Phase-Parameter Representation of Proton-Proton Scattering from 9.7 to 345 Mev*

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Results of gradient searches for phase parameters by means of an IBM-704 machine representing proton-proton scattering are reported on and the procedure used is described. The analysis made use of most available measurements. The number of "measurements" used by the machine was 541. In some cases each "measurement" was a composite of two or more measurements found in the literature. The fits join smoothly to the $^1S_0$ phase-shift energy curve below 9.7 Mev. A family of fits has been obtained employing as a starting point the extended source+spin-orbit potential representation below 150 Mev extrapolated to one or another of the several Stapp, Ypsilantis, Metropolis fits at 312 Mev. Other searches started out with phase parameters corresponding to the Gammel-Thaler phenomenological potential. Evidence is presented to the effect that the better fits of both families are essentially the same. Error limits of phase parameters derived are estimated by various procedures. Some of the accumulated evidence for the applicability of the one-pion exchange potential (OPEP) to the calculation of phase parameters for the higher L and J is described and the basis for previously published inferences regarding the spatial extension within which the major part of the potential is the OPEP is illustrated. In the case of the best fit, referred to as YLAM below, tests indicating the existence of potentials for separate J have been made showing therefore that there is no serious question regarding the energy rates of change of logarithmic derivatives of radial wave functions being negative. A brief discussion of the relationship of this requirement to the meson theory of nuclear forces is included.

INTRODUCTION

IN the absence of a satisfactory theory of nucleon-nucleon interactions the analysis of nucleon-nucleon scattering data has to fall back either on the employment of theoretically poorly founded concepts such as phenomenological potentials and boundary value treatments or else on the theoretically rigorous concept of real phase shifts and coupling parameters. Neglecting the very small effects of p-p and p-n bremsstrahlung, the latter approach is completely general below the threshold of meson production. The dispersion relations approach[1] may prove eventually adequate for the treatment of the problem, but it is insufficiently developed to yield more than very partial results in a meaningful manner. In the present report, therefore, the rigorous phase-parameter approach is employed in spite of its phenomenologic character. It will be seen that in spite of the strongly empirical character of the treatment it is capable of yielding results having direct implications for fundamental theory. The term "phase parameter" is used here in the sense of phase shifts as well as coupling parameters between states with the same total angular momentum $J\hbar$ but different orbital angular momenta $L\hbar$. The availability of phase parameters can be expected to further the formulation of basic nucleon-nucleon interaction theory as well as to aid the correlation of nucleon-nucleon scattering data with that on nucleon-nucleus scattering and on nuclear structure.

The analysis of scattering data at one energy usually yields many possible solutions.[2] Even though a proposal has been made[3] to employ five suitably chosen quantities at all angles together with the unitarity condition for the determination of the scattering matrix, the practical carrying out of the proposal appears difficult and involves the consideration of errors introduced by the unavailability of data at all scattering angles and especially those close to 0 and π in the center-of-mass system. On the other hand, some theoretical guides are available for the classification of the phase-parameter dependence on the energy E into reasonable and unreasonable categories. Thus any ordinary theory leads to the expectation of relative dominance of s-wave effects at low energies and to the setting in of phase parameters with increasing energy somewhat in the order of increasing L. On rather general hypotheses the limitations of which are partially discussed below the logarithmic derivative of a radial wave function is expected to decrease with E. The availability of such criteria indicates an advantage regarding uniqueness in a simultaneous fit to data at many energies. If, for example, the experimental errors at energy E_1 have produced a spurious fit, it is unlikely that such a fit will be reconcilable in terms of reasonable energy variations of phase parameters with a fit to data at energy E_2. The more energies that are used in the search the less chance there is for the survival of an essentially spurious fit at one energy in the process. The computational difficulty of handling many energies at

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[1] M. L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. 2, 226 (1957); H. P. Noyes, University of California Radiation Laboratory Report UCRL-5921-T (unpublished). The writers are grateful to Dr. Noyes for supplying them with a copy of his UCRL report.

[2] R. M. Thaler and J. Bengston, Phys. Rev. **94**, 679 (1954); R. M. Thaler, J. Bengston, and G. Breit, Phys. Rev. **94**, 683 (1954); H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957) referred to as SYM in text; M. H. Hull, Jr., and J. J. Shapiro, Phys. Rev. **109**, 846 (1958); M. H. MacGregor, Phys. Rev. **113**, 1559 (1959). These analyses are examples demonstrating the lack of uniqueness of the answers rather than an exhaustive list of references.

[3] L. Pusikov, R. Ryndin, and J. Smorodinsky, Nuclear Phys. **3**, 436 (1957); J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 592 (1957) [translation: Soviet Phys.-JETP **5**, 489 (1957)].

once may be expected to be more than offset by that of the consideration of many solutions obtained by fitting at one E at a time because the digital machine search guided by the many pieces of data in the correct direction more reliably. A least squares adjustment to data at one energy even if it is in the proximity of the physically correct fit may be unduly influenced by errors at the particular energy and as a result may deviate from the true answer appreciably. Such an effect may be expected to be less serious if data at several energies are used in one operation, the accidental accumulations of errors at different energies tending to compensate statistically.

II. MAIN SEARCH PROCEDURE AND ERROR ESTIMATES

The search for best phase-parameter values was carried out by the gradient method. Different sets of starting values have been used and the phase-parameter dependence on energy has been improved by a set of successive gradient searches. The relationship of the work of Marshak, Signell, and Zinn[4,5] to the extended source theory of pion-nucleon interaction[6] provided a point of departure for some of the searches with a partial theoretical basis. In these cases a modification of the Rochester potential made on empirical grounds, which will be mentioned somewhat more fully presently, has been used from 9 to 150 Mev while at the high end of the energy range one of the phase parameter fits of Stapp, Ypsilantis, and Metropolis (SYM) was used to anchor the phase-parameter versus energy curves. In between the curves were drawn in by eye. Another starting point was provided by the Gammel-Thaler potential which was used for this purpose without further modification.

In order to avoid a possible misunderstanding regarding the employment of 312-Mev data in searches employing the SYM solutions as starting points, it should be mentioned that the 312-Mev group of measurements has been used employing the observed values of different quantities at the energies at which they were performed rather than at a mean nominal energy. It was found necessary to do so. In fact, some of the earliest gradient searches indicated that if the 312-Mev group is used at its nominal energy, the fits to phase parameters tend to reproduce the data at different energies within the group of measurements in a questionable manner. For this reason in this energy group assigning data to one nominal energy has not been done and such assignments to a nominal energy have been avoided in most cases unless it was known or strongly surmised that the difference caused by energy lumping was of no significance.

The process of gradual improvement was carried out as follows. Any one of the phase parameters $\delta_p(E)$ with values $\delta_p^{(n-1)}(E)$ available from previous n searches was changed to

$$\delta_p^{(n)}(E)=\delta_p^{(n-1)}(E)+\sum_q a_{pq}^{(n)}f_{pq}^{(n)}(E), \qquad (1)$$

where the $f_{pq}^{(n)}$ are a set of conveniently chosen functions and the $a_{pq}^{(n)}$ are parameters to be adjusted by the gradient search. The $f_{pq}^{(n)}(E)$ are, in general, different at different stages in the succession of searches, although in some cases they are the same. The $a_{pq}^{(n)}$ form the many dimensional space in which the gradient method is used. The $f_{pq}^{(n)}$ will be sometimes referred to as the correction or else the expansion functions. The superscript (n) will often be dropped, when clarity does not suffer. The choice of the f_{pq} was dictated by practical requirements of computational simplicity and of emphasis on one or another energy region. The following forms have been found useful for different purposes. From the viewpoint of simplicity and, therefore, speed in computation as well as of possibility of enforcing approximate theoretical expectation at small E the form

$$f_{pq}(E)=(E/E_0)^{S+I(q-1)} \qquad (1.1)$$

has been useful. The quantity E_0, usually chosen around the middle of the energy range could have been absorbed in the a_{pq} but introducing it secures the absence of either large or small numbers for the f_{pq} when $E\cong E_0$. By assigning positive integral values to the q and suitable values to S and I convenient flexibility of the f_{pq} is obtained. In the simplification of vanishing Coulomb field, an approximation reasonably well justified for $L>0$ in the energy range considered, the low energy dependence of phase shifts is expected[7] to correspond to $S=(2L+1)/2$, $I=0$ in Eq. (1.1). These values were often not useful partly because the changes in the δ_p were not small enough to be sure of the applicability of a first-order perturbation formula and particularly because Eq. (1.1) usually gives large changes at high E which therefore influence seriously the adjustment of the a_{pq}. Values of $I=1$, $\frac{3}{2}$ with $S=1$ for the lower and $S=3$ for the higher L have often proved useful, the smaller S eliminating undue emphasis on high-energy data.

In some cases inspection of results indicated the desirability of leaving the δ_{pq} unchanged at a selected energy E_1. In such cases the form

$$f_{pq}(E)=(E/E_0)^{S+I(q-1)}[(E/E_f)-1] \qquad (1.2)$$

was often used, especially with $E_f=312$ Mev. In other

[4] P. S. Signell and R. E. Marshak, Phys. Rev. **109**, 1229 (1958). This paper is referred to as SM in the text.
[5] P. S. Signell, R. Zinn, and R. E. Marshak, Phys. Rev. Letters **1**, 416 (1958).
[6] G. F. Chew, Phys. Rev. **95**, 1669 (1954); S. Gartenhaus, Phys. Rev. **100**, 900 (1955); G. C. Wick, Revs. Modern Phys. **27**, 339 (1955); G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 and 1579 (1956).
[7] G. Breit, H. M. Thaxton, and L. Eisenbud, Phys. Rev. **55**, 1018 (1939), see especially p. 1060.

cases the form

$$f_{pq}(E)=2^m[(E-E_c)/E_0]^{S+I(q-1)} \times[1+(E-E_c)/E_0]^{-m} \quad (E>E_c) \quad (1.3)$$
$$=0, \quad (E<E_c)$$

was employed. It has the advantage of giving no change below the cutoff energy E_c and is especially suitable for the later stages of the general search when data below E_c have been satisfactorily reproduced. This correction function has proved useful also with $E_c=0$. For $S+I(q-1)<m$ this function decreases at large E and is suitable therefore for increasing the emphasis on data in a desired region of E without removing the guiding influence of neighboring regions.

In applying the correction functions a certain amount of personal judgment had to be used both regarding securing good fits without excessive calculation and in avoiding undue and physically improbable bumpiness of phase-parameter versus energy plots.

The gradient search procedure followed the general plan introduced in connection with pion-proton scattering by Fermi, Metropolis, and Alei.[8] The weighted sum of squares of deviations of the fitting curves from the data was minimized approximately by following the gradient in the space of the a_{pq}. The uncertainty in the values of the a_{pq} and the associated uncertainty in the values of the phase parameters was obtained by the same method as in the pion analysis of Anderson, Davidon, Glicksman, and Kruse.[9] This involved the computation of the matrix $\mathfrak{M}$ with elements

$$\mathfrak{M}_{pq,rs}=\sum_i w_i f_{pq} S_{ip} S_{ir} f_{rs}, \quad (2)$$

and its inverse. Here the weight of the ith datum y_i expressed in terms of the standard deviation in the measurement of y_i, Δy_i, is

$$w_i=1/(\Delta y_i)^2, \quad (2.1)$$

while the sensitivity of η_i, the expression for y_i in terms of the phase parameters, to the phase parameter δ_p is

$$S_{ip}=\partial\eta_i/\partial\delta_p. \quad (2.2)$$

The weighted sum of squares of deviations divided by the number of observations N will be referred to as D so that

$$D=\frac{1}{N}\sum_{i=1}^{N} w_i(\eta_i-y_i)^2. \quad (2.3)$$

Since N is large compared with the number of parameters pq that are determined by the search, the standard deviations of the a_{pq} are

$$(\Delta a_{pq})^2 \cong D(\mathfrak{M}^{-1})_{pq,pq}, \quad (2.4)$$

while the standard deviation of the phase parameters is obtainable as the square root of

$$\langle(\Delta\delta_p)^2\rangle \cong D\sum_{q,r}(\mathfrak{M}^{-1})_{pq,pr} f_{pq} f_{pr}. \quad (2.5)$$

It will be noted that the presence of systematic errors in the data increases D and accordingly also the $(\Delta\delta_p)^2$. In the last equation the statistical correlation coefficients between a_{pq} and a_{pr} are taken into account similarly to (2.4). The assumption that D has been minimized which is implicit in (2.5) is only approximately fulfilled but since a precise knowledge of the uncertainty limits in the phase parameters is not needed, this circumstance was disregarded.

