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Influence of Bose-Einstein Statistics on the Antiproton-Proton Annihilation Process*

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Recent observations of angular distributions of π mesons in $\bar{p}$-p annihilation indicate a deviation from the predictions of the usual Fermi statistical model. In order to shed light on these phenomena, a modification of the statistical model is studied. We retain the assumption that the transition rate into a given final state is proportional to the probability of finding N free π mesons in the reaction volume, but express this probability in terms of wave functions symmetrized with respect to particles of like charge. The justification of this assumption is discussed. The model reproduces the experimental results qualitatively, provided the radius of the interaction volume is between one-half and three-fourths of the pion Compton wavelength; the dependence of angular correlation effects on the value of the radius is rather sensitive. Quantitatively, there seems to remain some discrepancy, but we cannot say whether this is due to experimental uncertainties or to some other dynamic effects. In the absence of information on π-π interactions and of a fully satisfactory explanation of the mean pion multiplicity for annihilation, we wish to emphasize the preliminary nature of our results. We consider them, however, as an indication that the symmetrization effects discussed here may well play a major role in the analysis of angular distributions. It is pointed out that in this respect the energy dependence of the angular correlations may provide valuable clues for the validity of our model.

I. INTRODUCTION

RECENTLY a study has been made[1] in a propane bubble chamber of "hydrogenlike" annihilations of antiprotons of 1.05-Bev/c laboratory-system momentum, corresponding to an energy release of 2.1 Bev in the center-of-mass system. A hydrogenlike event is defined as one in which equal numbers of π^+ and π^- mesons are produced and in which no visible evaporation prongs appear.[2] The experiment indicates[1] that the distribution of the angle between pairs of pions (in the c.m.-system of $\bar{p}$-p) deviates from the prediction of the conventional statistical model. In particular it was found that there is a clear difference between the angular distribution for pion pairs of like charge and that for pairs of unlike charge. In the statistical model in its usual sense, there is no room for distinctions of this kind.

It is the purpose of this paper to indicate a simple refinement of the statistical model which could possibly explain the bulk of the effect, and which consists of taking into account the influence of the Bose-Einstein (BE) statistics for pions of like charge. As we show in what follows, such an interpretation appears to reproduce the experimental results qualitatively—provided, however, that the radius of the volume of strong interactions is about $\frac{3}{4}$ times the π Compton wavelength, which is a physically reasonable order of magnitude. The dependence of the angular effects on the interaction radius appears to be a sensitive one. Hence, it would seem that such effects may provide valuable information on the annihilation mechanism.

It should be stressed from the outset, however, that results of this study should not be construed to imply that detailed dynamical effects (such as, for example, π-π interactions) are definitely negligible in the discussion of the kind of phenomena considered here. The present stage of both our experimental and our theoretical knowledge of the annihilation process seems to us to be far too early to make such categorical statements. In the concluding remarks (Sec. IV), we briefly discuss the dependence of the BE effect on the available energy for annihilation. This gives one instance of how further experimental study may reveal whether or not the present considerations provide substantially the correct approach to the problem. It may directly be noted, however, that the symmetrization effects which we shall now outline are relevant regardless of whether π-π interactions are large or small.

For the statement of our ideas, it is helpful to recall first what the assumptions of the usual statistical model (SM) are. For definiteness, consider the system

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[1] G. Goldhaber, W. B. Fowler, S. Goldhaber, T. F. Hoang, T. E. Kalogeropoulos, and W. M. Powell, Phys. Rev. Letters 3, 181 (1959).

[2] All center-of-mass transformations were made on the assumption that the struck proton is at rest. From the known annihilation cross sections in carbon and hydrogen and from the π-multiplicity distribution, it was deduced that about 85% of the hydrogenlike events correspond to annihilations on hydrogen.

enclosed in a large box with volume V and with periodic boundary conditions. A first assumption of the SM is that the rate of annihilation into any given $N\pi$ state is proportional to $P_N(\Omega)$, given by

$$P_N(\Omega)=(\Omega/V)^N. \tag{1}$$

Here Ω is the "reaction volume" in which the statistical mixture of states is supposed to be produced. For what follows,[3] it is helpful to interpret $P_N(\Omega)$ as the probability to find N free pions in the reaction volume:

$$P_N(\Omega)=\int_\Omega\cdots\int_\Omega d\mathbf{r}_1\cdots d\mathbf{r}_N|\phi_N|^2, \tag{2}$$

where

$$\phi_N=\frac{1}{V^{N/2}}\exp\left(i\sum_{m=1}^{N}\mathbf{p}_m\cdot\mathbf{r}_m\right). \tag{3}$$

Thus, according to the SM, the total rate R_N of annihilation is given by

$$R_N=C_NP_N(\Omega)F_N(W). \tag{4}$$

Here $F_N(W)$ is the Lorentz-invariant phase space introduced by Srivastava and Sudarshan.[4]

$$F_N(W)=\left(\frac{V}{8\pi^3}\right)^N\int\cdots\int\frac{d\mathbf{p}_1\cdots d\mathbf{p}_N}{\omega_1\cdots\omega_N}\times\delta\left(W-\sum_{i=1}^{N}\omega_i\right)\delta\left(\sum_{i=1}^{N}\mathbf{p}_i\right), \tag{5}$$

$$\omega_i=(\mathbf{p}_i^2+\mu^2)^{\frac{1}{2}},$$

where W is the available annihilation energy and μ is the π mass. These authors noted that this invariant form lends itself conveniently to the derivation of relations recursive in N. This circumstance was also employed by Kalogeropoulos,[5] and is likewise used in what follows.

The factor C_N in Eq. (4) does not depend on W, and is usually taken to be

$$C_N=\text{const}\times n(N)/N!, \tag{6}$$

where $n(N)$ is the total number of $I=0$, 1 states of the $N\pi$ system. Hence the SM assumptions are a constant transition rate into a given $N\pi$ configuration, and equal weight for all allowed I-spin states.

Thus the SM takes only incompletely into account the various conservation laws and asymmetries to which the system is subjected. In particular, angular-momentum conservation is neglected in this version of the SM.[6] Furthermore, BE statistics is rather cursorily taken care of by the factor $(N!)^{-1}$ in Eq. (5). It is this last aspect of the SM that we refine here. We again assume proportionality of R_N to $P_N(\Omega)$ given by Eq. (2), but employ suitable symmetrized wave functions instead of ϕ_N given by Eq. (3).

Rigorously, R_N is the incoherent sum of transitions into the various $I=0$, 1 states. For given I, these states can be characterized by distinct spatial symmetries.[7] For a specific charge partition of the final products, such as, for example, for $N=4$:

$$\bar{p}+p\rightarrow 2\pi^++2\pi^-, \tag{7}$$

the rate $R_4(2^+,2^-)$ is, of course, distinct from R_4, the latter being the sum over all charge channels for $N=4$. To get $R_4(2^+,2^-)$ we must first project out that part of each I state which refers to the given charge partition and then sum the corresponding charge-partition probability over the I states.

All of these projections have in common the property of symmetry between particles of like charge. They are distinguished (always for a given charge partition) by additional properties of symmetry and (or) antisymmetry between particles of unlike charge.[7] The problem that we study is characterized as follows. We again take free-particle states for the given charge partition and we assume that the summing over the isotopic spin states tends to cancel the additional symmetry or antisymmetry properties just mentioned. Hence we approximately describe $P_N(\Omega)$ by introducing in Eq. (2) an expression for ϕ_N, which is symmetrized with respect to the sets of particles of like charge only. This paper is devoted to a discussion of four charged-pion stars from this point of view. Here the simplest contributions come from reaction (7); it is assumed that in addition only the channels $(2^+,2^-,1^0)$ and $(2^+,2^-,2^0)$ contribute (see, further, Sec. III).

Once the free-particle assumption is introduced, it becomes, of course, a decidable proposition to find out actually how good is the assumption of a SM with BE symmetrization between like particles. Let us first note that this last assumption is certainly not rigorously satisfied. This can be seen as follows. Suppose we ignore isotopic spin conservation altogether and then give all possible final isotopic spin states $(I=0, 1, \cdots, N)$ equal weight. The number of projections for the charge partition $N=n_++n_-+n_0$ is then

$$n'(n_+,n_-,n_0)=N!/n_+!\,n_-!\,n_0!. \tag{8}$$

Now it is physically obvious that if all these states have equal weight, the net result will be just the BE-symmetrization effect between like particles and nothing

[3] For a further discussion of this interpretation, see R. H. Milburn, Revs. Modern Phys. **27**, 1 (1955). It does not affect any subsequent argument if one takes $P_N(\Omega)=(\Omega/V)^{N-1}$, as is sometimes done.

[4] P. P. Srivastava and G. Sudarshan, Phys. Rev. **110**, 765 (1958).

[5] T. Kalogeropoulos, thesis, Lawrence Radiation Laboratory Report UCRL-8677, March 6, 1959 (unpublished).

[6] For an attempt to incorporate this, see LeRoy F. Cook, thesis, Lawrence Radiation Laboratory Report UCRL-8841, July 31, 1959 (unpublished).

