective quantum gravitational cutoff might be $\sim(\gamma\hbar c^{-3})^{\frac{1}{2}}$ on dimensional grounds. This only differs from the classical $a$ by $\alpha^{\frac{1}{2}} \equiv (e^2/4\pi\hbar c)^{\frac{1}{2}}$ and would not affect the discussion in text.

[13] This formula, $m = m_0(e^2/e_r^2)^{9/4\nu}$ does not make clear wath relation between $m$, $m_0$, and $e_r$ might be expected, since the estimate of Eq. (5.7) fails to determine $e$ sufficiently.

the bare mass distribution remains a $\delta$-function) is not clear. Finally, it should be emphasized that we have assumed in the above discussion the simplest possibility that the gravitational effects on the quantum theory can be summarized in terms of a cutoff of the type considered by Landau.

## APPENDIX

In this Appendix, we will show that the results of Sec. 3 are independent of the model one uses for the $\delta^3(\mathbf{r})$ source functions. In order to see the general structure of the solutions, we will first examine the model of a shell distribution, $\delta(r-\epsilon)/4\pi r^2$. For simplicity, we consider the neutral case in detail, where

$$-8\nabla^2\chi = m_0\delta(r-\epsilon)/(4\pi r^2\chi(r))$$
$$= [m_0/\chi(\epsilon)]\delta(r-\epsilon)/4\pi\epsilon^2. \quad (A.1)$$

The exterior solution (for $r \geqslant \epsilon$) of course has the form $\chi(r) = 1 + [m_0/\chi(\epsilon)](1/32\pi r)$ with the total mass $m = m_0/\chi(\epsilon)$ determined by solving $\chi(\epsilon) = 1 + m_0/(32\pi\epsilon\chi(\epsilon))$ for $\chi(\epsilon)$. One obtains

$$m = \tfrac{1}{2}\{-\epsilon + [\epsilon^2 + 2m_0\epsilon]^{\frac{1}{2}}\}, \quad (A.2)$$

precisely the result of Sec. 3. Note that only the exterior solution was used in obtaining Eq. (A.2), suggesting that in the limit $\epsilon \to 0$ the result is model independent. The inner solution is a constant $\chi(r) = \chi(\epsilon)$. It is the jump discontinuity in the derivative of $\chi(r)$ at $r = \epsilon$ that reproduces the shell distribution. For small $\epsilon$ one finds

$$\chi(r) = \chi(\epsilon) \sim 1 + (1/32\pi)(m_0/2\epsilon)^{\frac{1}{2}}, \quad r \leqslant \epsilon. \quad (A.3)$$

The above result, that $m = 0$ in the point limit, is indeed independent of the particular model one uses for the mass distribution. The generalization of Eq. (5.4) (with $e = 0$) for the case of an extended source $m_0\rho(r)$ with $\int d^3r\,\rho(r) = 1$ is

$$E \equiv m = m_0 \int d^3r\, \rho(r)/\chi(r). \quad (A.4)$$

If $\epsilon$ represents the radius of the matter distribution, the integral extends only between 0 and $\epsilon$. As was discussed in Sec. 5, $\chi(r)$ takes its minimum value at the boundary $r = \epsilon$ and so

$$m \leqslant [m_0/\chi(\epsilon)] \int_0^\epsilon d^3r\, \rho(r) = m_0/[1 + m/32\pi\epsilon]. \quad (A.5)$$

In the last member of Eq. (A.5), we have used the general form of $\chi(r)$ which holds outside any source. Equation (A.5) states that $m$ is less than or equal to the value of Eq. (A.2) and hence $m = 0$ when $\epsilon = 0$.

For the charged particle, a similar analysis holds. Again $m = 2e$ in the point limit, independent of the shape of the mass and charge distributions. This may again be shown from the generalized version of Eq. (5.4)

$$m = m_0 \int_0^\epsilon d^3r\, \rho/\chi + e^2 \int d^3r [\nabla(1/\nabla^2)\bar{\rho}]^2\chi^{-3}, \quad (A.6)$$

with the mass density $\rho$ and charge density $\bar{\rho}$ both normalized to unity. The second term on the right-hand side may be rewritten as

$$e^2 \int_0^\epsilon d^3r [\nabla(1/\nabla^2)\bar{\rho}]^2\chi^{-3} + \int_\epsilon^\infty (e/4\pi r)^2$$
$$\times [(1 + m/32\pi r)^2 - (e/16\pi r)^2]^{-\frac{3}{2}} d^3r. \quad (A.7)$$

In the second term of Eq. (A.7), we have inserted the general exterior solution for $\mathcal{E}$ and $\chi$; this integral may be trivially evaluated. For the first term one has

$$e^2 \int_0^\epsilon d^3r [\nabla(1/\nabla^2)\bar{\rho}]^2\chi^{-3}$$
$$\leqslant e^2\chi^{-3}(\epsilon) \int_0^\epsilon d^3r [\nabla(1/\nabla^2)\bar{\rho}]^2, \quad (A.8)$$

where $\chi^2(\epsilon) = (1 + m/32\pi\epsilon)^2 - (e/16\pi\epsilon)^2$. The right-hand side integral goes as $1/\epsilon$ provided one considers distributions such that $\bar{\rho} \leqslant A/\epsilon^3$. (This restriction is required to avoid models where $\epsilon$ is not a reasonable measure of the radius of the distribution.) Thus,

$$e^2 \int_0^\epsilon d^3r [\nabla(1/\nabla^2)\bar{\rho}]^2\chi^{-3} \leqslant e^2 B/\epsilon\chi^3(\epsilon), \quad (A.9)$$

where $B$ is independent of $\epsilon$. The inequality arising from Eq. (A.6), when Eqs. (A.9) and (A.7) are inserted, reads

$$[(m^2 - 4e^2) + 32\pi\epsilon(m + 16m_0)]$$
$$\times [(m^2 - 4e^2) + (32\pi)^2\epsilon^2 + 64\pi m\epsilon]$$
$$\leqslant 16(32\pi)e^2 B\epsilon^2. \quad (A.10)$$

In the limit $\epsilon = 0$, we obtain the result of Sec. 3, i.e., $m = 2e$.