It would have been possible to calculate the probable statistical averages

$$\langle\Delta\delta_p\Delta\delta_q\rangle \cong D\sum_{r,s}(\mathfrak{M}^{-1})_{pr,qs} f_{pr} f_{qs}, \quad (2.6)$$

which determine the correlations of errors of different phase parameters. On account of the additional work involved, these correlation coefficients have not been calculated. They are not known to be small and it would not be justifiable to interpret the error bands calculated by means of (2.5) in terms of independent errors for different δ_p. Three main assumptions are needed for the applicability of the statistical error estimates as follows. (A) The errors of individual measurements are uncorrelated. This assumption is only partially satisfied since, for example, all cross sections of a group of observers may be incorrect by the same factor arising in the current measurement. (B) The fit is supposed to be a good approximation to a least squares fit. While the fits used are only approximately least square fits, the error introduced thereby is estimated to be small. (C) The number of observations is sufficiently large to make the well-known factor $N/(N-n-1)$ replaceable by unity. Here n is the number of adjustable parameters in the fit. This assumption is well satisfied in the applications made below.

The assumption of Gaussian distributions for the measurement errors is not directly essential, the primary meaning of a standard deviation in the present case being that of the root mean square. If one interprets the Δy_i in this sense, then the same result applies as in reference 9 and hence in the notation used here

$$\langle\Delta a_{pr}\Delta a_{qs}\rangle=(\mathfrak{M}^{-1})_{pr,qs},$$

where the $\langle\,\rangle$ indicate the statistical average. The employment of such a formula with the nominal errors from experimental papers used for computing $\mathfrak{M}$ would not take into account the fact that mean square deviation around a fitted δ_p, E curve is greater than the nominal error. The factor D in (2.5) and (2.6) may be regarded as providing the desired correction, calculated on the basis of the whole statistical sample.

Since the matrix $\mathfrak{M}$ makes it possible to carry through a least squares calculation on the assumption of constant S_{ip}, an attempt was made to do so. Usually

[8] E. Fermi, N. Metropolis, and E. F. Alei, Phys. Rev. **95**, 1581 (1954).

[9] H. L. Anderson, W. C. Davidon, M. Glicksman, and U. E. Kruse, Phys. Rev. **100**, 279 (1955).

this method did not secure convergence to a minimum, the probable reason being the variability of the S_{ip}, the dependence of the η_i on the δ_p being nonlinear. This was the case even when all $\eta_i - y_i$ were artificially decreased to about 1/100 or less of their values so as to approach the minimum by smaller steps within which the S_{ip} could be hoped to be sufficiently constant. This circumstance did not interfere with the gradient method because a small step along the negative of the gradient usually decreases D and because the S_{ip} can be recalculated when necessary without going through the time-consuming step of matrix inversion. In some cases, however, the least squares method proved useful in verifying that no significant change results on continuing gradient searches close to their final values.

From (2.3), after n searches,

$$\partial D/\partial a_{pq} = (2/N) \sum_i w_i [\eta_i^{(n-1)} - y_i] S_{ip}^{(n-1)} f_{pq}^{(n-1)}, \quad (3)$$

gives the sensitivity of D to the individual a_{pq}. The components of the unit vector along the gradient may therefore be calculated as

$$\delta a_{pq} = (\partial D/\partial a_{pq}) / [\sum_{p,q} |\partial D/\partial a_{pq}|^2]^{\frac{1}{2}}, \quad (3.1)$$

with $\partial D/\partial a_{pq}$ available from (3). The search took place in a direction opposite to the gradient, the standard convention of defining the positive direction of a gradient of a function along the line of most rapid increase of the function being adhered to. It is realized that the a_{pq} could be multiplied by arbitrary constants and that the direction of the gradient would then be changed. No attempt was made to find the most rational set of multipliers for the a_{pq}, the only immediate purpose being to decrease D reasonably rapidly rather than to study the most effective way of doing so.

The earlier searches attempted to determine the phase parameters by terminating the values of L and J for which the phase parameters were searched at a reasonably large number such as $J=6$ without any theoretical guide except for reasonableness of energy variation. These searches have been successful in the sense of producing marked improvements in D but gave very wiggly angular distribution curves. This experience was interpreted as indicating that the inclusion of the higher L and J is essential. Since a search with too many parameters is impractical on account of the large consumption of machine time and the increase in the error bandwidths of the parameters, it was assumed that the one-pion exchange potential[10–12] (OPEP) may be used for the higher L and J. Its employment removed the wiggliness and enabled further decreases in D to be made. Some experimentation on the values of L and J at which the validity of the OPEP may be assumed to hold took place and has been used[12] for estimates of distances beyond which the OPEP is the principal interaction. A partial but incomplete account of this experimentation is found below in connection with the description of search designations.

III. SPECIAL METHODS AND DATA USED

It is very difficult and probably impossible to be sure that all relevant portions of the many dimensional phase-parameter space have been explored and even that all relevant paths in a given search have been tried. Given the end results of two searches 1 and 2 which have resulted from different starting points, the question arises as to whether the end values are different because a path leading from one to another has remained unexplored and whether along this path there might exist a D smaller than that for either 1 or 2. A partial answer is obtained by calculating D for values of the δ_p linearly interpolated between those for cases I and II, viz.,

$$\delta_p^{(\xi)} = \delta_p^{(\mathrm{I})} + \xi(\delta_p^{(\mathrm{II})} - \delta_p^{(\mathrm{I})}), \quad (4)$$

where the dependence of the δ on the energy is not shown. In some cases the variation of ξ from 0 to 1 produces an improvement in D and shows that the fits are essentially the same in the sense that a suitable path, had it been found at an earlier stage, would have led to the improved D. This procedure will be referred to as *the ξ variation.*

Regarding ξ as a parameter which is being adjusted to produce the best fit to experiment, the standard deviation in ξ is $\Delta\xi$ such that

$$|D(\xi_0 \pm \Delta\xi) - D(\xi_0)| \cong D(\xi_0)/N. \quad (4.1)$$

It is assumed in this approximate equation that the error distributions are Gaussian which is questionable but it is probable that the approximation is not very poor.

One of the difficulties of the data analysis is the probable presence of systematic errors in the measurements indicated by the occasional disagreement of sets of data with the fits which is outside the nominal errors given in the experimental papers. Regarding the results of the search procedure as a way of furnishing a smooth interpolation between measured values of the $\delta_p(E)$ such deviations are improbable and the inclusion of data giving the large deviations in the analysis increases the width of the error belts. If one were sure that certain data are definitely inferior than indicated by the nominal errors, their weight could be appropriately decreased. In the absence of such knowledge and in view of the danger of introducing subjective

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[10] M. Taketani, S. Nakamura, and M. Sasaki, Progr. Theoret. Phys. (Kyoto) **6**, 581 (1951). J. Iwadare, S. Otsuki, R. Tamagaki, and W. Watari, Progr. Theoret. Phys. (Kyoto) **16**, 455 (1956); Suppl. Progr. Theoret. Phys. (Kyoto) **3**, 32 (1956). S. Otsuki, Progr. Theoret. Phys. (Kyoto) **20**, 171 (1958); R. Tamagaki, Progr. Theoret. Phys. (Kyoto) **20**, 505 (1958).

[11] M. J. Moravcsik, P. Cziffra, M. H. MacGregor, and H. P. Stapp, Bull. Am. Phys. Soc. **4**, 49 (1959); P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959).

[12] G. Breit and M. H. Hull, Jr., Nuclear Phys. **15**, 216 (1960).

criteria[13] practically all available data have been used. As a result, the knowledge of the δ_p attainable in an energy region within which there are data consistent with each other and with good nominal accuracy may be inadequate. In such cases a local improvement in the values of the δ_p may be attempted and the procedure can also give an improvement in the apparent accuracy of the δ_p in the energy regions selected on the basis of apparent good quality of the data. It is assumed in these cases that the general trend of the δ_p versus E plot is correctly given by the search as a whole. The function $\delta_p(E)$ was therefore changed by adding an adjustable constant which was determined by a search for the best value of the constant employing only the data in the selected energy region. This procedure would be questionable in wide energy regions where at least a systematic change of general slope of the plot would have to be included. The energy regions were usually narrow and therefore the procedure could even be simplified by employing a linear representation for each δ_p. This procedure will be referred to as *the parallel shift adjustment*. The practical object of this procedure is to provide a few energy regions within which the δ_p are known with more certainty than that available from the employment of the matrix method. For a fit of a given type the general knowledge of the δ_p is also improved because a "reasonable" curve may be drawn between the selected regions. The improvement in accuracy thus achieved is valid only in the context of a given type of fit, however, it being essential in the procedure to assume the type of energy variation furnished by the fit as a whole.

The parallel shift adjustment has been used for the determination of error limits in some of the final results. In these cases all data in the energy range covered by one shift have been used and the energy regions have been chosen in such a way as to leave no gaps in the whole region covered so as not to underestimate the error. In a few cases energy ranges within which the data are especially complete regarding variety of experimental quantities have been treated by the parallel shift adjustment.

In the later calculations of the OPEP phase parameters for p-p data the mass of the neutral pion was used and the relativistic effects were also included, although some of the earlier calculations employed the charged pion mass without relativistic corrections with not very different results. Nevertheless, for the sake of definiteness the more final calculations and the later comparisons with n-p data were made uniformly as just mentioned for p-p and on the following basis for n-p cases. The $T=1$ state for n-p have the isotopic spin function

$$\chi_0=(1/\sqrt{2})(\gamma_1\delta_2+\gamma_2\delta_1), \tag{5}$$

in the usual notation with 1, 2 referring to the nucleons. The $(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)$ part of the interaction energy consists of the parts $\tau_{1\xi}\tau_{2\xi}+\tau_{1\eta}\tau_{2\eta}$ and $\tau_{1\zeta}\tau_{2\zeta}$. The first has to do with charged and the second with neutral pion exchange. Since $(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\chi_0=\chi_0$ and $\tau_{1\zeta}\tau_{2\zeta}\chi_0=-\chi_0$, it follows that $(\tau_{1\xi}\tau_{2\xi}+\tau_{1\eta}\tau_{2\eta})\chi_0=2\chi_0$ and therefore the phase parameters enter in the combination

$$2\delta(m_{\pi^+})-\delta(m_{\pi^0}), \quad (T=1) \tag{5.1}$$

which replaces $\delta(m_{\pi^0})$ for the p-p case. For T=0 the isotopic spin function is

$$\tilde{\chi}_0=(1/\sqrt{2})(\gamma_1\delta_2-\gamma_2\delta_1), \tag{5.2}$$

such that $(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2+3)\tilde{\chi}_0=0$. For it, therefore, $(\tau_{1\xi}\tau_{2\xi}+\tau_{1\eta}\tau_{2\eta}+2)\tilde{\chi}_0=0$ and the combination is

$$-[2\delta(m_{\pi^+})+\delta(m_{\pi^0})], \quad (T=0) \tag{5.3}$$

which replaces $-3\delta(m_\pi)$ of calculations making no distinction between charged and uncharged pion masses. The assumption is made here that the pion-nucleon coupling constant is the same for charged and neutral pions. Formally this assumption is consistent with a possible interpretation of charge independence. Since the physical pions have different masses, the propriety of this treatment is not certain, the complete symmetry of the theory being applicable presumably at an earlier stage than that of employing the physical pion masses. In view of the difficulty of formulating a complete treatment caused by unavoidable inaccuracies especially in connection with n-p data and the convenience of Eqs. (5.1), (5.3) in digital machine computation, no attempt to refine this treatment was made. The formulas for the computation of the first-order effects of the OPEP are as in.[11,12] In the second of the two references the difference between the relativistic and nonrelativistic approximations is discussed and consists in the replacement of Mc^2 by the relativistic energy of one of the particles in the rest system. The n-p calculations which gave the phase-parameter error limits shown in the present paper have been made employing Eqs. (5.1) and (5.3).

It is believed that the analysis into relativistic and nonrelativistic effects[12] is desirable if higher orders of interaction than the OPEP are to be treated and if a local potential is used. In the calculations as carried out so far it makes no difference, however, whether[11] or[12] is used. The effects of successive L, J states have been added until further terms made a negligible difference.

The criterion for neglecting contributions of the higher J was that the last phase shift included should be less than 0.0005 radian. It was found by trial that the changes in the quantities to be compared with experiment became very small at this stage and decreased rapidly as J was varied. Addition of OPEP parameters with values between 0.0005 and 0.00005 changes quantities compared with experiment by $\sim$1/200 of the experimental uncertainty. From 210

[13] The employment of an external criterion in ascertaining a set of weights determined from a comparison with an interpolated function such as has been used by M. C. Yovits, R. L. Smith, M. H. Hull, J. Bengston, and G. Breit, Phys. Rev. **85**, 540 (1952), has been borne in mind but has not been carried out so far.