[7] See A. Pais, Ann. Phys. (to be published).

else.[8] But if we adhere to isotopic spin conservation and only consider equal weights for $I=0,1$ states, the number of projections of the charge partition (n_+,n_-,n_0) is in general smaller than n', and therefore some symmetries other than that of like-particle kind may remain.

Even so, the approximation is perhaps not too bad. In Appendix I we discuss this in a little more detail; there it is shown that for $N=4$ the assumption of equal weight for the projections of the charge partition (7) into the various isotopic spin states happens to give *exactly* the BE effect between like particles only. It is then shown, again for $N=4$, that the SM assumption of equal weight for the isotopic spin states [rather than for the projection (7)] leads to a small deviation from the pure like-particle-only effect. For the case of $N=5$, 6, no such detailed studies have been performed, but it is made plausible that there also the present picture may be a reasonable approximation.

Thus it would appear that, as a first orientation at least, the present assumption of BE symmetrization is not much less well-founded than any other aspect of statistical considerations in this domain. We repeat, however, that we consider this work as an orienting approach rather than as a definitive answer and wish to give one more reason for this reservation. Of course, an adequate model should at the same time give a reasonable account of all combined aspects of the annihilation process, especially also of the mean multiplicity. The usual SM needs a radius of $\sim 2.5\ \hbar/\mu c$ to account for multiplicities.[9] Such a large radius is devoid of direct physical meaning. As we argue in Sec. IV, the inclusion of the BE effect tends to decrease this value of the radius, but at least in the way we proceed here, we cannot hope to fit the multiplicities with a value $\sim 0.75\ \hbar/\mu c$ for the radius, which was quoted above in connection with the angular-correlation effect. Until this problem is resolved, our results must be considered as tentative. Possibly improved angular-momentum considerations may here bridge the gap, or, perhaps the presence of a π-π interaction is making itself felt.[10]

II. STATISTICAL MODEL WITH BE-CORRELATIONS

A. The Correlation Function

As an orientation, consider first the case of $N=2$ with two identical particles, having momenta $\mathbf{p}_1$, $\mathbf{p}_2$. The corresponding $P_2(\Omega)$ plays an important role in what follows and is denoted by $\psi(12)$. Thus we can write

$$\psi(12)=\int\int|\phi^S(1,2)|^2 d\mathbf{r}_1 d\mathbf{r}_2, \tag{9}$$

where we integrate twice over a sphere $\Omega=4\pi\rho^3/3$, and

$$\phi^S(1,2)=(1/2^{\frac{1}{2}}V)\{\exp[i(\mathbf{p}_1\cdot\mathbf{r}_1+\mathbf{p}_2\cdot\mathbf{r}_2)] +\exp[i(\mathbf{p}_2\cdot\mathbf{r}_1+\mathbf{p}_1\cdot\mathbf{r}_2)]\}. \tag{10}$$

Thus, on integration we obtain[11]

$$\psi(12)\approx 1+9\left(\frac{\cos t}{t^2}-\frac{\sin t}{t^3}\right)^2,\ t=|\mathbf{p}_1-\mathbf{p}_2|\rho,\ \text{(sphere)}. \tag{11}$$

Evidently $\psi(12)$ as defined by Eqs. (9) and (10) no longer depends only on the size of the interaction volume Ω but also on its shape. It is premature to discuss this shape dependence in any detail, but one point is of some computational interest, namely that $\psi(12)$ for a spherical model, given by Eq. (11), differs very little from $\psi(12)$ for a Gaussian-shaped volume:

$$\psi(12)=\int\int|\phi^S(1,2)|^2\exp[-(r_1^2+r_2^2)/2\lambda]d\mathbf{r}_1 d\mathbf{r}_2$$
$$\approx 1+\exp(-s^2),\ s=|\mathbf{p}_1-\mathbf{p}_2|\lambda^{\frac{1}{2}},\ \text{(Gaussian)}, \tag{12}$$

where we integrate twice over all space. This well-known property of the Fourier transform of a sphere relative to that of a Gaussian is shown in Fig. 1 where the two curves refer to a ratio of ρ to $\lambda^{\frac{1}{2}}$ given by

$$\rho=2.15\lambda^{\frac{1}{2}}. \tag{13}$$

The Gaussian model simplifies some computations to follow and therefore we shall adopt it from here on. However, we shall continue to refer to the "radius" ρ of the interaction volume—by which we mean the quantity related to λ by Eq. (13).

In one further respect we have used an argument of convenience to simplify the calculations as much as possible before reverting to numerical evaluation techniques. Instead of Eq. (12) we have actually used its relativistic counterpart,

$$\psi(12)=1+e^{-\lambda x_{12}}, \tag{14a}$$

where

$$x_{12}=(\mathbf{p}_1-\mathbf{p}_2)^2-(\omega_1-\omega_2)^2. \tag{14b}$$

This is indeed convenient because we have to deal with integrals of the type (5) but with a number of ψ functions—the "correlation functions"—entering into the integrand. Thus the relativistic scalar form of $\psi(x)$ makes it possible to make simplifying Lorentz transformations on the integrand. Of course, it must be asked how much difference it makes to use Eq. (14) as

[8] Proof: if all states have equal weight, we can as well choose a set of base states that have the following properties: (a) they have the desired BE symmetry to begin with; (b) they are orthogonal; (c) their number is just equal to n'. For an example of such a set of states for $N=4$, see Appendix I.

[9] See for example O. Chamberlain, G. Goldhaber, L. Jauneau, T. Kalogeropoulos, E. Segrè, and R. Silberberg, Phys. Rev. **113**, 1615 (1959).

[10] E. Eberle, Nuovo cimento 8, 619 (1958); T. Gôto, Nuovo cimento 8, 625 (1958); and F. Cerulus, Nuovo cimento 14, 827 (1959).

[11] From here on we use the symbol $\approx$ to denote equality apart from such constant factors that do not affect the angular correlations under consideration.

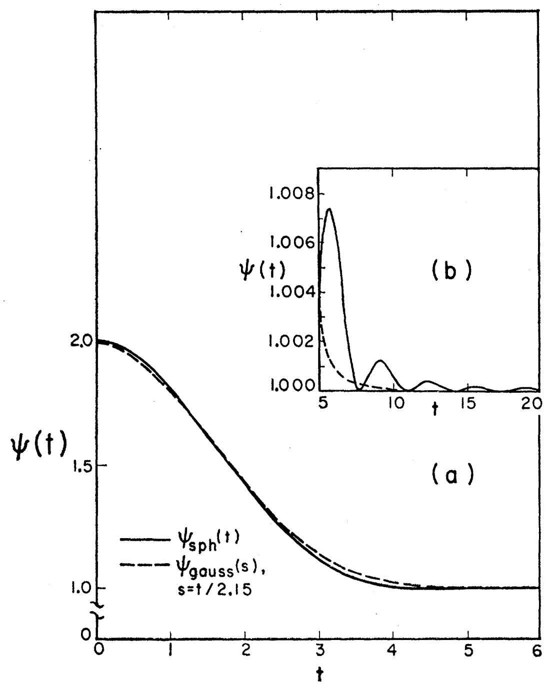

FIG. 1. Evaluation of the correlation functions as a function of the argument. Here $\psi_{\text{sph}}{}^{(t)}$ and $\psi_{\text{gauss}}{}^{(s)}$ correspond to the spherical and Gaussian models, respectively. As can be seen from the figure, the curves corresponding to the two models differ by about 2% at most. Note that the insert [Fig. 1(b)] is enlarged by a factor of 100 vertically and is reduced by a factor of 5 horizontally.

compared to Eq. (12). In two regions—$|\mathbf{p}_1|$, $|\mathbf{p}_2|\ll\mu$ and $|\mathbf{p}_1|\simeq|\mathbf{p}_2|$—the difference is small. As the momentum distribution in annihilation is fairly sharply peaked (certainly for $N=5, 6$), it follows that the replacement of Eq. (12) by Eq. (14) cannot change the results drastically. We have made a numerical check of this, which is mentioned below.

Instead of Eq. (4), for R_N we now have

$$R_N\approx\int\cdots\int\frac{d\mathbf{p}_1\cdots d\mathbf{p}_N}{\omega_1\cdots\omega_N}P_N(\rho,\,\mathbf{p}_1\cdots\mathbf{p}_N)\times\delta(W-\sum\omega_i)\delta(\sum\mathbf{p}_i).\quad(15)$$

For the case of reaction (7), we must symmetrize separately with respect to two pairs of particles, and hence P_4 is a product of two correlation functions ψ. The same is true for the channel $(2^+,2^-,1^0)$, while for $(2^+,2^-,2^0)$, P_6 is the product of three correlation functions.

Thus we see immediately that the deviations in angular correlations due to the expression (15) as compared to the usual SM must vanish in two limiting cases. First, ψ approaches a constant for $\rho\to 0$ [see Eqs. (13) and (14)] and we are back to the SM result—for small interaction volume, the BE correlations have no opportunity to develop. Second, for $\rho\to\infty$ (or rather if Ω tends to V), it follows from Eq. (2) that the BE effects will be confined more and more to such configurations where two participating momenta are more and more equal to each other. Hence, the weight of the configurations affected by the BE effect gets smaller and smaller and can be ignored in the limit considered, so that also for $\rho\to\infty$ we reach the SM values. Hence an optimum finite ρ exists for which the BE effects are most marked. This is shown quantitatively below.