Table I. Quantities and energies used in addition to those from the collection by Hess.

Quantity	Energy (incident, Mev)	Angular range (degrees, center-of-mass system)	Source	Number of angles	Remarks
σ	9.69	10.03–89.85	a	26	
σ	14.16	18.07–89.79	b	16	Values of σ at 70.2° and 110.2° averaged and run at 70°.
σ	25.63	10.07–89.61	c	23	
σ	39.40	8.08–89.40	d	27	
σ	46.00	45.5	e	1	
P	46.00	45.5	e	1	
σ	44.66	90	f	1	Run at 46 Mev
σ	56 00	45.6, 90.0	e,f	2	
P	56.00	45.6	e	1	
σ	66.00	20.4 –71.0	e	11	
P	66.00	20.4 –71.0	e	11	
σ	68.3	10.18–88.98	g	25	
σ	78.0	45.8	e	1	
P	78.0	45.8	e	1	
σ	95	20.6 –86.40	e	14	
P	95	20.6 –86.40	e	14	
σ	98	10.20–81.40	h	14	When two values for same angle were given, they were averaged and experimental error adjusted statistically.
P	98	10.20–81.40	h	14	
D	98	20.5 –61.30	i	5	
σ	102,107	30.8 –66.45	e	3	Data at 102 Mev and 107 Mev averaged and run at 104.5 Mev.
P	102,107	30.8 –66.45	e	3	
σ	118	20.6 –88.2	e	16	
P	118	20.6 –88.2	e	16	
σ	127,137	31.10–66.80	e	3	Data at 127 Mev and 137 Mev averaged and run at 133 Mev.
P	127,137	31.1 –66.8	e	3	
P	133	36.17–88	j	7	
R	140	15 –40	k	6	
R	142	24 –90	l	8	
σ	142	5.19–90	h	20	When two or more values for the same angle were given, they were averaged and the error adjusted.
P	142	5.19–90	h	28	
σ	147	4.13–87.50	e	24	Values at same angle or angles differing by 1° or less were averaged and the error adjusted.
P	147	4.13–87.50	e	24	
D	143	12.40–85	m	8	Run at 147 Mev.
A	210	30 –90	n	7	
R	210	30 –90	o	7	
R	310	22.3 –80.1	p	6	Run at 312 Mev
D	310	23.0 –80.5	p	6	Run at 312 Mev
A	316	25.4 –76.3	q	3	Run at 312 Mev

[a] L. H. Johnston and D. E. Young, Phys. Rev. **116**, 989 (1959). The writers are indebted to Professor Johnston for supplying them with these and other data before publication.

[b] S. Kikuchi, J. Sanada, S. Sawa, I. Hayashi, K. Nisimura, and K. Fukanaga, J. Phys. Soc. (Japan) (to be published). The writers are indebted to the authors of this paper for supplying them with information before publication.

[c] T. H. Jeong, L. H. Johnston, D. E. Young, and C. N. Waddell, Phys. Rev. **118**, 1080 (1960).

[d] L. H. Johnston and D. A. Swenson, Phys. Rev. **111**, 212 (1958).

[e] J. N. Palmieri, A. M. Cormack, N. F. Ramsey, and R. Wilson, Ann. Phys. **5**, 299 (1958). In the work quoted a distinction is made between relative and absolute errors, the latter being concerned with a constant factor for $\sigma(\theta)$ at each energy. In the earliest stages of the present work the data were used as published and in the fits the Harvard points were systematically high. Consultation with Professor Wilson indicated that he considered lowering the experimental values by ~4.5%, the stated absolute error, reasonable in the light of his knowledge of data in this energy region as a whole. Searches YRB1, YLA, YRB2, YLAM(1) were made in their later stages in this manner and employing relative errors only. For search YLAM on receiving additional advice from Professor Wilson, the $\sigma(\theta)$ at 66, 118, 133, and 147 Mev were lowered by slightly different amounts from 3 to 4.5%, these being partly determined by comparison with data at 68.3 Mev.[g] For the final determination of the mean square error, D, data were used as published and the error for weighting was obtained by adding the relative and absolute errors in quadrature. For YLAM this procedure reduced D from 1.77 [lowered $\sigma(\theta)$, relative errors only] to 1.49 (published data, absolute errors included), with $\sigma(\theta)$ at 147 Mev giving by itself a D of 2.11, a value only somewhat in excess of the average 1.49. The reduction in D from 1.77 to 1.49 is caused by the inclusion of absolute errors in the weight ng.

[f] L. H. Johnston and Y. S. Tsai, Phys. Rev. **115**, 1293 (1959).

[g] L. H. Johnston (private communication), and Bull. Am. Phys. Soc. **4**, 252 (1959).

[h] A. E. Taylor, E. Wood, and L. Bird, Nuclear Phys. **16**, 320 (1960).

[i] E. Thorndike and T. Ophel (private communication).

[j] J. Dickson and D. Salter, Nature **173**, 946 (1954).

[k] E. Thorndike, J. Lefrancois, and R. Wilson (private communication).

[l] L. Bird, D. N. Edwards, B. Rose, A. E. Taylor, and E. Wood, Phys. Rev. Letters **4**, 302 (1960), and reference h.

[m] C. F. Hwang, T. R. Ophel, E. H. Thorndike, R. Wilson, and N. F. Ramsey, Phys. Rev. Letters **2**, 310 (1959), and private communication regarding a few additional data from the Harvard group.

[n] A. England, W. Gibson, E. Heer, and T. Tinlot, Bull. Am. Phys. Soc. **5**, 76 (1960), and private communication regarding slight improvements in data from Dr. England.

[o] J. Tinlot, E. Heer, A. England, and W. Gibson, Bull. Am. Phys. Soc. **4**, 252 (1959).

[p] O. Chamberlain, E. Segrè, R. D. Tripp, C. Wiegand, and T. Ypsilantis Phys. Rev. **105**, 288 (1957).

[q] J. Simmons, Phys. Rev. **104**, 416 (1956).

to 345 Mev the maximum J included was 12, from 118 to 172 Mev it was 10, from 39.4 to 104.5 Mev between 6 and 8, from 9.7 to 25.6 Mev it was 4.

The number of separate data used by the machine was 541. In this count values that have been lumped into one were counted as one datum. One-hundred and twenty-seven supplementary data were used to obtain lumped values. In addition to the values collected by Hess[14] the data used were as in Table I.

The data taken from the collection by Hess and comments regarding the way they were used are shown in Table II.

For reasons of economy of machine time, some of the data listed by Hess[14] have not been used in the analyses reported here. In general, if two or more sets of data of the same type were available in the same energy range, the more complete set was preferred, or the set having the smallest reported experimental uncertainty was used. If the latter circumstance prevailed, as in the case of the 10-Mev energy region, the data with larger uncertainties would have had little influence on the gradient direction.

Data in the energy range 9.6–345 Mev listed by Hess and not included in the present analysis are listed below with references to data sources in the designation used by Hess. Cross-section measurements at 9.7 and 9.85 Mev, C16; at 10 Mev, W3 (data in this case were reported only in graphical form); five measurements of $\sigma(90°)$, C10, between 18 and 32 Mev (most of these are plotted in Fig. 18); cross-section measurements at 30.14 Mev, F3 and at 31.8 Mev, C9 (the data at 29.4 Mev listed in Table II were arbitrarily selected as representative of this energy range; in point of fact the data at 31.8 Mev appear to fit in better with the analysis); at 300 Mev, C13 were omitted in favor of the improved data at 330 Mev listed in Table II. Measurements of the average cross section over the angular range 40°–90° c.m. at 75 and 105 Mev, B4, and over the angular range 20°–90° at 160, 230, and 330 Mev, C13, could not be conveniently used by the gradient program. Polarization data at 130 Mev, B14, were omitted in favor of more complete data listed at 127, 137, and 133 Mev in Table I; the data at 240 Mev, B10, appear not to have been reported except in the form of a rough graph in the Rochester Conference[15]; the polarization data at 314 Mev, M8, were omitted in favor of data listed in Table II at 310 and 315 Mev.

Data below 9.7 Mev have not been explicitly used, except for the earliest searches. They have been taken into account indirectly through an adjustment of the K_0 dependence on E which was made to be in agreement with that obtained at $E<9.7$ Mev from an analysis of the data employing ${}^1S_0$ waves only.[16]

[14] W. N. Hess, Revs. Modern Phys. **30**, 368 (1958).

[15] *Proceedings of the Fifth Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, New York, 1955).

[16] M. C. Yovits, R. L. Smith, M. H. Hull, J. Bengston, and G. Breit, Phys. Rev. **85**, 540 (1952).

This was accomplished by choices of correction functions for K_0 which produced only small changes at low E. The assumption of negligible smallness of effects of phase parameters with $L>0$ implicit in[16] is supported by the present analysis for $E>9.7$ Mev. In the case of YRB type fits the agreement with data below 9.7 Mev was secured by the adjustment of the range constant resulting in a 16% increase of the range of the singlet-even potential.[17] For searches YLA and YLAM the starting point furnished by the Gammel-Thaler potential is in good agreement with the required energy variation.

TABLE II. Treatment of data given by Hess.[a] Where Hess gives renormalized values of $\sigma(\theta)$, these are used.

Quantity	E, incident, Mev	Angular range degrees, c.m. system	Number of angles	Remarks
σ	18.2	30 –90	8	
σ	19.8	14 –90	16	The slightly changed values of $\sigma(\theta)$ given by Burkig, Richardson, and Schrank,[b] and by Royden and Wright,[c] were used and averaged when two values at same angle are given.
σ	29.4	24 –87.3	9	These data run as representative of the 30-Mev results for most of the work. *Omitted* from final calculations.
σ	78.5	90	1	Run at 78 Mev.
σ	95	90	1	Other data due to Kruse listed by Hess were used in much of the work, and are shown in the figures. In final runs, only the 90° datum was used.
σ	120	63 –89.2	4	Run at 118 Mev.
σ	134	90	1	Run at 133 Mev.
σ	147	25 –75	5	
σ	147	90	1	Average of data due to Cassels and Pickavance.
σ	164	90	1	Run at 172 Mev.
σ	170,174	9.6 –62.3	8	Data for same angles at 170 and 174 Mev averaged and all data run at 172 Mev.
P	170,174	20.8 –82.47	9	
P	210	13.7 –83.03	12	
σ	240,250, 260	8.7 –90	19	All data run at 250 Mev, and averaged if angles differed by <2° for $\theta>25°$.
P	276	19.3 –90	7	
P	310	6.5 –21.7	7	Data run at 312 Mev, values at 21.7° and 21.6° were averaged.
P	315	21.6 –89.4	6[d]	
σ	330	4.67–29.7	18	Values at same angle or angles differing by 3° for $\theta>30°$ were averaged.
σ	345	11.30–88.33	12	

[a] See reference 14.
[b] J. W. Burkig, J. R. Richardson, and G. E. Schrank, Phys. Rev. **113**, 290 (1959).
[c] H. N. Royden and B. T. Wright, Phys. Rev. **113**, 294 (1959).
[d] Hess lists for 315-Mev measurements of P at seven angles for his reference Y2. The measurement at 89.4° was inadvertently omitted. The 21.7° and 21.6° data were averaged, respectively, from the 310- and 315-Mev entries of Hess.

IV. PARAMETRIZATION AND SEARCHES MADE

The phase parameters have been used in the following convention. For singlet states the phase shift K_L is defined by the asymptotic form of the radial function $r\mathfrak{F}_L$ being such that

$$\mathfrak{F}_L \sim \sin[kr - L\pi/2 - \eta \ln 2kr + \arg\Gamma(L+1+i\eta) + K_L], \quad (6)$$

where

$$k = [ME_{\text{lab}}/2\hbar^2]^{\frac{1}{2}}, \quad \eta = e^2/\hbar v, \quad (6.1)$$

and v is the velocity of the incident proton in the laboratory system[18] calculated relativistically. Nonrelativistically v could be used as the relative velocity in either system. If k is calculated without reference to relativity from the incident energy in the case of a target at rest, its relativistic value is obtained. The Coulomb wave sometimes denoted as ψ^c has been included in all of the p-p calculations. For triplet uncoupled states the phase shift is defined as in (6) and is denoted by $\delta^L{}_J$. For triplet coupled states the matrix U which enters the calculation[19] of the amplitudes is parametrized as[20]

$$U_J = \begin{pmatrix} (1-\rho_J{}^2)^{\frac{1}{2}} \exp(2i\theta^{J-1}{}_J) & i\rho_J \exp[i(\theta^{J-1}{}_J + \theta^{J+1}{}_J)] \\ i\rho_J \exp[i(\theta^{J-1}{}_J + \theta^{J+1}{}_J) & (1-\rho_J{}^2)^{\frac{1}{2}} \exp(2i\theta^{J+1}{}_J) \end{pmatrix}. \quad (6.2)$$

In the symbols $\theta^L{}_J$ with $L=J\pm 1$ the L refers to orbital angular momentum $L\hbar$ of the two coupled channels. The relationship to the parameters used by Blatt and Biedenharn is

$$\begin{aligned} \theta^{J-1}{}_J + \theta^{J+1}{}_J &= \delta_\alpha + \delta_\beta, \\ \tan(\theta^{J-1}{}_J - \theta^{J+1}{}_J) &= (\cos 2\epsilon) \tan(\delta_\alpha - \delta_\beta), \quad (6.3) \\ \rho_J &= (\sin 2\epsilon) \sin(\delta_\alpha - \delta_\beta), \end{aligned}$$

and the relation to the "nuclear bar" quantities of SYM is

$$\rho_J = \sin 2\bar{\epsilon}_J, \quad \theta^{J-1}{}_J = \bar{\delta}_{J-1}, \quad \theta^{J+1}{}_J = \bar{\delta}_{J+1}. \quad (6.4)$$

For the convenience of readers it may be noted that in Eq. (3.15) of SYM the ϵ_j in $\cos 2\epsilon_j$, $\sin 2\epsilon_j$, the ϵ should be $\bar{\epsilon}$ (a misprint). The p-p searches and the corresponding D values are summarized in Table III, the footnotes to which contain a brief description of each search.

The pion-nucleon coupling constant employed for the OPEP values was $g_0{}^2=14$ in the searches recorded in Table III. This value was used as an approximation to the values obtained by adjusting $g_0{}^2$ for best fit to data

[17] C. R. Fischer, K. D. Pyatt, M. H. Hull, and G. Breit, Bull. Am. Phys. Soc. **3**, 183 (1958).
[18] G. Breit, Phys. Rev. **99**, 1581 (1955).

[19] G. Breit, J. B. Ehrman, and M. H. Hull, Phys. Rev. **97**, 1051 (1955).
[20] This parametrization was found convenient in connection with a qualitative unpublished consideration of the effect of meson production on nucleon polarization by one of the authors, according to which the contribution to $P(\theta)$ caused by phase shifts associated with the threshold of a new meson channel may be enhanced. [*Proceedings of the Sixth Annual Rochester Conference on High-Energy Nuclear Physics, April, 1956* (Interscience Publishers, New York, 1956), p. II-26.] After the appearance of the work by H. P. Stapp, T. J. Ypsilantis, and N. Metropolis containing a closely related parametrization, the relationship of the two was noticed and kindly communicated by Dr. J. Shapiro.

TABLE III. Values of D for p-p gradient searches.

Search designation	Initial D	Final D
YRB1[a]	15.9	3.2
YRB2[b]	30.7	3.6
YRB3[c]	27.4	3.1
YLA[d]	13.9	2.26
YLAM(1)[e]	(13.9) 3.11	2.08
YLAM(2)[f]	15.34	1.77 (1.49)

[a] Search YRB1 was made employing as a starting point up to 150 Mev the Signell-Marshak potential but with a 16% longer range of the singlet even potential[17] in order to produce agreement with low-energy data. From 150 Mev on to higher energies the starting curves were joined to Solution 1 of SYM. The shortened range of the spin-orbit potential [see reference 5 and M. H. Hull, K. D. Pyatt, C. R. Fischer, and G. Breit, Phys. Rev. Letters **2**, 264 (1959); these calculations have been performed independently of those by Signell, Zinn, and Marshak, the realization of the questionable nature of the long-range spin-orbit potential having arisen in connection with a paper by G. Breit, in Phys. Rev. **111**, 652 (1958)] was not used to start an independent search. While theoretically more reasonable the discrepancies between its predictions and experiments were large enough to make a change to it as a new starting point of doubtful value.

[b] Same starting point as for YRB1 except for replacement of Solution 1 by Solution 2 of SYM.

[c] Same starting point as for YRB1 except for replacement of Solution 1 by Solution 3 of SYM.

[d] The phase parameters of J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957), were used as starting values.

[e] Continuation of YLA with OPEP values of the 3-4 group of phase parameters ($\delta^F{}_3,\theta^F{}_4,\rho_4,K_4$) which were released, however, in later stages for $E>150$ Mev. The $^3H_{4,5,6}$ were used in the OPEP approximation. The values of the 3-4 group parameters at the end of the YLA search were close to their OPEP values. The replacement of searched values of the 3-4 group by OPEP values increased D to 3.11 at the start of YLAM(1). Up to this point in the succession of searches the charged pion mass and nonrelativistic treatment of the OPEP have been used. For $E>150$ Mev, the parameters $\delta^F{}_3$, $\theta^F{}_4$, ρ_4, K_4 were searched while $^3H_4$ and parameters with higher L or J were not searched.

[f] Insertion of mass of π^0 in place of that of π^+ and relativistic formula for OPEP phase parameters and largely the inclusion of additional data produced an increase of D to 15.34. Further searches decreased it to 1.77. The parameters searched were K^0, $\delta^P{}_0$, $\delta^P{}_1$, $\theta^P{}_2$, ρ_2, $\theta^F{}_2$. All other parameters were kept at their one-pion values below 150 Mev. For $E>150$ Mev the parameters $\delta^F{}_3$, $\theta^F{}_4$, ρ_4 were also searched for but not K_4 because parallel shift calculations indicated considerable stability of phase shifts searched so that searching K_4 would probably make no important difference from a practical standpoint. The results of this search are referred to as YLAM in the graphs. With Harvard data used as published and absolute errors included in data weighting as explained in footnote e to Table I, the value of D for YLAM is 1.49 rather than 1.77. The addition of recent $P(\theta)$ data at 142 Mev, with little searching, increased D to 1.68. This value could probably be lowered somewhat by further searches. The values of $P(\theta)$ at 142 Mev, however, have a small stated error and do not fit in with other measurements in this energy range exceptionally well. The increase of D to 1.68 is largely due to these circumstances.

and in agreement with the pion physics value.[21] In spite of repeated attempts, it proved difficult to lower D significantly for YRB2. No claim can be made regarding there being no path in the phase-parameter space leading from YRB2 to YLA or to YRB1 with a monotonic decrease of D but it appears probable in view of the present experience that if such a path exists it is a circuitous or long one. An inspection of the graphs to be presented later shows significant qualitative differences between YRB2 and the other fits. The Berkeley experience[11] showing a close connection of SYM Solutions 1 and 3 would indicate that YRB1 and YRB3 are part of the same depression of the D surface in the phase-parameter space and that they represent essentially the same fit. YRB3 continued, in fact, to improve and it appears likely that it would lead to much the same results as the continuation of YRB1. A ξ variation between YRB3 and YLAM(1) gave a smooth decrease of the mean square error D from its YRB3 value to that for YLAM (1) demonstrating the essential relationship between the fits.

A ξ variation between YRB2 and YLAM(1) gave a minimum, maximum, and lower minimum of D at $\xi=-0.9$, -0.65, 0, respectively, demonstrating existence of a ridge of the D surface along path of variation.

There is also evidence that the continuation of

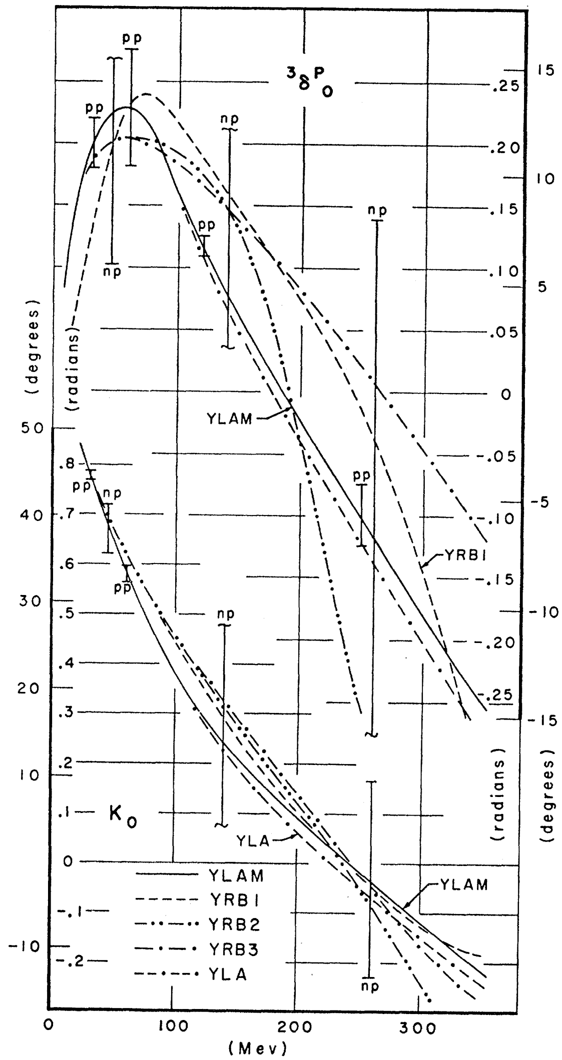

FIG. 1. Phase shifts K_0 for $^1S_0$ state and $\delta^P{}_0$ for $^3P_0$ plotted against energy. In this and other figures the notation described in Sec. 4 is used. The association of a curve with the search that gave it is shown in the curve designation key reproduced in all figures. Full lines, for example, are used for search YLAM. Error bars correspond to limits $\pm\Delta\delta_p$ as determined from Eq. (2.5) in the text and are distinguished by "p-p" and "n-p" marks depending on the scattering data used. In both cases YLAM phase parameters have been used in the error evaluation.

[21] H. A. Bethe and F. de Hoffman, *Mesons and Fields* (Row, Peterson and Company, New York, 1955), Vol. II, see Sec. 42, 43. G. F. Chew, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, California, 1959), Vol. 9, p. 29, reviews the determination of the renormalized coupling constant, f^2 [related to $g_0{}^2$ by $f^2=g_0{}^2(m_\pi/2M)^2$] with the aid of dispersion relations for the pion-nucleon interaction. H. J. Schnitzer and G. Salzman, Phys. Rev. **113**, 1153 (1959), contains a recent determination of f^2 from pion data.

TABLE IV. Displacements of YLAM values in selected energy regions. The entry is given by the number and (in parentheses) the power of 10 by which it should be multiplied. The standard deviations are listed under the numbers to which they refer.

			Shifts and standard deviations of shifted values in radians (ρ_2 excepted) for phase parameters as below[b]						
Case	Energy (Mev)	Quantities	K_0	$\delta^P{}_0$	$\delta^P{}_1$	$\theta^P{}_2$	ρ_2	$\theta^F{}_2$	K_2
(1)	9.69		−1.7(−4)	−1.0(−4)	−1.3(−4)	−3.4(−4)	−1.0(−4)	−5.2(−4)	−1.1(−3)
(2)	95	σ	1.8(−3)	5.2(−4)	−4.6(−3)	−5.2(−3)	−3.5(−3)	1.1(−3)	−5.6(−3)
	98	σ,P,D	±2.8(−2)	±4.2(−2)	±7.1(−3)	±6.4(−3)	±1.3(−2)	±6.0(−3)	±9.1(−3)
(3)	140	R							
	142	P,σ	2.3(−4)	−1.5(−3)	−9.8(−3)	1.3(−2)	4.0(−3)	−2.5(−3)	−1.8(−3)
	147[a]	σ,P,D	±1.3(−2)	±1.1(−2)	±4.7(−3)	±3.1(−3)	±5.1(−3)	±3.8(−3)	±4.9(−3)
(4)	210	P,R,A							
			1.6(−2)	2.3(−3)	−1.0(−3)	3.1(−4)	1.3(−2)	2.5(−3)	−6.0(−3)
	250	σ	±2.8(−2)	±6.4(−2)	±1.7(−2)	±1.5(−2)	±1.7(−2)	±2.9(−2)	±1.3(−2)
(5)	312	P,R,A,D							
	330	σ	−3.8(−3)	−1.5(−3)	−5.6(−3)	−5.7(−4)	−3.0(−3)	1.6(−3)	3.0(−3)
	345	σ	±4.3(−2)	±3.9(−2)	±2.6(−2)	±1.4(−2)	±2.3(−2)	±1.5(−2)	±1.6(−2)

[a] Published values of σ and published absolute errors.
[b] In cases (4) and (5) $\delta^F{}_3$, $\theta^F{}_4$, and ρ_4 were included in the search. The shifts of these quantities were respectively, 1(−5), 2.9(−3), 5.1(−3) for case (4) and 2.0(−3), −2.0(−3), 2(−5) for (5). The standard deviations of the shifted quantities are, respectively, ±1.6(−2), ±1.6(−2), ±1.1(−2) for case (4) and ±1.3(−2), ±9.0(−3), ±1.4(−2) for case (5).