We use $\Phi_N{}^l(y)$ and $\Phi_N{}^u(y)$ to denote the distribution in $y=\cos\theta$ of pion pairs of like and unlike charge, respectively (θ is the pair angle in the $\bar{p}p$ c.m. frame). For $\rho\to 0$, both these functions approach the common limit of the SM distribution denoted by $\Phi^{\text{SM}}(y)$. The ratio of pairs emitted in the backward hemisphere to those in the forward is denoted by γ. Specifically, γ^l, γ^u, and γ^{SM} denotes this ratio for the cases of like pairs, unlike pairs, and the statistical model without correlation functions, respectively. In the following discussion ψ means the relativistic expression (14) except in Eq. (29).

B. Calculation of the Correlation Effects

1. $\bar{p}+p\to 2\pi^++2\pi^-$

We have

$$R_4(2^+,2^-)\approx\int\frac{d\mathbf{p}_1\cdots d\mathbf{p}_4}{\omega_1\cdots\omega_4}\psi(12)\psi(34)\times\delta\left(W-\sum_1^4\omega_i\right)\delta\left(\sum_1^4\mathbf{p}_i\right).\quad(16)$$

To find $\Phi_4{}^l$ we integrate only over $\mathbf{p}_3$, $\mathbf{p}_4$, $|\mathbf{p}_1|$, $|\mathbf{p}_2|$. The integration over the 3, 4 variables is simplified by going to the system where $\mathbf{p}_3+\mathbf{p}_4=0$ and using invariance arguments. The result is

$$\Phi_4{}^l(y)\approx\int\int p_1p_2\psi(12)d\omega_1d\omega_2\times F_2(W_{12})\psi(W_{12}{}^2-4\mu^2),\quad(17\text{a})$$

where

$$W_{12}{}^2=(W-\omega_1-\omega_2)^2-(\mathbf{p}_1+\mathbf{p}_2)^2.\quad(17\text{b})$$

Here

$$F_2(W)=2\pi[1-4\mu^2/W^2]^{\frac{1}{2}}\quad(18)$$

is the two-body phase space. The (ω_1,ω_2) integration is bounded by

$$\omega_1\geq\mu,\quad\omega_2\geq\mu,\quad W_{12}{}^2\geq 4\mu^2.\quad(19)$$

To find $\Phi_4{}^u$ we integrate in Eq. (16) over all variables except the angle between particles 1 and 3. The result is

$$\Phi_4{}^u(y)\approx\int\int p_1p_2d\omega_1d\omega_2\,F_2(W_{12})Z(\mathbf{p}_1,\mathbf{p}_2,\xi),\quad(20)$$

where

$$Z(\mathbf{p}_1,\mathbf{p}_2,\xi)=1+\frac{e^{-2\lambda A_1}}{2\lambda|\mathbf{B}_1|\xi}\sinh(2\lambda|\mathbf{B}_1|\xi)+\frac{e^{-2\lambda A_2}}{2\lambda|\mathbf{B}_2|\xi}\times\sinh(2\lambda|\mathbf{B}_2|\xi)+\frac{e^{-2\lambda(A_1+A_2)}}{2\lambda|\mathbf{B}_1+\mathbf{B}_2|\xi}\times\sinh(2\lambda|\mathbf{B}_1+\mathbf{B}_2|\xi). \quad (21)$$

Here we have

$$\xi=[\tfrac{1}{4}W_{12}^2-\mu^2]^{\frac{1}{2}}, \quad (22)$$

$$2A_i=W\omega_i-(\omega_1\omega_2-\mathbf{p}_1\cdot\mathbf{p}_2)-3\mu^2, \quad (23a)$$

$$\mathbf{B}_i=\epsilon_i\mathbf{p}_i+\epsilon_i(\mathbf{p}_1+\mathbf{p}_2)\left[\frac{\omega_i}{W_{12}}+\frac{\mathbf{p}_i\cdot(\mathbf{p}_1+\mathbf{p}_2)}{(\mathbf{p}_1+\mathbf{p}_2)^2}\times\left(\frac{W-\omega_1-\omega_2}{W_{12}}-1\right)\right], \quad (23b)$$

and $i=1, 2$, $\epsilon_1=-1$, and $\epsilon_2=+1$. The integration limits are again given by Eq. (19).

2. $\bar{p}+p\rightarrow 2\pi^++2\pi^-+\pi^0$

We start from

$$R_5(2^+,2^-,1^0)\approx\int\cdots\int\frac{d\mathbf{p}_1\cdots d\mathbf{p}_5}{\omega_1\cdots\omega_5}\psi(12)\psi(34)\times\delta\left(W-\sum_1^5\omega_i\right)\delta\left(\sum_1^5\mathbf{p}_i\right). \quad (24)$$

To find Φ_5^l, integrate over all variables except the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$. The integration over the (3,4,5) variables is best performed in the (3,4,5) rest system, and one finds

$$\Phi_5^l(y)\approx\int_\mu\int_\mu p_1p_2d\omega_1d\omega_2\,\psi(12)F_{2+1}(W_{12}), \quad (25)$$

where

$$W_{12}^2\geq 9\mu^2.$$

Here $F_{2+1}(W_{12})$ is the three-particle phase space for two like plus one distinct particle. We have

$$F_{2+1}(W)=\int\frac{d\mathbf{p}_3d\mathbf{p}_4d\mathbf{p}_5}{\omega_3\omega_4\omega_5}\psi(34)\delta\left(W-\sum_3^5\omega_i\right)\delta\left(\sum_3^5\mathbf{p}_i\right),$$

which can be reduced further by integrating in the (3,4) rest system. This yields

$$F_{2+1}(W)=\int p_5d\omega_5\,F_2(W_5)\psi(W_5^2-4\mu^2),\qquad \mu\leq\omega_5\leq(W^2-3\mu^2)/2W, \quad (26)$$

and

$$W_5^2=W^2+\mu^2-2W\omega_5.$$

Proceeding in a similar way with Φ_5^u, one gets

$$\Phi_5^u\approx\int\int p_1p_2d\omega_1d\omega_2\int p_5d\omega_5\times F_2\{W_5(W_{12})\}Z(\mathbf{p}_1,\mathbf{p}_2,\xi'), \quad (27)$$

where

$$W_5^2(W_{12})=W_{12}^2+\mu^2-2W_{12}\omega_5.$$

The bounds in the (1,2,5) integrations are again as given in Eqs. (25) and (26), Z is as defined in Eq. (21), and we have

$$\xi'=[\tfrac{1}{4}W_5^2(W_{12})-\mu^2]^{\frac{1}{2}}. \quad (28)$$

We next give an expression for Φ_5^l where the nonrelativistic form (12) of the correlation function is used which we shall here label ψ_{NR}. The starting point is again Eq. (24), with ψ replaced by ψ_{NR}. Thus we have

$$\Phi_5{}^l{}_{\mathrm{NR}}(y)\approx\int\int\frac{d\mathbf{p}_1d\mathbf{p}_2}{\omega_1\omega_2}\psi_{\mathrm{NR}}(12)G(W_{12}),$$

where

$$W_{12}^2\geq 9\mu^2.$$

Here G has the form (in the 3, 4 rest system):

$$G(W_{12})=\int\frac{d\mathbf{p}_5}{\omega_5}\int\frac{d\mathbf{p}_3'd\mathbf{p}_4'}{\omega_3'\omega_4'}\delta(\mathbf{p}_3'+\mathbf{p}_4')\times\delta(W_{125}-\omega_3'-\omega_4')\psi_{\mathrm{NR}}(t),$$

where

$$W_{125}=(W-\omega_1-\omega_2-\omega_5)^2-(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_5)^2,$$

$$\psi_{\mathrm{NR}}(t)=1+\exp(-\lambda t^2),$$

$$t^2=4\mathbf{p}'_3{}^2\left[1+\frac{v^2z^2}{1-v^2}\right],$$

$$v=\left|\frac{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_5}{W-(\omega_1+\omega_2+\omega_5)}\right|,$$

and z is the cosine of the angle between $\mathbf{p}_3'$ and $(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_5)$. After some further transformations, we get

$$\Phi_5{}^l{}_{\mathrm{NR}}(y)\approx\int\int\frac{d\mathbf{p}_1d\mathbf{p}_2}{\omega_1\omega_2}\psi_{\mathrm{NR}}(12)\int_{-1}^{+1}dx\int p_5d\omega_5\times F_2(W_{125})\bar{\psi}(q),$$

where

$$\bar{\psi}(q)=\int_{-1}^{+1}dz[1+\exp(-\lambda\eta^2)],$$

$$\eta^2=4q^2\left[1+\frac{v^2}{1-v^2}z^2\right], \quad (29)$$

$$q^2=\tfrac{1}{4}W_{125}^2-\mu^2,$$

and x is the cosine of the angle between $\mathbf{p}_5$ and $\mathbf{p}_1+\mathbf{p}_2$.

The bounds of the (ω_5,x) domain are given by

$$W_{12}^2-3\mu^2-2\omega_5(W-\omega_1-\omega_2)-2p_5|\mathbf{p}_1+\mathbf{p}_2|x\geq 0,$$

while the limits on the (ω_1,ω_2) domain are again as indicated in Eq. (25).

In the next section we shall discuss the relation between Eqs. (25) and (29) from a numerical point of view. Here we only note the considerable advantage that the use of an invariant correlation function brings with it in simplifying the integrals.

3. $\bar{p}+p\rightarrow 2\pi^++2\pi^-+2\pi^0$

In this case, the starting point is

$$R_6(2^+,2^-,2^0)\approx\int\cdots\int\frac{d\mathbf{p}_1\cdots d\mathbf{p}_6}{\omega_1\cdots\omega_6}\psi(12)\psi(34)\psi(56)\times\delta\left(W-\sum_1^6\omega^i\right)\delta\left(\sum_1^6\mathbf{p}_i\right). \quad (30)$$

It will be obvious that

$$\Phi_6{}^l(y)\approx\int_\mu\int_\mu p_1p_2d\omega_1d\omega_2\,F_{2+2}(W_{12})\psi(12), \quad (31)$$

with

$$W_{12}^2\geq 16\mu^2,$$

where $F_{2+2}(W)$ is the four-particle phase space for two pairs of like particles. Thus we have

$$F_{2+2}(W)=\int_{-1}^{+1}dy\,\Phi_4{}^l(y,W), \quad (32)$$

where $\Phi_4{}^l$ is given by Eq. (17). For $\Phi_6{}^u(y)$ one finds, after some transformations,

$$\Phi_6{}^u(y)=\int_\mu\int_\mu p_1p_2d\omega_1d\omega_2\int_\mu\int_\mu p_3p_4d\omega_3d\omega_4\int_{-1}^{+1}dx\times\psi(34)F_2(W_{12}')Z(\mathbf{p}_1,\mathbf{p}_2,\xi''), \quad (33)$$

with

$$x=\cos\theta_{34},$$

$$(W_{12}')^2=W_{12}^2+2\mu^2+2(\omega_3\omega_4-p_3p_4x)-2W_{12}(\omega_3+\omega_4), \quad (34)$$

and

$$\xi''=\left[\tfrac{1}{4}(W_{12}')^2-\mu^2\right]^{\frac{1}{2}}.$$

In Eq. (33) the respective integration domains are further bounded by

$$W_{12}'\geq 2\mu,\quad\text{for}\quad(\omega_3,\omega_4),$$

and

$$W_{12}\geq 4\mu,\quad\text{for}\quad(\omega_1,\omega_2). \quad (35)$$

III. NUMERICAL RESULTS AND COMPARISON WITH EXPERIMENT

To compare the effect of the BE correlation functions with experiment, we have to evaluate the contribution of each charge channel to the sample under consideration. Four-prong events of the type $(2^+,2^-,n^0)$ consist (at the energy of the experiment under discussion) of four charge channels, namely $(2^+,2^-)$, $(2^+,2^-,1^0)$, $(2^+,2^-,2^0)$, and $(2^+,2^-,3^0)$. As has been shown in Sec. II, the complexity of the integration involved in evaluating the distribution functions of the pion-pair angles, $\Phi(y)$, increases with the number of particles participating. For seven pions, a new type of correlation enters the problem, namely, that between three identical bosons. The experimental indication is that the contribution of seven pions to the $(2^+,2^-,n^0)$ sample is no more than $7\pm5\%$. We have thus restricted our calculations to the first three charge channels only. For each of the three channels we evaluated the functions $\Phi^l(y)$ and $\Phi^u(y)$. To investigate the behavior of $\Phi(y)$ as a function of the radius of interaction, ρ, we evaluated Φ for six values of ρ, i.e., $\rho=0$, 0.3, 0.5, 0.75, 1, and 2 (in units of $\hbar/\mu c$).

A. Numerical Evaluation

The distribution functions of the pion-pair angles were numerically integrated either by the Simpson-rule technique or the Monte Carlo method, depending on the complexity of the problem.

Wherever possible, the symmetry properties of the integrand were used. Each function $\Phi(y)$ was evaluated at equally spaced intervals of $y=\cos\theta$ for each of the above-mentioned ρ values. The functions $\Phi_4{}^l$, $\Phi_4{}^u$, $\Phi_5{}^l$, $\Phi_5{}^u$ and $\Phi_6{}^l$ were evaluated on an IBM-650 computer by the Simpson-rule technique.[12] The more involved integrations of the functions, $\Phi_6{}^u$ and $\Phi_5{}^l{}_{\mathrm{NR}}$ were performed by the Monte Carlo method on the IBM 704 (see Appendix II).

B. The Pion-Pair Angle-Distribution Functions, $\Phi(y)$

Here we will illustrate the deviation of the functions Φ^l and Φ^u which include BE correlation effects, from the one obtained from the conventional statistical model Φ^{SM}. In Fig. 2 we show Φ^l, Φ^u, and Φ^{SM} for one particular radius, $\rho=0.75$. As can be seen from a comparison of Figs. 2(a), 2(b), and 2(c) corresponding to 4, 5, and 6 pions, respectively, the variation of Φ^{SM} towards greater isotropy with increasing N is very marked. The BE correlation effects become somewhat less pronounced as N increases.

In Fig. 3 we display the ratios $\Phi_N{}^l/\Phi_N{}^{\mathrm{SM}}$, and $\Phi_N{}^u/\Phi_N{}^{\mathrm{SM}}$. These ratios indicate, perhaps even more

[12] The $\Phi_4{}^l$, $\Phi_4{}^u$ functions were calculated to 2%, the $\Phi_5{}^l$ to 3%, and the $\Phi_5{}^u$ to 5% accuracy. Several individual y values for $\Phi_5{}^u$ were evaluated to a 2% accuracy as a check on the calculations. For the function $\Phi_6{}^l$ [Eqs. (31)] the calculation was carried out in two steps. First the function $F_{2+2}(W)$ [Eq. (32)] was computed by the 2% Simpson rule and fitted with an expansion. Second, $\Phi_6{}^l$ was computed by using the above expansion with the 2% Simpson rule. As a cross check, several values of $\Phi_6{}^l$ were also computed by the Monte Carlo method (see Appendix II).

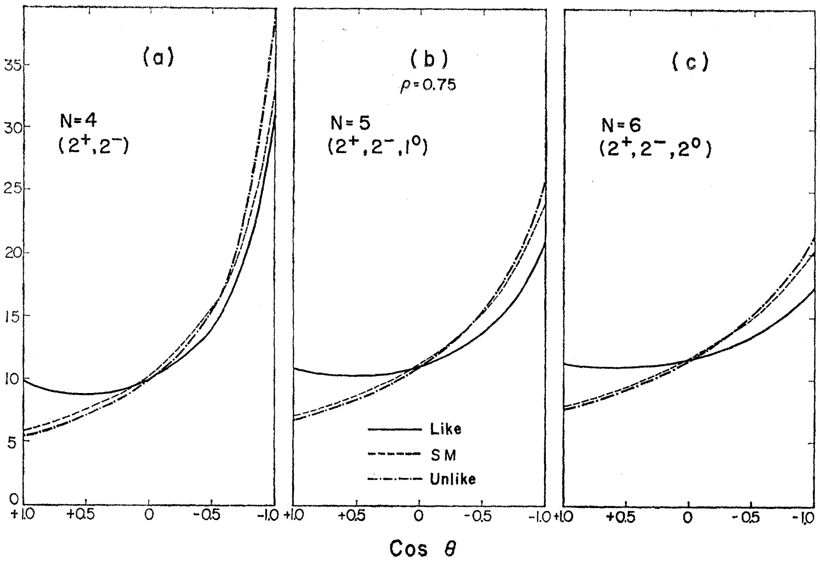

FIG. 2. The distribution functions of pion-pair angles. The functions $\Phi_N{}^l(\cos\theta)$ and $\Phi_N{}^u(\cos\theta)$ referring to the distributions of angles between pion pairs are plotted for like and unlike pions, respectively. We illustrate the behavior of the functions for $\rho=0.75\ \hbar/\mu c$. Also shown, for comparison, is Φ^{SM}, the distribution without correlation functions. All curves are normalized to the same area with arbitrary units for Φ. Figures 2(a), 2(b), and 2(c) refer to $N=4$, $N=5$, and $N=6$, respectively.

clearly than the distribution functions themselves, the effect of the BE correlation functions on the statistical model. It is interesting to note that these ratios for $N=4$, 5, and 6 fall rather close together. The BE correlation functions for like pions have the effect of raising the distribution for small pair angles and lowering it for large pair angles. For the unlike pions, the converse is true.