YRB1 would agree with YLAM. The reasons for this belief are as follows. At a preliminary stage of the work the ξ variation procedure described in Sec. III was used between YRB1 and YLA and a minimum was found between the two fits. Had the search path been taken to begin with in the direction of this minimum and then led over YLA, the search would have ended on YLAM. Secondly, in the energy range 100–200 Mev the main differences on the basis of absolute values between YRB1 and YLAM are regarding the values of K_0 and $\delta^P{}_0$. Parallel shift adjustments of YRB1 employing in the local search $R(\theta)$ at 140 Mev; σ and P at 142 Mev; σ, P and $D(\theta)$ at 147 Mev were made. Three gradients were used. The first employed all of the above data with usual weights, the second increased the relative weight of $R(\theta)$ and $D(\theta)$ by a factor of about 2.5, and the third used regular weights again. The agreement with $D(\theta)$ was much improved, the local χ^2 decreasing by factor $\sim$7 and fits to P at 142 and 147 Mev improved also. Both K_0 and $\delta^P{}_0$ moved markedly toward their YLAM values. The difference from YLAM decreased by factor $\sim$4 for K_0 and by $\sim$4.5 for $\delta^P{}_0$. Finally a ξ variation was run between YLAM and YRB1 with the following results. The mean square error, D, varied smoothly between its YRB1 and YLAM values as ξ was changed, going through a minimum value $\sim$0.002 less than D_{YLAM} for a point about 1/50 of the distance from YLAM to YRB1. The standard deviation in ξ according to Eq. (4.1) was calculated to be $\Delta\xi=0.037$. If one obtains uncertainties for the phase parameters as $\Delta\delta_p=\Delta\xi(\delta_p{}^{\text{II}}-\delta_p{}^{\text{I}})$, the values of this quantity at 9.69 Mev are 0.0007, 0.002, 0.0009, 0.000006, 0.0004, 0.00006, 0 radian, respectively, for K_0, $\delta^P{}_0$, $\delta^P{}_1$, $\theta^P{}_2$, ρ_2, K_2, $\theta^F{}_2$. At 95 Mev, the values of $\Delta\delta_p$ for the same phase parameters in the same order are 0.002, 0.002, 0.0002, 0.0006, 0.0008, 0.0005, 0.0001; at 147 Mev they are 0.002, 0.003, 0.00002, 0.0008, 0.0002, 0.0003, 0.0002; at 210 Mev they are 0.00004, 0.003, 0.0004, 0.0009, 0.0006, 0.0004, 0.0002, with additional values of $\Delta\delta_p$ for $\delta^F{}_3$, $\theta^F{}_4$, ρ_4 of 0.0008, 0.0004, 0.0006; at 312 Mev the $\Delta\delta_p$ are 0.0004, 0.0006, 0.001, 0.0002, 0.001, 0.0007, 0.0003, 0.0005, 0.00008, 0.0007, where values for the three extra phase parameters included at 210 Mev are also given.

Parallel shift adjustments give through the resulting displacements an idea of the stability of the fit to different emphasis assigned to different energy regions, the shift being in fact caused by the neglect of all but the data used in the parallel shift adjustment. For YLAM the shifts and the nature of data used in them are summarized in Table IV. Cross-section data at 147 Mev have been used as published in these parallel shift adjustments with data weighting determined by compounding absolute and relative errors in quadrature. A related discussion of treatment of these data is included in footnote e to Table I. The shifts are seen to be negligibly small in some cases such as K_0 at low E and to fall within the error limits obtained in the general procedure of error determination which included all data. The standard deviation in the determination of the shifted value is recorded after the $\pm$ following the shift entry when available.

The results obtained for the phase parameters are shown in Figs. 1-7 inclusive. The best fit YLAM is always shown by a full drawn curve and the conventions for designating other fits have been used consistently in other figures as well, with minor exceptions noted in figure legends.

Comparison with experiment is presented in Figs. 7–19 inclusive. In all cases the figures use experimental *data as published* even though in a few cases the data have been adjusted for the YLAM search as described in footnote (e) to Table I. Most of the figures are self-explanatory. It will be noted that in many cases the

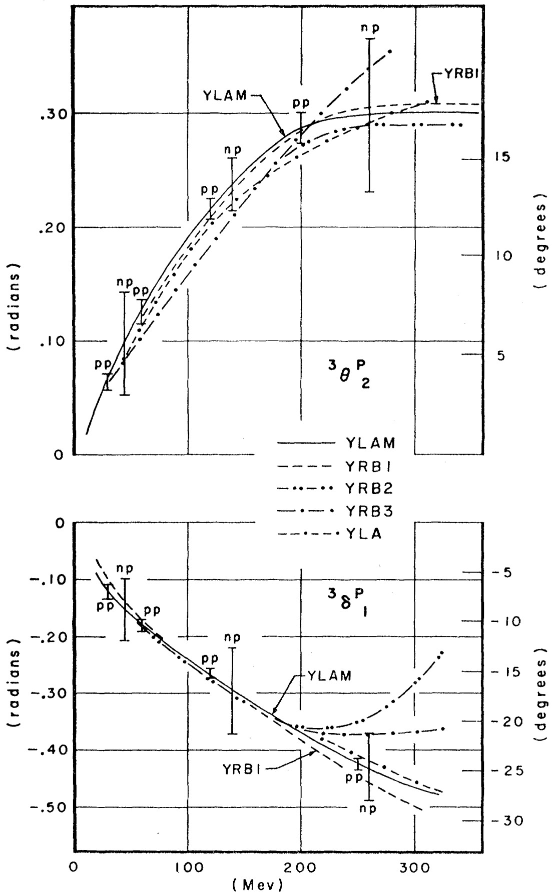

FIG. 2. Phase shift $\delta^P{}_1$ for $^3P_1$ state and phase parameter $\theta^P{}_2$ mainly concerned with $^3P_2$ plotted against energy. Conventions regarding error bars are as in Fig. 1.

quality of the representation is not obviously different between different fits if the comparison is made for one quantity at one energy. In addition to comments contained in figure legends, the following matters may be noted. According to Fig. 4 the phase parameters $\delta^F{}_3$ and $\theta^F{}_2$ approach zero for fits YRB2 and YRB3 at a much higher energy than for the other fits. There is some difficulty in accounting for such a shape of the graphs and they appear to offer some evidence against the results of the YRB2 and YRB3 search series.

No error bars are shown for YLAM values of K_4 in Fig. 6 because this phase shift was not searched in the YLAM series, it having been found in work on search series YRB1 and YRB3 that these searches yielded values of K_4 sufficiently close to those obtainable from the OPEP to justify the direct employment of the latter. A more complete discussion is included in footnote f to Table III.

With reference to Fig. 9 it should be stated that the 98-Mev data were lumped with those at 95 Mev in searches other than YLAM. They have been plotted separately in Fig. 9 so as to secure a minimum alteration of experimental material in presenting comparison with calculation. With reference to Fig. 11 it may be of interest that in the early searches difficulty has been experienced in securing agreement with data on polarization shown in this figure simultaneously with reproducing the absolute value of $\sigma(\theta)$ in the data which show a higher $\sigma(\theta)$ than calculated at 66, 95, 118, and 147 Mev in Figs. 8 and 9. Since other $\sigma(\theta)$ data allow a simultaneous fit to polarization, the searches stabilized in the manner presented in preceding figures. The answer appeared reasonable since the relative angular distributions (ratios of σ at different θ for fixed E) at the energies just mentioned are supposed to be more accurate than the absolute values.

The progressively poorer fit for YRB2 to $P(\theta)$ apparent in Figs. 11, 12, and 13 becomes pronounced

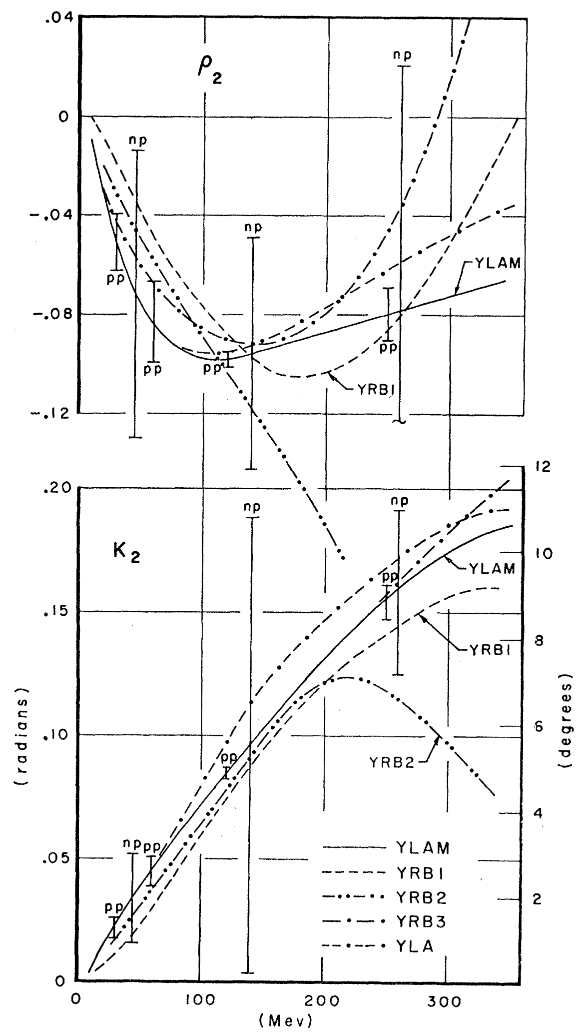

FIG. 3. Phase shift K_2 for $^1D_2$ state and coupling parameter ρ_2 between $^3F_2$ and $^3P_2$ states. Conventions regarding error bars are as in Fig. 1. Large errors in n-p case illustrate insensitivity of existing n-p data to some phase parameters.

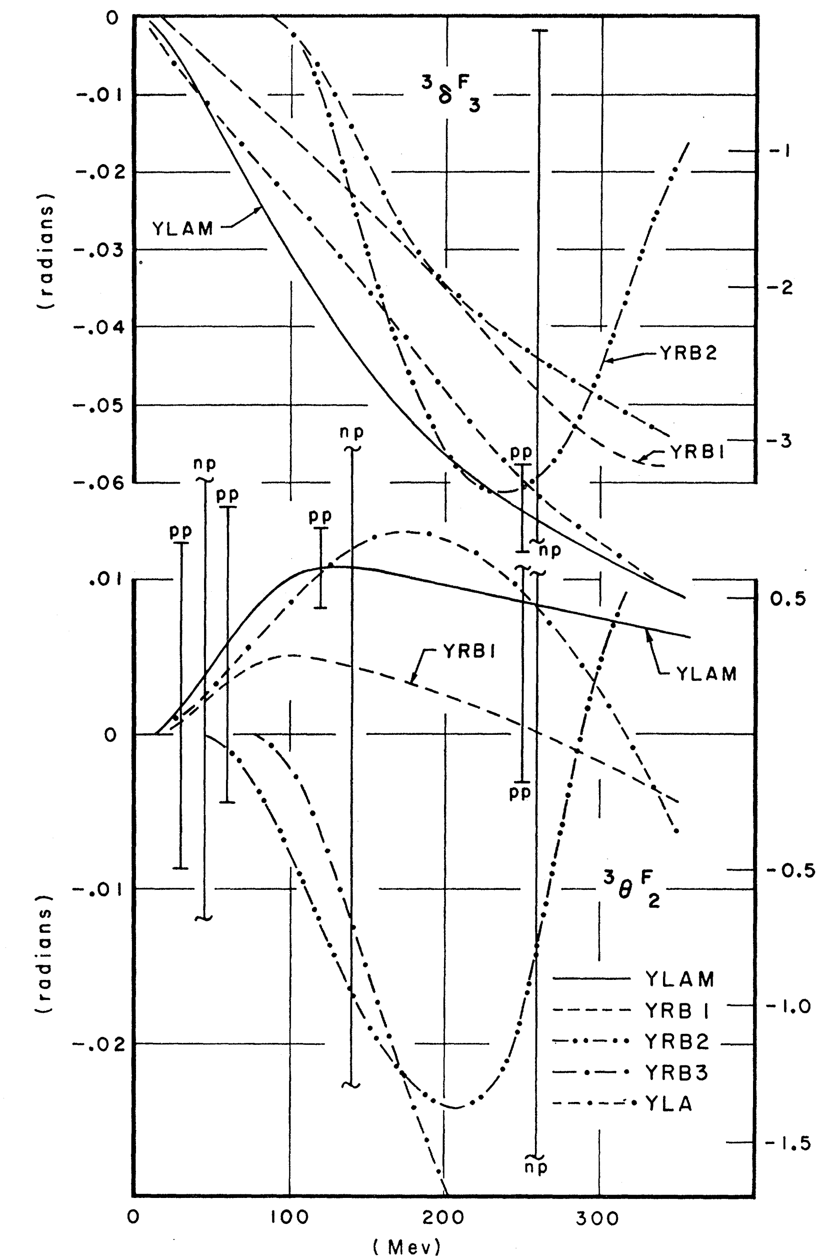

FIG. 4. Phase parameters $\theta^F{}_2$ for $^3F_2$ state and $\delta^F{}_3$ for $^3F_3$ plotted against energy. Conventions regarding error bars are as in Fig. 1. At small E the values of these phase parameters are small and poorly determined by the data. They are shown as zero.

at 276 Mev as seen in Fig. 14. The disagreement with experiment at 312 Mev around the maximum of the YRB2 curve is also pronounced. This feature of YRB2 appears to be systematic and definite enough to justify classifying the fit as improbable. In the same connection it will be observed from graphs of $\sigma(45°)$ in Fig. 18 that YRB2 values have an improbable energy trend when compared with experiment at this and other energies.