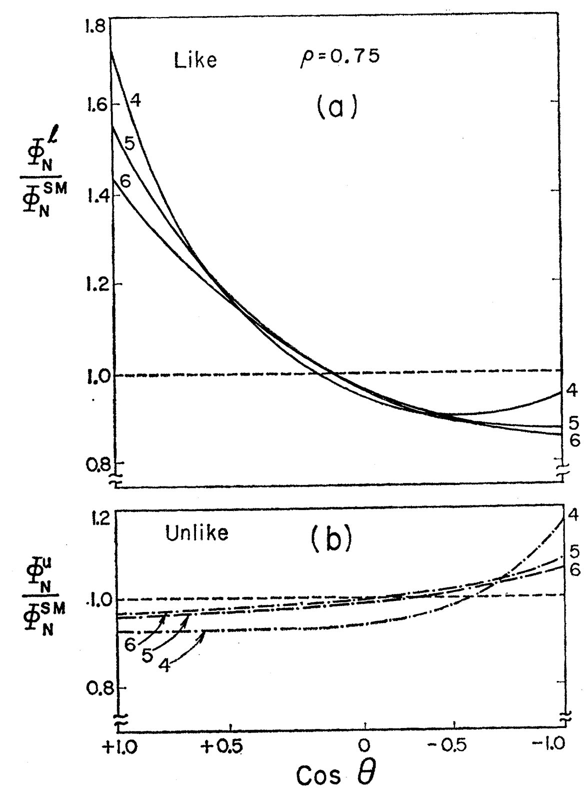

FIG. 3. The ratios $\Phi_N/\Phi_N{}^{SM}$ for like and unlike pions.

The experimental observations with which we wish to compare our results have been expressed by a quantity, γ. The ratio γ is defined as the ratio of the number of pion-pair angles greater than 90 deg to the number of pion pair angles less than 90 deg. Thus we obtain

$$\gamma_N{}^J=\int_{-1}^{0}\Phi_N{}^J(y)dy\bigg/\int_{0}^{1}\Phi_N{}^J(y)dy,$$

where J corresponds to l, u, and SM, respectively. The ratio γ gives a convenient quantitative measure of the modifications occurring in the SM by the introduction of the BE correlation functions as a function of ρ. In Fig. 4, we present γ for $N=4$, 5, and 6 for both the like and unlike correlation functions. It is evident that the maximum effect of the correlation function occurs for values of ρ between $\frac{1}{2}$ and $\frac{3}{4}\ \hbar/\mu c$.

C. Comparison between Invariant and Noninvariant Correlation Functions

The relativistic scalar form of the correlation function $\psi(x)$ [Eq. (14)] facilitates the calculations of rather involved integrations. In order to test the validity of this approximation, we performed the calculation also in the nonrelativistic form for two selected cases, i.e., $\Phi_4{}^l$ and $\Phi_5{}^l$. The result for five pions [see Eq. (29) for $\Phi_5{}^l{}_{NR}$] is illustrated in Fig. 5 in which we show both $\gamma_5{}^l$ (relativistic) and $\gamma_5{}^l{}_{NR}$. As can be seen, the qualitative features of the distributions are similar. Results of the two calculations differ by 10% from each other at $\rho=0.75$ with the relativistic form deviating more from the statistical model. The corresponding calculations performed for four pions (not given in this paper) give essentially the same results as for five pions.

D. Comparison with Experiment

To enable us to compare the calculated distribution functions with the experimental data, we need to know the relative weights of the contributing charge channels. The latter were obtained from the experimental data by determining both the average number of neutral pions produced in $2\pi^+2\pi^-$ stars and the distribution of the energy not accounted for by the charged pions ($E_{\rm miss}$). A detailed discussion of the method is given below.

(a) The average missing energy per event $\langle E_{\rm miss}\rangle$ was determined for the sample $(2^+,2^-,n^0)$.

(b) The average neutral-pion energy $\langle E_{\pi^0}\rangle$ was estimated from the experimental-average charged-pion energy. Here we assume that for stars of a given multiplicity the charged and neutral pions have the same energy spectrum. A small correction which lowers $\langle E_{\pi^0}\rangle$ has been applied. This correction arises from the fact that $\langle E_{\pi^\pm}\rangle$ was obtained from stars with four, five, six, and seven pions, while $\langle E_{\pi^0}\rangle$ comes from stars with five, six, and seven pions.

(c) The average number of neutral pions is thus $\langle n^0\rangle = \langle E_{\rm miss}\rangle/\langle E_{\pi^0}\rangle$. The experimental result is $\langle n^0\rangle = 1.15\pm0.1$. This value is in excellent agreement with the one obtained from a direct count of electron pairs produced from the conversion of the π^0-decay γ rays, viz., $\langle n_0\rangle = 1.1\pm0.1$.[13]

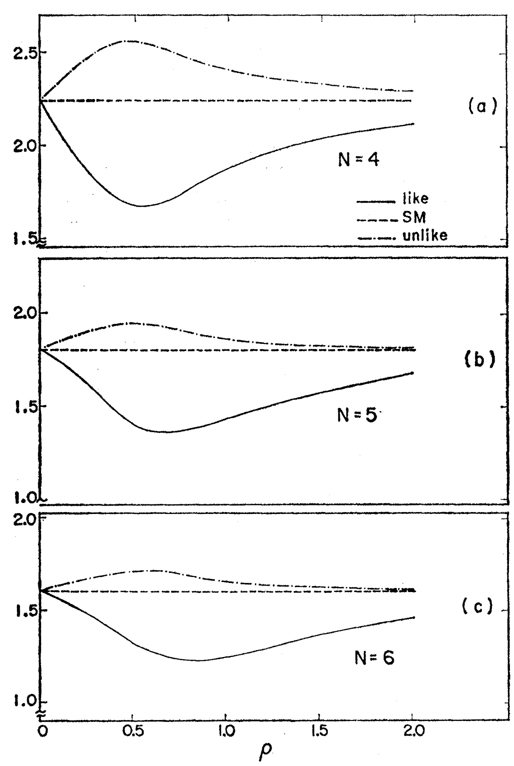

FIG. 4. The ratio γ for like and unlike pions as a function of radius ρ. Here ρ is given in unit of $\hbar/\mu c$. All calculations correspond to four charged pions ($2\pi^+$ and $2\pi^-$) with zero, one, or two π^0 mesons (i.e., $N=4$, 5, and 6, respectively).

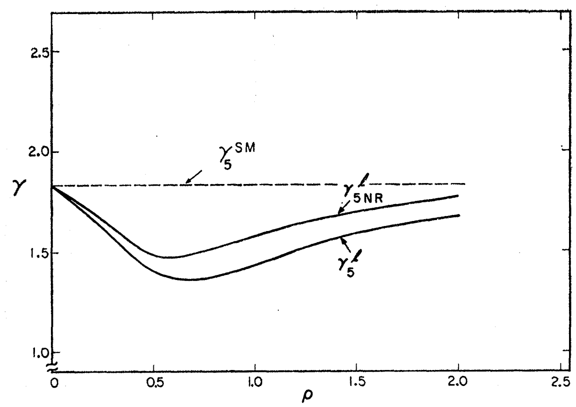

FIG. 5. A comparison of the γ distributions calculated with the relativistic and nonrelativistic correlation functions, respectively. Here $\gamma_5{}^l{}_{\rm NR}$ refers to the distribution obtained from the nonrelativistic correlation function, whereas $\gamma_5{}^l$ refers to the relativistic one.

(d) The experimental distribution of $E_{\rm miss}$ corresponds to a folded distribution of the energy in neutral pions. If one could unfold this distribution completely, it would determine the weights of the corresponding charge channels uniquely. The experimental errors in the momentum determination and the fluctuation in neutral-pion energies do not permit such a complete unfolding. It is possible, however, to set narrow limits for the two end points of the distribution. From these we obtain the corresponding weights $S_4=0.15\pm0.05$ and $S_7=0.07\pm0.05$ for $N=4$ and $N=7$, respectively.

(e) To solve for the weights S_5 and S_6, we used the two equations:

$$\sum_{N=4}^{7} S_N=1 \quad \text{and} \quad S_5+2S_6+3S_7=\langle n^0\rangle.$$

In these equations we allow S_4, S_7, and $\langle n^0\rangle$ to vary within their quoted uncertainty, imposing the constraint that only one maximum can occur in the multiplicity distribution.

Finally, as these calculations for Φ have not been extended to seven pions, we have added the seven-pion contribution to that from six pions. The ratios of the resulting weights are $S_4{:}S_5{:}(S_6+S_7)=0.15{:}0.60{:}0.25$, with limiting values of 0.10:0.70:0.20 and 0.20:0.50:0.30, respectively. Fortunately $\Phi_{\rm av}$, given by $\Phi_{\rm av} = S_4\Phi_4+S_5\Phi_5+(S_6+S_7)\Phi_6$, is very insensitive to which of the above sets is chosen. In Fig. 6, the experimental distribution is compared with that calculated for $\rho=0.75$. The dashed curve gives the result of the SM.

[13] Rein Silberberg, thesis, University of California Radiation Laboratory Report UCRL-9183, May, 1960 (unpublished).