Results of experiments to obtain $\sigma(\theta)$ at $\theta=45°$ and 90° in addition to those covered in angular distribution measurements are included in Fig. 18. Except as noted in Tables I and II these data have not been included in the searches on account of the extra machine time their inclusion would have involved. They are plotted for comparison with other data and the curves calculated from the search determined values of the phase parameters. The additional data used are as in Table V.

In Figs. 20, 21 a few comparisons are made with fits obtained by means of potentials. The "YRB1 start" are essentially for the SM potential below 150 Mev except for the change in the range of the singlet potential previously mentioned. There are also some additional slight differences in the values of phase parameters as published by SM and calculated at

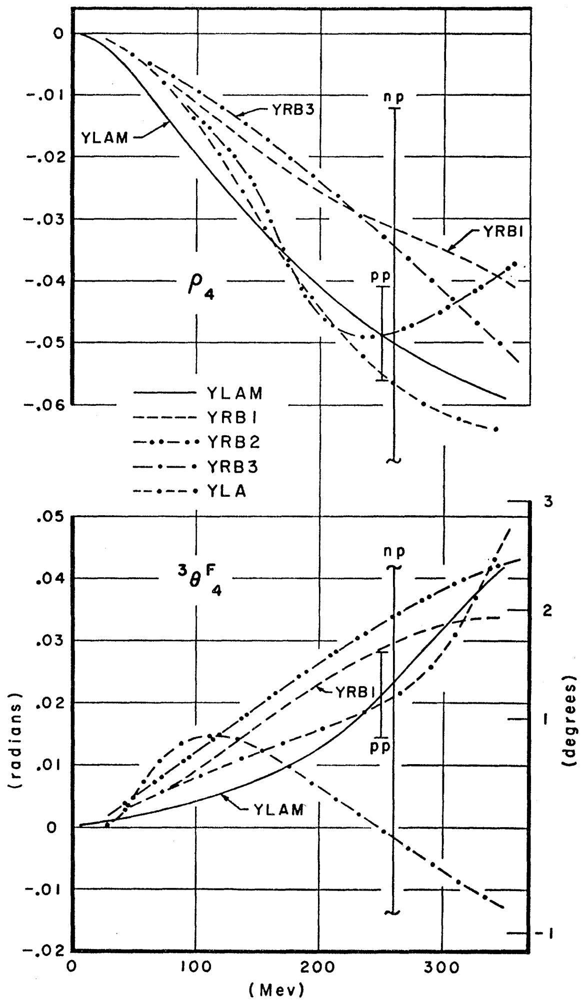

FIG. 5. Phase parameters $\theta^F{}_4$ and ρ_4, the "phase shift" for $^3F_4$ and the coupling parameter to $^3H_4$, plotted against energy. Conventions regarding error bars are as in Fig. 1.

Yale. These are probably at least partly due to different ways of using the numerical tables of the Gartenhaus potential. These figures have not been selected to show worst cases of agreement of the popular potentials with data. Thus, for example, the SM potential with singlet range adjusted to fit low E data is in poor agreement with differential cross-section data giving values $\sim$0.5 of the experimental at $\theta=60°$ and values $\sim$20% above the experimental at $\theta=20°$. It gives a maximum of $P(\theta)$ of $\sim$0.5 of the experimental at 312 Mev. These disagreements have been pointed out in slides at the London Conference[22] by one of the authors on the basis of collaborative work reported on here. As has been mentioned in the talk referred to, these comparisons may not be quite fair to the SM potential because the potential has been primarily intended for $E<150$ Mev and because the calculations showing disagreement employed some modifications of the SM original prescription. Nevertheless, the disagreements show the necessity for improvement which has prompted the present work.

V. DISCUSSION

The YLAM fit agrees very well with that of Gammel and Thaler (GT) for K_0, $\delta^P{}_0$ and $\delta^P{}_1$. For K_0 the GT values are low in comparison with YLAM by $\sim$0.025 radian at 200 Mev and 0.020 radian at 300 Mev. For $\delta^P{}_0$ they are low by 0.010 radian at 200 Mev and 0.014 at 300. There is also a noticeable difference in the position of the maximum of $\delta^P{}_0$, which occurs at a 10-Mev lower energy for GT than for YLAM. For $\delta^P{}_1$ the two curves cross at $\sim$170 Mev with GT running above YLAM above the crossing. There appears to be a second crossing at about 340 Mev. For $\theta^P{}_2$ GT is consistently low by $\sim$0.02 radian from 170 Mev to 340 Mev, the agreement becoming better on an absolute basis at lower E. For ρ_2 the agreement on a relative basis is poor, GT being high by 0.039 at 340 Mev while YLAM gives -0.06. The ρ_2 versus E plots cross at $\sim$147 Mev with GT running low by -0.0085 at 70 Mev in a total of ~ -0.09. For K_2 fit GT is high by $\sim$0.077 at 250 Mev while YLAM gives 0.155 at this E.

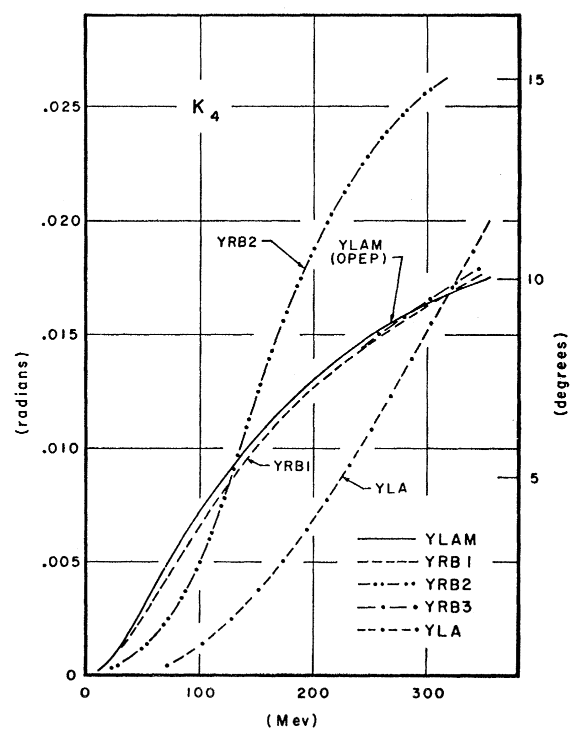

FIG. 6. Phase shift K_4 for $^1G_4$ state plotted against energy. No error bars are shown for reason explained in the text.

[22] G. Breit, International Conference on Nuclear Forces and the Few Nucleon Problem, University College, London, July, 1959 (to be published). This report was based on work done in collaboration with M. H. Hull, Jr., K. D. Pyatt, Jr., C. R. Fischer, K. Lassila, and T. Degges.

The fractional deviation decreases at low E but at 100 Mev it still amounts to ~ 0.022 in a YLAM value of 0.072. For $\theta^F{}_2$ the GT value is about 2.7 times that for YLAM at 150 Mev but there is a crossing at ~ 310 Mev.

Signell and Marshak's values for K_0 are higher than those for YLAM by 0.02, 0.04, 0.11, 0.12, 0.14 radian at 18, 40, 100, 150, 300 Mev, respectively. For $\delta^P{}_0$

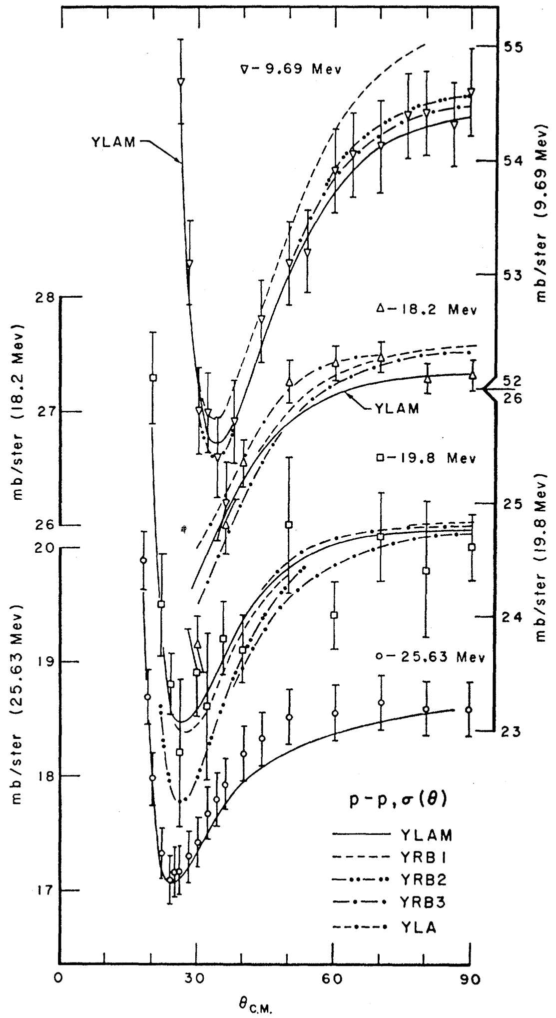

FIG. 7. Representations of the proton-proton differential scattering cross section, $\sigma(\theta)$, at 9.69, 18.2, 19.8, and 25.63 Mev as a function of the center-of-mass scattering angle, θ, provided by phase parameters of Figs. 1–6 combined with OPEP values for phase parameters as described in the text. The curves are being compared with experiment. The same conventions regarding full and dashed lines with gradient searches is used as in preceding figures. Data sources for this and succeeding figures are listed in Tables I and II. Different designations of experimental points are used to distinguish between different energies. Data at 25.63 Mev became available after completion of all searches other than YLAM and are compared with that fit only. Cross-section data at 14.16 Mev mentioned in footnote[b] to Table I, which were available only for late work on YLAM, are not compared with prediction in this figure. However, the data are well fitted by YLAM: the weighted mean square error for these data alone is 0.55.

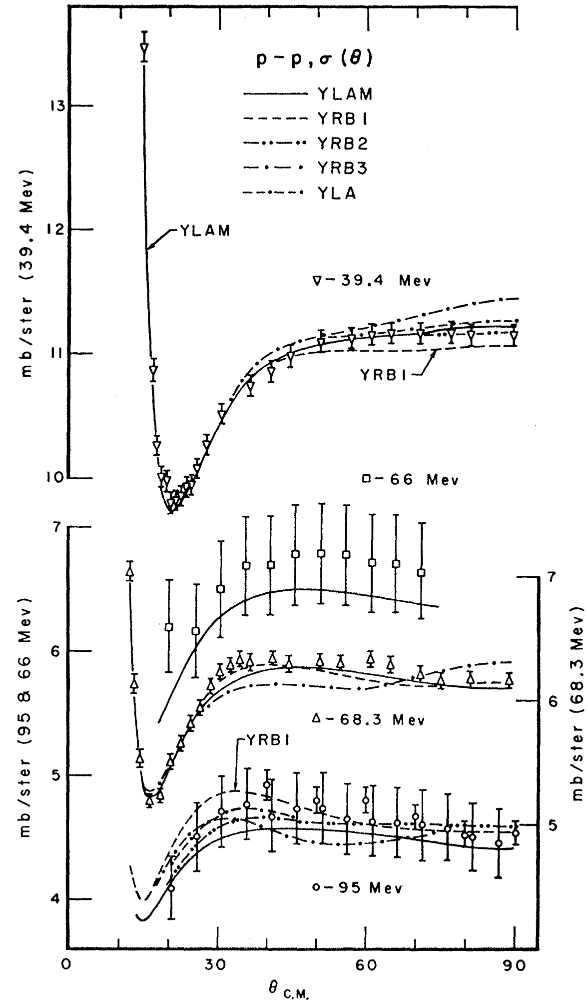

FIG. 8. Representations of the proton-proton differential scattering cross section, $\sigma(\theta)$, at 39.4, 66, 68.3, and 95 Mev compared with experiment. Data at 66 Mev were omitted in searches other than YLAM. At 66 and 95 Mev the fit YLAM is seen to fall consistently below the measured values. The *data* shown in this and other figures are directly *as published*, even though adjustments of the Harvard data have been made for probable shifts in the YLAM search.

the values in their Table I show a qualitatively different trend from those for YLAM giving a value low by 0.072 radian at 40 Mev and values of 0.28, 0.27, 0.17 at 100, 150, 300 Mev, respectively. For $\delta^P{}_1$ there is fair agreement at 100 and 150 Mev while at 40 Mev their value is high by 0.03 radian and by ~ 0.11 radian at 300 Mev. For K_2 the SM values are low in comparison with YLAM by ~ 0.012 at 40 Mev, high by ~ 0.027 radian at 300 Mev and the crossover is at ~ 210 Mev. In view of adjustments of core radii considered by Signell and Marshak and by Signell, Zinn, and Marshak in relation to avoiding bound $^3P_2$ and $^1P_1$ states and for fitting purposes, it does not appear appropriate to be making a more detailed comparison. But it may do no harm to state that even though search YRB1 is related to the SM potential, its end result as well as that of YLAM differ appreciably from their starting points.