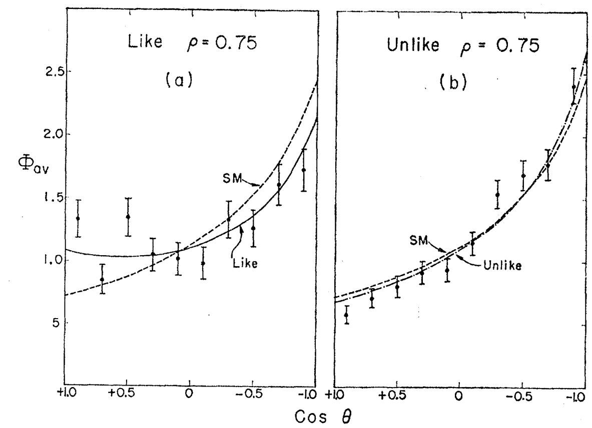

FIG. 6. The functions $\Phi_{av}(\cos\theta)$ computed at $\rho=0.75$ are compared with the experimental distribution of angles between pion pairs. Figures 6(a) and 6(b) give the distributions for like and unlike pions respectively. Also shown in each is the curve for $\Phi_{av}^{SM}(\cos\theta)$, the statistical distribution, without the effect of correlation functions. Here Φ_{av} represents an average of Φ_4, Φ_5, and Φ_6, weighted according to the individual charge channels. The experimental data comes from reference 1 (see also Table I, footnote a).

It is clear from this figure that the fit to the experimental data is improved for both like and unlike pions by the introduction of BE correlation functions.

In Tables I and II we give the experimentally determined values for γ together with a series of γ values calculated for various radii of interaction. An inspection of Table I, which lists also γ^{SM}, shows again that the bulk of the experimentally observed deviations from the SM can be accounted for by our calculations with a reasonable choice of ρ (i.e., ρ between $\frac{1}{2}$ to $\frac{3}{4}$ of $\hbar/\mu c$).[14] It cannot be concluded now whether the remaining discrepancy between experimental results and the SM including BE correlation effects, as evaluated here, is due to experimental uncertainty or to inadequacies of our model.

IV. CONCLUDING REMARKS

We have seen that the BE symmetrization leads to a fairly satisfactory possible interpretation of the observed angular distributions. We believe that this conclusion is of importance for the assessment of evidence for the existence of the strength of possible π-π interactions. The least the present results indicate is that if one wishes to extract information about such interactions from annihilation phenomena, such kinematic symmetry effects as here discussed must always be taken into account.

It may be asked whether further information can lead to arguments for or against the model here employed. Several possibilities exist for getting such information. In the first place one may study six- and higher-prong stars by the same method. Secondly, if the BE symmetrization is the major source for the deviations from the usual SM, this implies a specific dependence of quantities like γ^u, γ^l on the available annihilation energy, W. For the case $N=4$, this dependence is shown in Fig. 7. Here we have computed γ_4^l as a function of ρ for various values of W, the available energy in the center-of-mass system. We have chosen for W the energies 1.88, 2.5, and 4.4 Bev corresponding to $\bar{p}$ laboratory momenta of 0, 2.25, and 6 Bev/c, respectively. It can be seen from Fig. 7 that the correlation effects occur at smaller values of the radius as the energy increases. If a radius of interaction is a meaningful quantity for the annihilation and does not depend critically on the incident antiproton energy, it might be expected that the correlation effects due to BE statistics will decrease at higher bombarding energy. Studies of correlation effects as a function of W may thus be a test for the ideas discussed in this paper. Of course, with increasing W, the relative fraction of four-pion annihilations will decrease. It is therefore indicated that if one wishes to pursue the annihilation process in more detail, an unambiguous separation into the various individual multiplicities will become quite imperative. Only if this is done will curves like those of Fig. 7 and similar ones for other given N be of any use.

Finally, a comment may be made about the question of the mean pion multiplicity. It has been suggested by various people that the high ρ value obtained from the SM may be reduced by taking into account the

TABLE I. Comparison between the experimental values for γ^l and γ^u and the corresponding values derived by use of the BE correlation functions for $\rho=0.5$ and 0.75. Also shown is the value for the usual Fermi SM. All the theoretical values have been averaged over the four-, five-, and six-pion distributions as discussed in the text.

	γ_{expt}	γ_{av} $\rho=0.5$	γ_{av} $\rho=0.75$	γ_{av}^{SM}
Like	1.23 ± 0.10[a]	1.41	1.38	1.80
Unlike	2.18 ± 0.12	1.95	1.91	

[a] The experimental data quoted in this paper is essentially the same as given in reference 1. A small improvement in the available data has, however, been incorporated involving (1) some additional events, namely a total of 752 like and 1504 unlike pion pairs coming from $(2^+,2^-,n^0)$ stars have been used, and (2) a complete recalculation of all the center-of-mass momentum and angle values making use of the known incident beam momentum $P_{\bar{p}}=1.05$ Bev/c rather than the measured value for each individual annihilation event.

TABLE II. List of computed $(\gamma^l)_{av}$ and $(\gamma^u)_{av}$ values. The values for $\rho=0.5$ and 0.75 are repeated here for clarity.

ρ $(\hbar/\mu c)$	$(\gamma^l)_{av}$	$(\gamma^u)_{av}$
0.3	1.57	1.91
0.5	1.41	1.95
0.75	1.38	1.91
1.0	1.44	1.87
2.0	1.66	1.79

[14] It should be noted that the γ distribution, calculated by using the noninvariant form of the correlation function $\psi(x)$, will probably give a poorer fit to the experimental data than the invariant form. This is illustrated in Fig. 5.

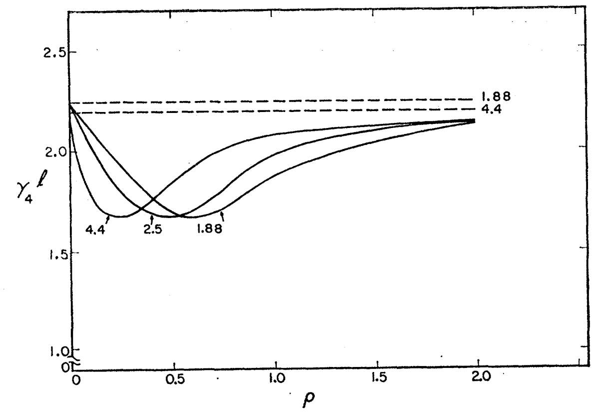

FIG. 7. The distribution of γ_4^l as a function of ρ for various incident $\bar{p}$ energies. The energies in the center-of-mass system are $W=1.88$, 2.5, and 4.4 Bev and the curves are labeled with W. These correspond to $\bar{p}$-p collisions at laboratory momenta 0, 2.25, and 6 Bev/c, respectively. The dotted curves refer to γ_4^{SM}.

existence of π isobars. These isobars are often thought of as pseudoparticles compounded of two (or more) π mesons and with prescribed spin and angular momentum. It is clear on qualitative grounds that the existence of such structures would reduce the ρ value for given average multiplicity. Again on qualitative grounds it follows that under the same conditions the present model also will lead to a reduction in the ρ value. This is because correlated pairs are somewhere between pseudo two-body systems and totally free pairs. Preliminary estimates indicate, however, that the BE effect seems to be insufficient by itself to lead to the right multiplicity for $\rho \sim 0.75\ \hbar/\mu c$. It must be added, however, that several points are at present not quite clear to us. In particular, it may be asked whether the use of the factor $(N!)^{-1}$ occurring in Eq. (6) is indeed a proper way to deal with the question of indistinguishability. This particular N dependence plays a sizable role, of course, in the theoretical determination of average multiplicities. Thus a further study of the effect of BE symmetries is needed in conjunction with improved considerations on angular momentum and on the possible role of strong π-π forces.

ACKNOWLEDGMENT

We wish to thank Dr. T. Kalogeropoulos for his important contributions in the initial stages of this work and also Dr. J. Lepore for many helpful discussions. We are grateful to Mrs. Marjory Simmons for help in the coding of all the numerical evaluations, and to Mr. B. Boyd, Mr. M. Guinan, and Mrs. F. Simon for help with graphical integrations. One of us (A.P.) would also like to thank Professor E. McMillan and the staff of the Lawrence Radiation Laboratory for the kind hospitality extended to him during his stay in Berkeley.

APPENDICES

I. Derivation of the BE Correlation Functions for $N=4$

We shall here give a rather detailed discussion for the case $N=4$ of the validity of the use of wave functions symmetrized with respect to like particles only. For the case $N=5$, 6, we shall content ourselves with some qualitative remarks.

Thus we consider the charge partition $(2^+,2^-)$. First note that if we would ignore isotopic spin (as described in the Introduction), we would have, by Eq. (8), six independent and orthogonal $(2^+,2^-)$ states for a given momentum configuration. Their spatial wave functions can be chosen as follows. Define $[ijkl]$ by

$$[ijkl]=\exp[i(\mathbf{p}_i\cdot\mathbf{x}_1+\mathbf{p}_j\cdot\mathbf{x}_2+\mathbf{p}_k\cdot\mathbf{x}_3+\mathbf{p}_l\cdot\mathbf{x}_4)]. \quad \text{(A1)}$$

Next define a symbol of the $[ijkl]$ type, with a bar over two letters to mean symmetrization with respect to the two momenta marked by a bar *and* symmetrization with respect to the two remaining momenta. For example,

$$[\bar{3}1\bar{4}2]=[3142]+[4132]+[3241]+[4231]. \quad \text{(A2)}$$

The six functions,

$$\begin{aligned} \xi_1&=[\bar{1}\bar{2}34], & \xi_4&=[3\bar{1}\bar{2}4],\\ \xi_2&=[\bar{1}34\bar{2}], & \xi_5&=[3\bar{1}4\bar{2}],\\ \xi_3&=[\bar{1}3\bar{2}4], & \xi_6&=[34\bar{1}\bar{2}], \end{aligned} \quad \text{(A3)}$$

form a complete orthogonal set of spatial functions spanning the configuration $(\mathbf{p}_1,\cdots,\mathbf{p}_4)$. (We referred to this set earlier in footnote 8.) Note that we have

$$\int_\Omega |\xi_i|^2 d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4 \approx \psi(12)\psi(34), \quad i=1,\cdots,6, \quad \text{(A4)}$$

where $\psi(12)$ is given by Eq. (11). Thus, as already stated for general N in the Introduction,[15] it is trivially correct that equal weight for all six states just means like particle symmetry and nothing else.