Good fits to *p-p* data employing a static potential have recently been obtained by Bryan,[23] although they are not as good as those for YLAM. The relatively good reproduction of polarization data is in accord with the relatively good agreement of ^3P parameters for the two fits. A marked disagreement is present for K_0. Taking YLAM as a reference standard Bryan's K_0 is high by 0.06 radian at 40, 0.11 at 150, 0.09 at 210, 0.08 at 310 Mev. At the same energies the deviations of Bryan's values from those for YLAM are, respectively, -0.056, 0.020, 0.020, 0.002 for $\delta^P{}_0$; 0.027, 0.015, 0.018, -0.008 for $\delta^P{}_1$; -0.028, -0.023, -0.023, -0.016 for $\theta^P{}_2$; 0.030, -0.011, -0.020, -0.019 for ρ_2; -0.016, -0.017, -0.002, 0.040 for K_2; -0.0014, 0.0066, 0.012, 0.017 for $\theta^F{}_2$. In absolute value Bryan's $\delta^F{}_3$ is smaller than the same quantity for YLAM by roughly 12% at 310 Mev, 25% at 150 and a factor ~ 4 at 40 Mev. On the other hand, his $\theta^F{}_4$ exceeds that for YLAM by a factor ~ 2 at 310 Mev and a somewhat smaller one in energy region down to 150 Mev while at 40 Mev his value is much smaller than that for YLAM. For ρ_4 there is very good agreement between the two fits.

The GT K_0 agrees much better with YLAM than Bryan's. At high E both GT and Bryan give higher K_2 than YLAM but below 210 Mev the Bryan fit runs

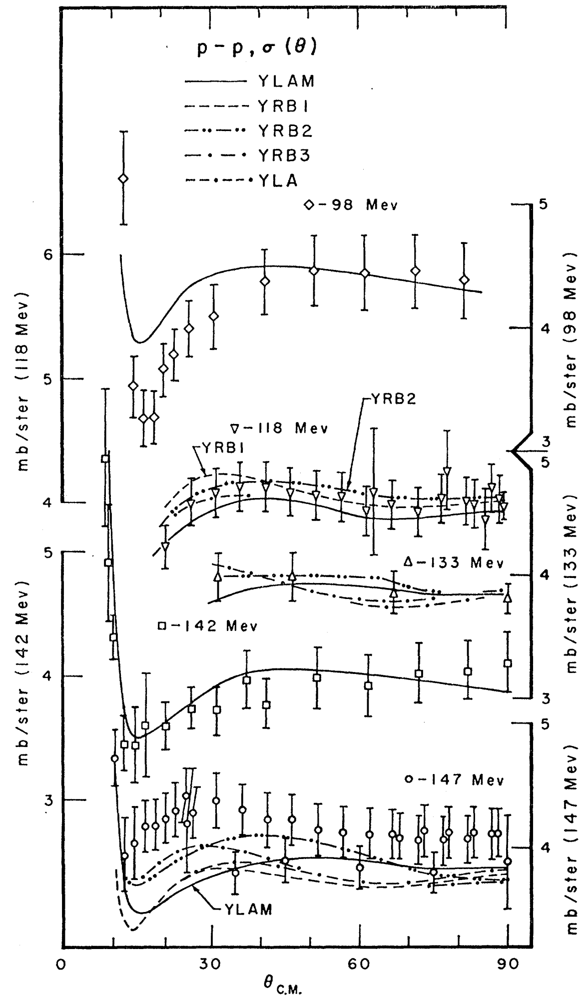

FIG. 9. Representations of the proton-proton differential scattering cross section $\sigma(\theta)$ at 98, 118, 133, 142, and 147 Mev compared with experiment. At 98 Mev the observed interference minimum is deeper than predicted. At 95 (Fig. 8), 118 and especially at 147 Mev the published Harvard data fall above the calculated curves, although the ratios at different angles are in good agreement with prediction. The Harwell data at 35°, 45°, 60°, and 75° fall on the YLAM curve within their standard deviations. The Harwell 25° point is on the general level of the Harvard data but has a large standard deviation. At 90° data from both laboratories have been averaged.

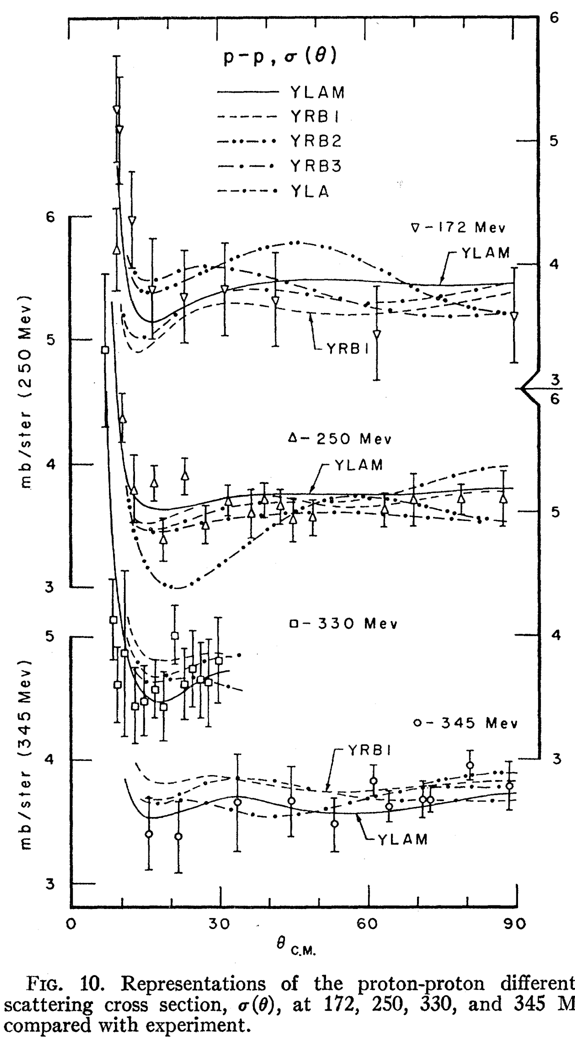

FIG. 10. Representations of the proton-proton differential scattering cross section, $\sigma(\theta)$, at 172, 250, 330, and 345 Mev compared with experiment.

[23] R. A. Bryan, Bull. Am. Phys. Soc. 5, 35 (1960), and preprint of paper to be published. The writers are grateful to Dr. Bryan for the communication of his results before publication and for supplying them with additional information regarding his work.

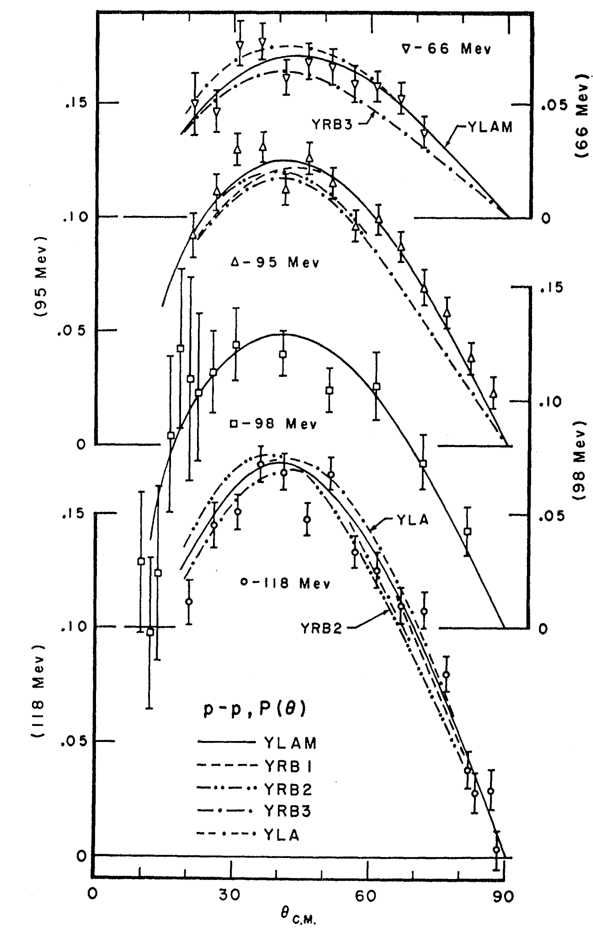

FIG. 11. Representations of the proton-proton polarization, $P(\theta)$, at 66, 95, 98, and 118 Mev compared with experiment. Data at 98 Mev became available after completion of all searches other than YLAM.

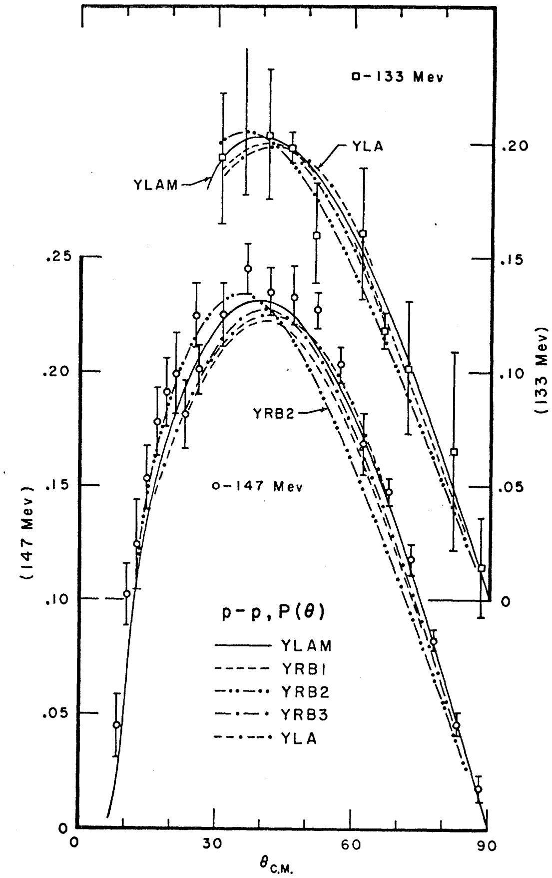

FIG. 12. Representations of the proton-proton polarization, $P(\theta)$, at 133 and 147 Mev compared with experiment. The experimental point at 147 Mev (Harvard) at 6.2° with $P=-0.004 \pm 0.014$ is not shown. It agrees with YLAM within the error limits. The point by the same observers at 4.13° with $P=-0.120 \pm 0.040$ disagrees with calculated values of -0.0003, -0.0008, -0.0011 for YLAM, YRB1, YRB2. It may be remarked that the angle is probably the smallest at which a measurement of P has been attempted and that special difficulties may be expected to enter.

below while the GT fit runs above YLAM down to 60 Mev, below which the GT K_2 is also below that for YLAM. For $\delta^P{}_0$ and $\delta^P{}_1$ YLAM values are between those for GT and Bryan at most E. For $\theta^P{}_2$ the agreement between GT and Bryan is better at most energies than that with YLAM, both potentials giving values below those for YLAM. From 150 Mev down to $E=10$ Mev GT and YLAM begin to agree much better ending in very good agreement at the lowest E while Bryan's value is definitely low. For ρ_2 YLAM is between the other two fits at the higher E, while at low E the Bryan fit deviates from the closely agreeing GT and YLAM giving a relatively small absolute value of this parameter. For $\theta^F{}_2$ there is a consistent disagreement of YLAM with the values calculated from potentials, in both cases YLAM giving the smaller values above ~ 100 Mev. At 40 Mev, however, Bryan's value is again below that for YLAM. The smallness of absolute values of Bryan's phase parameters at the lower E can be associated with the very short range of the V_{LS} potential used by him. From the comparisons just enumerated there is some likelihood that, while Bryan's general intention of emphasizing the effect of p waves[24] in comparison with those for higher L in accounting for polarization effects may be correct, it may have been carried too far. There is also the possibility[24] that not all of the effects attributed to V_{LS} are caused by the term customarily employed and that it may in effect be