Next consider the $I=0, 1$ states for $N=4$. By the methods described in reference 7, we may label these states by their correlation numbers and divide them in the classes: (4), (31), (22), (211). The number of states pertaining to these classes is 1, 3, 2, and 3, respectively.[16] This totality of 9 states may be chosen as an orthogonal set. We must now project out the $(2^+,2^-)$ parts of these nine states. Observe that the three states of class (211) have projection null. This is easily seen from the Young tableaux corresponding to these states, which imply antisymmetry between the coordinates of three of the four particles, a condition that leads to identically vanishing wave functions for the charge partition in hand. Thus, the number of orthogonal isotopic spin projections for $I=0, 1$ is

[15] See also footnote 8.
[16] See reference 7, Eq. (10).

equal to $9-3=6$. Hence these six projections must be related to the functions $\xi_1,\cdots,\xi_6$ of Eq. (A3) by a unitary transformation. Thus equal weight for these projections again gives us the BE effect between like particles only, a result mentioned in the Introduction.

We verify this explicitly by constructing the various isotopic spin projections of $(2^+,2^-)$, a procedure that is also helpful for the rest of the argument. To do this, we construct first the following operators. Let $\boldsymbol{\phi}(i)$ denote the π field (an isotopic vector) at the point $\mathbf{x}_i$.[17] Put

$$A_{ijkl}=(\boldsymbol{\phi}(i)\cdot\boldsymbol{\phi}(j))(\boldsymbol{\phi}(k)\cdot\boldsymbol{\phi}(l)),$$
and
$$\mathbf{B}_{ijkl}=(\boldsymbol{\phi}(i)\cdot\boldsymbol{\phi}(j))(\boldsymbol{\phi}(k)\times\boldsymbol{\phi}(l)), \tag{A5}$$

and consider the nine operators:

$$\begin{aligned}
O_1^{(4)}&=A_{1234}+A_{1324}+A_{1423},\\
O_1^{(22)}&=2A_{1234}-A_{1324}-A_{1423},\\
O_2^{(22)}&=A_{1324}-A_{1423},\\
\mathbf{O}_1^{(31)}&=-2\mathbf{B}_{1234}+\mathbf{B}_{1324}+3\mathbf{B}_{1423}+\mathbf{B}_{2314}+3\mathbf{B}_{2413},\\
\mathbf{O}_2^{(31)}&=\mathbf{B}_{1324}+\mathbf{B}_{1423}-\mathbf{B}_{2314}-\mathbf{B}_{2413}-2\mathbf{B}_{3412},\\
\mathbf{O}_3^{(31)}&=\mathbf{B}_{1234}+\mathbf{B}_{1324}+\mathbf{B}_{2314},\\
\mathbf{O}_1^{(211)}&=-2\mathbf{B}_{1234}+\mathbf{B}_{1324}-\mathbf{B}_{1423}+\mathbf{B}_{2314}-\mathbf{B}_{2413},\\
\mathbf{O}_2^{(211)}&=-3\mathbf{B}_{1324}+\mathbf{B}_{1423}+3\mathbf{B}_{2314}-\mathbf{B}_{2413}-2\mathbf{B}_{3412},\\
\mathbf{O}_3^{(211)}&=\mathbf{B}_{1423}-\mathbf{B}_{2413}+\mathbf{B}_{3412}.
\end{aligned} \tag{A6}$$

These O operators have just the isotopic spin and symmetry properties required by the classes that label them by superscript; the subscript distinguishes the isotopic spin states within each class. The construction of this set of operators was first given by Halpern[18] and was also recently discussed elsewhere.[19] It is a simple matter to derive from these O operators the wave functions for the $(2^+,2^-)$ system. One imagines the $\boldsymbol{\phi}(i)$ to be Fourier-expanded and picks off in all possible ways the contributions to the momentum configurations $\mathbf{p}_1,\cdots,\mathbf{p}_4$ for the charge channel $(2^+,2^-)$. Evidently this procedure guarantees the combined BE symmetry with regard to charge *and* space coordinates. Proceeding in this way, one finds that the operators $\mathbf{O}_i^{(211)}$ for $i=1$, 2, and 3, give the zero result mentioned earlier. The nonzero functions are written generally as $\Psi(1234)$, where the arguments shall refer to the momentum labels now. It is convenient to write

$$\Psi(1234)=\Lambda(1234)+\Lambda(1324)+\Lambda(1432). \tag{A7}$$

One finds[20]

$$\begin{aligned}
\Lambda_1^{(4)}(1234)&=[\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6]\times1/24^{\frac12},\\
\Lambda_1^{(22)}(1234)&=(\pi_1^+\pi_2^+\pi_3^-\pi_4^-+\pi_1^-\pi_2^-\pi_3^+\pi_4^+)\times[2\xi_1-\xi_2-\xi_3-\xi_4-\xi_5+2\xi_6]\times1/48^{\frac12},\\
\Lambda_2^{(22)}(1234)&=[-\xi_2+\xi_3-\xi_4+\xi_5]\times1/16^{\frac12},\\
\Lambda_1^{(31)}(1234)&=[2\xi_1+\xi_2-\xi_3-\xi_4+\xi_5-2\xi_6]\times1/48^{\frac12},\\
\Lambda_2^{(31)}(1234)&=(\pi_1^+\pi_2^+\pi_3^-\pi_4^--\pi_1^-\pi_2^-\pi_3^+\pi_4^+)\times[\xi_2+\xi_3-\xi_4-\xi_5]\times1/16^{\frac12},\\
\Lambda_3^{(31)}(1234)&=[\xi_1-\xi_2+\xi_3+\xi_4-\xi_5-\xi_6]\times1/24^{\frac12}.
\end{aligned} \tag{A8}$$

Here $\pi_i^\pm$ denotes the amplitude for a $\pi^\pm$ meson with momentum $\mathbf{p}_i$. The space wave functions ξ_i are as defined in Eq. (A3). Now if the projections $\Psi(1234)$ into $(2^+,2^-)$ [given by Eqs. (A7) and (A8)] all have equal weight, the total rate of transition into a given momentum configuration is equal to

$$\sum_{\text{six states}}\int|\Psi(1234)|^2, \tag{A9}$$

where the summation is taken over the six states of different symmetry, and the integral means summation over charge and integration over space coordinates. Performing all these operations, one sees that all cross terms of the type $\xi_i^*\xi_j$ cancel. Hence, using also Eq. (A4), we have verified the property of pure BE symmetry between like particles only, which was proved previously on general grounds.

Let us next see what happens if we give equal weight to the various isotopic spin states, rather than to their $(2^+,2^-)$ projections, as is required in the SM. This means that we must weigh each of the six terms in Eq. (A9) with the branching ratio (or correlation coefficient) that gives the relative weight of the $(2^+,2^-)$ part in the mixture of (4^0), $(1^+,1^-,2^0)$, $(2^+,2^-)$ out of

[17] Explicit time dependence is not needed for what follows.
[18] F. Halpern, Ann. Phys. 7, 146 (1959); see Appendix.
[19] See reference 7, Appendix.
[20] This is apart from a common normalization factor $6^{-\frac12}$.

which the isotopic spin states are built up. These relative weights, which can be read off from Table IV, $N=4$ of reference 7, are 8/15 for class (4), 4/5 for class (31), and 1/3 for class (22). Hence it follows that now the over-all rate of transition into a given momentum configuration becomes, in the SM,

$$\sim\int \xi_1{}^*\xi_1+\frac{1}{54}\,\mathrm{Re}\int \xi_1{}^*(3\xi_2+\xi_3+3\xi_4+5\xi_5-18\xi_6).$$

The first integral just corresponds to pure BE symmetry between like particles. The second term constitutes a correction to this. To evaluate the magnitude of this correction term, we have computed $\Phi_4{}^l(y)$ for the case $\rho=0.75$ at three selected y values. The integrations, in which 80 individual integrals were involved, were performed on the IBM-704 computer. The results show that for $y=+1$, 0, and -1, the correction terms are all positive and amount to a 2%, 2.7%, and 2.1% change in $\Phi_4{}^l$, respectively. Since the corrections are all positive, the effect on γ is even smaller.

For N greater than four, we have not made such a detailed analysis, but merely note the following. Evidently something special happens for $N=4$ in that the number of $(2^+,2^-)$ states is equal to the actual number of projections of the orthogonal $I=0$, 1 states into this charge channel regardless of isotopic spin restriction to $I=0$, 1. This accounts to a large extent for the fact that the BE assumption is particularly good in this case. For $N=5$, 6, this equality is no longer true. Equation (8) tells us that the total number of $(2^+,2^-,1^0)$, $(2^+,2^-,2^0)$ states is equal to 30 and 90, respectively, wheras the number of orthogonal projections from $I=0$ and 1 is equal to 21 and 51, respectively.[21] As the latter numbers are of the same order as the former ones, it is at least plausible that most of the additional symmetries between unlike particles will cancel out, so that also here the assumption of like particle symmetry only may be a reasonable approximation.

II. The Monte Carlo Method Used for Multiple Integrals in Phase Space

For the evaluation of the n-fold integrals with $n>4$ occurring in the various expressions in many-particle phase space, we have used a Monte Carlo method (MCM) of integration. These calculations were coded in FORTRAN by Marjory Simmons and were evaluated with the IBM-704 computer of the University of California Computer Center in Berkeley.

An n-fold integral corresponds to a volume in $n+1$ dimensional space. This volume can be expressed as the average height of the function multiplied by the "area" of the domain of integration. Here the domain extends over n dimensions and contains all the permissible values of the variables. The MCM used here consists of generating n random numbers which, after suitable normalizations, correspond to a point in n-dimensional space. The essence of the method is just to ascertain whether this "point" lies inside (= success) or outside (= failure) of the domain of integration. For each "success" point, we then compute the value of the integrand. The sum over the values of the integrand divided by the number of "tries," N, for a sufficiently large N converges to a number proportional to the desired integral.

To be more specific, let us consider an n-fold integral and write

$$\Phi(\alpha,\beta)=\int\cdots\int_{\text{Domain } D(x_1,\cdots,x_n;\alpha)} f(x_1,\cdots,x_n;\alpha,\beta) \times dx_1\cdots dx_n. \quad \text{(A10)}$$

Here the integrations are to be carried out over an n-dimensional domain $D(x_1,\cdots,x_n;\alpha)$, where we have $f(x_1,\cdots,x_n;\alpha,\beta)\geq 0$. The n variables, x_k, are limited by known upper and lower bounds: $x_k{}^{\min}\leq x_k\leq x_k{}^{\max}$; for $k=1,\cdots,n$ ($x_k{}^{\min}$ and $x_k{}^{\max}$ are constants). The domain $D(x_1,\cdots,x_n;\alpha)$ over which the integration is carried out is an n-dimensional volume which is thus contained in, but is in general smaller than, $D^{\max}$, where

$$D^{\max}=\prod_{k=1}^{n}(x_k{}^{\max}-x_k{}^{\min}). \quad \text{(A11)}$$

We will designate the n variables in an n-dimensional point by $\mathbf{x}(\equiv x_1,\cdots,x_n)$.

Our procedure can be best understood if we now consider a specific example—say the function $\Phi_6{}^u(y)$ [Eqs. (33), (34), (35)]. This function is given by a five-fold integral. The parameters α and β are now $y(=\cos\theta)$ and ρ, respectively, as defined in the text. To obtain a distribution in y, we need to evaluate the integral for several values of the parameter y. We chose seven distinct values for y, and thus need seven five-fold integrals. In addition, we need a distribution in ρ. We chose six values of ρ, giving us 42 five-fold integrals in all. Fortunately we were able to evaluate all 42 integrals at the same time, because of the following two circumstances. First, the domains are independent of the parameter, ρ. Second, we can order the domains as a function of the parameter y in such a fashion that each domain contains all the subsequent domains,

$$D(\mathbf{x};y_1)\supset D(\mathbf{x};y_2)\supset\cdots\supset D(\mathbf{x};y_m).$$

Thus, when a given "point" $\mathbf{x}$ lies outside the ith domain $D(\mathbf{x};y_i)$, we know it will also lie outside all the subsequent domains, $i+1,\cdots,m$.

We will proceed in describing the MCM by giving a sequence of steps that correspond crudely to the logic

[21] These last two numbers follow from reference 7, Table I. There are no null projections in these instances.

TABLE III. Details on the evaluation of the function $\Phi_6{}^u$ by the Monte Carlo method, for a total number of "tries" $N=67\,000$ corresponding to 4.5 hr on the IBM 704. Errors are given for six of the 42 integrals performed.

y	Successful tries	Percent error for $\rho=0$	Percent error for $\rho=0.75$
-1	8188	1.4	1.2
0	6019	1.3	1.5
$+1$	4833	1.7	1.8

followed during the computations. This sequence was repeated a large number of times ($N\approx 10^5$), where each repetition represents another "try." The steps are:

(a) Generate n random numbers $r_1,\cdots,r_n$ with the property $0\leq r_i\leq 1$.

(b) Compute a set of random variables x_k with these random numbers according to

$$x_k=x_k{}^{\min}+(x_k{}^{\max}-x_k{}^{\min})r_k,\quad k=1,\cdots,n. \tag{A12}$$

This (ordered) set represents an n-dimensional point $\mathbf{x}$ chosen at random in the domain $D^{\max}$ defined in Eq. (A11).

(c) Test whether the point $\mathbf{x}$ is contained in $D(\mathbf{x};y_1)$. If it is, this try is counted as a "success" for the domain $D(\mathbf{x};y_1)$; proceed to step (d). If it is not, this try is considered as a "failure" for $D(\mathbf{x};y_1)$ and for all subsequent domains $D(\mathbf{x};y_1)i=2,\cdots,m$. Start at step (a) again.

(d) Test whether the point $\mathbf{x}$ is contained in $D(\mathbf{x};y_i)$ for $i=l-1$. This try is counted a "success" for the domains $D(\mathbf{x};y_i)i=2,\cdots,l-1$ and a "failure" from $i=l,\cdots,m$, where $D(\mathbf{x};y_l)$ is the first domain that does not contain the point $\mathbf{x}$. (Here $l-1=m$ means that the point is contained in all domains.) Proceed to step (e).

(e) Compute $f(\mathbf{x};y_i,\rho_j)$, the integrand at the point $\mathbf{x}$ for the values of the parameters $i=1,\cdots,l-1$ and ρ_j for $j=1,\cdots,\nu$. Cumulate the integrands in an array of $m\nu$ numbers. Repeat at step (a).

Let $N_s(y_i)$ be the number of successful tries for $y=y_i$. Then

$$\bar{f}(y_i,\rho_j)=\left[\sum_{p=1}^{N}f_p(\mathbf{x};y_i,\rho_j)\right]\Big/N_s(y_i) \tag{A13}$$

is the average value of the integrand, where we use the convention that whenever $\mathbf{x}$ is not inside the domain, f_p is set equal to zero. The value of the function is given by

$$\begin{aligned}\Phi_6{}^u(y_i,\rho_j)&=\lim_{N\to\infty}\bar{D}(y_i)\bar{f}(y_i,\rho_i)\\&=\lim_{N\to\infty}\left(D^{\max}\frac{N_s(y_i)}{N}\right)\left[\frac{1}{N_s(y_i)}\sum_{p=1}^{N}f_p(\mathbf{x};y_i,\rho_j)\right]\\&=\lim_{N\to\infty}D^{\max}\left[\sum_{p=1}^{N}f_p(\mathbf{x};y_i,\rho_j)\right]\Big/N.\end{aligned} \tag{A14}$$

The practical question is: After how many tries, N, has the above expression converged sufficiently close to its limiting value?

To answer this question, we have evaluated the variance. In terms of the variance, the statistical errors in the MCM have well-defined meanings.[22] Let us define $\Phi_\mu{}^{N_0}$ as the μth approximate solution obtained after a consecutive set of N_0 tries given by:

$$\Phi_\mu{}^{N_0}=\frac{1}{N_0}\sum_{p=(\mu-1)N_0+1}^{\mu N_0}f_p,$$

where $N_0=N/\lambda$. Then the solution after N tries, Φ^N, is given by

$$\Phi^N=\frac{1}{N}\sum_{p=1}^{N}f_p=\frac{1}{\lambda}\sum_{\mu=1}^{\lambda}\Phi_\mu{}^{N_0}.$$

The variance of N tries is obtained in the following manner. We choose N_0 large enough that the set $\Phi_\mu{}^{N_0}$, $\mu=1,\cdots,\lambda$ can be considered to have a Gaussian distribution and yet each $\Phi_\mu{}^{N_0}$ is not expected to be a good approximation to Φ. The variance of N_0 tries is obtained by plotting $\Phi_\mu{}^{N_0}$, $\mu=1,\cdots\lambda$. The variance $\sigma_N{}^2$ of N tries is then obtained from the variance of N_0 tries by the expression $\sigma_N{}^2=[\sigma(N_0)]^2/\lambda$.

To give a qualitative feeling for the time on the IBM 704 involved in such calculations, we quote some examples herewith. The 42 integrals for the function $\Phi_6{}^u(y,\rho)$ were evaluated in 4.5 hours for a total number of tries, $N=67\,000$. Here we chose $N_0=1000$ thus giving us $\lambda=67$ points for the evaluation of the variance. The resulting number of successful tries and errors are given in Table III.

[22] See, for example, F. Cerulus and R. Hagedorn, Nuovo cimento **9**, 659 (1958).