[24] The connection between the range of V_{LS} and the empirically desirable relative suppression of V_{LS} effects in states with high L has been discussed in connection with a comparison of the Signell-Marshak and Gammel-Thaler ranges with theory in G. Breit, Phys. Rev. **111**, 652 (1958), see pp. 662, 663, with a suggestion that the physical effects may not be covered by the introduction of a static V_{LS}, and that the empirical evidence is only for smallness of obvious V_{LS} effects in $T=1$, $L>1$ states which could be explained by special circumstances in the production of P waves leading to an apparent V_{LS} in $L=1$ states, with possible absence in all but p states; they have been used therefore for p states only in Hull, Pyatt, Fischer, and Breit, see reference a of Table III, and the lack of clear evidence for effects of V_{LS} has been again mentioned by Bryan.[23]

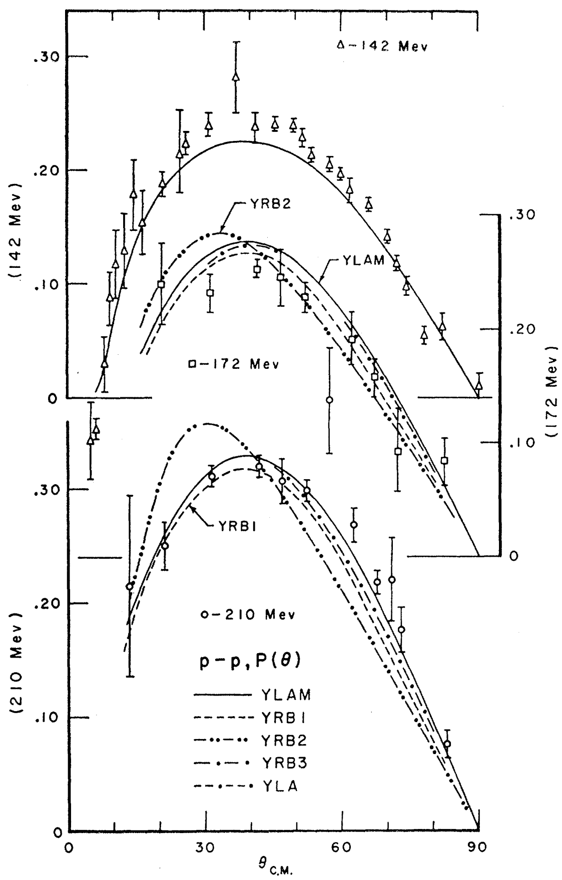

FIG. 13. Representations of the proton-proton polarization, $P(\theta)$, at 142, 172 and 210 Mev compared with experiment. Data at 142 Mev became available after completion of all searches other than YLAM and have been included in relatively few YLAM searches. These data are relatively high around 40°. There appears to be also a definite disagreement of the experimental $P=-0.027\pm0.009$ at 6.23°. In view of the difficulty of measurements of P at such small angles and the presence of similar disagreements at other angles and the agreement of YLAM with other P data, the presence of some additional experimental errors might perhaps be suspected.

different for states with different J. An analysis of YLAM values into S_{12} and $\mathbf{L}\cdot\mathbf{S}$ terms has not been made so far and a more definite interpretation of this point has to be postponed.

The comparison of $\delta^F{}_3$, $\theta^F{}_4$, and ρ_4 is probably less meaningful than that of the lower L and J partly because in all of these cases one-pion values are used below 150 Mev and partly because searches on individual parameters with high L and J are likely to be less significant. There is apparently no general simple interpretation of the relationships between the phase parameters for these cases.

The possibility of determining the best value of the pion-nucleon coupling constant by a fit to data of the collection of effects of all L and J above a certain minimum has been used for tests of charge independence and has been referred to previously. It may appear surprising that relatively high accuracy results in the value of the coupling constant even though there is considerable flexibility in the values of non-one-pion phase parameters. The authors do not have a complete explanation of this fact. A partial explanation is as follows. Dividing the phase parameters schematically into low L, J and high L, J categories the rôle of the first category is somewhat similar to that of long wavelength terms in a Fourier expansion and the second to that of the short wavelength parts of the analysis. If, taking an extreme example, one had to analyze by means of a Fourier series a function of the type $a+bf(x)$ with an even and known $f(x)$, and if $f(x)$ should be wiggly and representable by a series starting with a relatively high harmonic, a somewhat similar situation would be obtained. Since the form of $f(x)$ is supposedly known, the parameter b is determinable accurately and a quadrature will furnish it independently of whether a is known accurately or not. The example is not a fair representation of the actual situation because

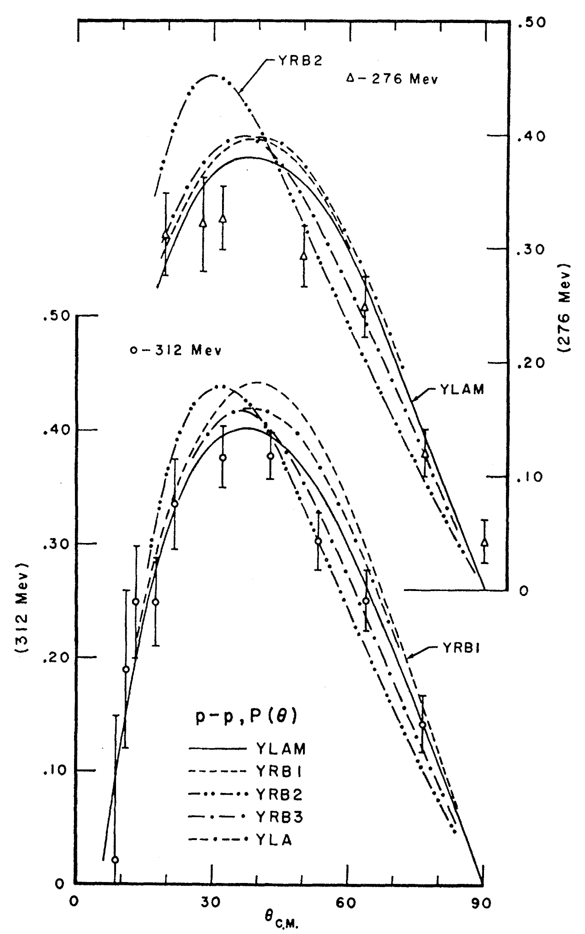

FIG. 14. Representations of the proton-proton polarization, $P(\theta)$, at 276 and 312 Mev compared with experiment. An experimental point at 6.5° with $P=-0.21\pm0.27$ is not shown. It agrees with YLAM within the limits of error.

for the latter a simple quadrature cannot be used and also because in the actual case the functions to be represented are smooth. The rôle of the one-pion part is, nevertheless, that of representing an oscillatory function with a short average wavelength and a least squares fit is an approximation to the quadrature procedure. The oscillatory part arises as a result of subtraction of contributions of effects of low L from the actual functions. The coefficients of the low L contributions are determined by the long wavelength features of the functions and reasonable accuracy in their values is not surprising. Fair accuracy in g^2 which is analogous to b in the example is, therefore, not altogether surprising.

In the earlier development of YRB1 one-pion values have been used for ρ_6 and all other phase parameters with $L>5$. The $L=5$ parameters $\theta^H{}_4$, $\delta^H{}_5$ and $\theta^H{}_6$ were used in the search employing their one-pion values as a starting point. As the search progressed these parameters oscillated around their one-pion values. For this reason in later work one-pion values were used

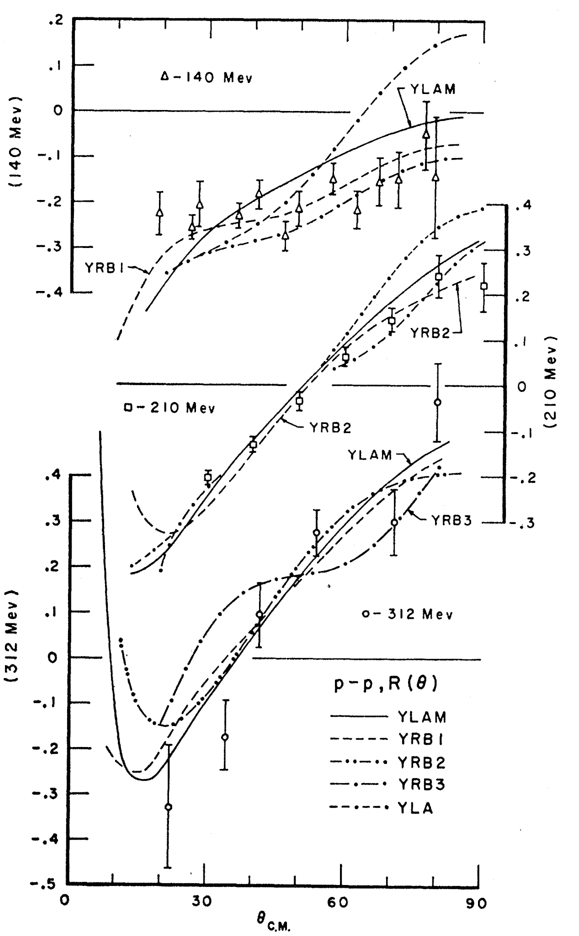

FIG. 15. Representations of the proton-proton rotation of polarization parameter, $R(\theta)$, at 140, 210, and 312 Mev compared with experiment.

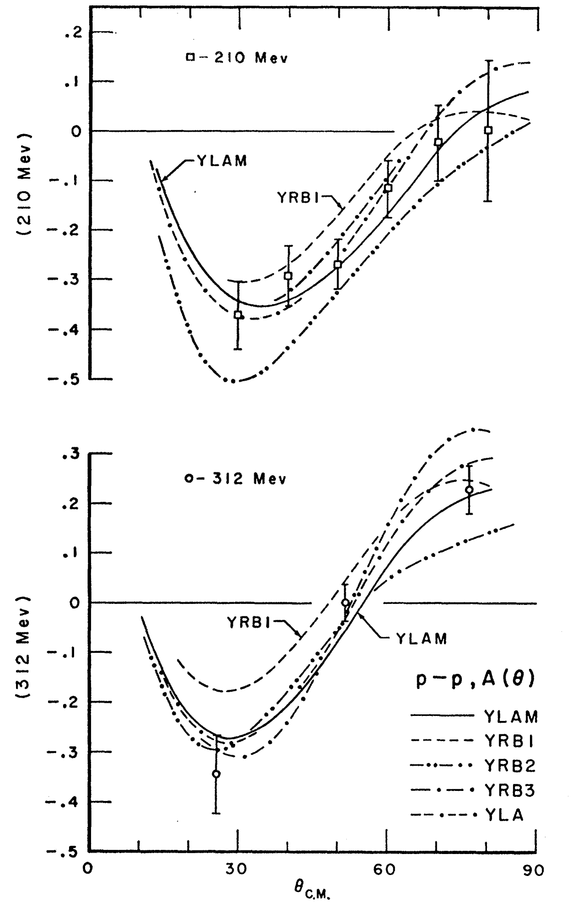

FIG. 16. Representations of the proton-proton triple scattering parameter $A(\theta)$ at 210 and 312 Mev compared with experiment.

for these parameters so that $L\geq 5$ became the criterion for applicability of the OPEP in the searches. The experience with the $L=5$ parameters speaks somewhat in favor of the physical reality of OPEP. In view of the apparent applicability of this potential, it was thought safe to use OPEP values for the 3–4 group for $E<150$ Mev as in search YLAM.

The values $\delta^F{}_3$, $\theta^F{}_4$, and ρ_4, determined by the YLAM search, fit in well with the OPEP values of these quantities although they are not identical with them at the highest energies considered. A marked difference exists for $\theta^F{}_4$, which has the YLAM value 0.040 ± 0.007 as compared with the OPEP value 0.017 at 340 Mev. For ρ_4 the YLAM value is again higher than that for OPEP but barely outside the error limit. For $\delta^F{}_3$ the situation is intermediate between those just mentioned. The difference between the YLAM and one-pion values varies roughly linearly with $E-150$ Mev for $\delta^F{}_3$ and $\theta^F{}_4$ in the $E>150$ Mev region, but with a smooth join at 150 Mev. For ρ_4 the plots cross at $\sim$225 Mev. It is thus seen that there is general consistency of the assumption that one-pion values are applicable to the $L=4$ parameters below 150 Mev.

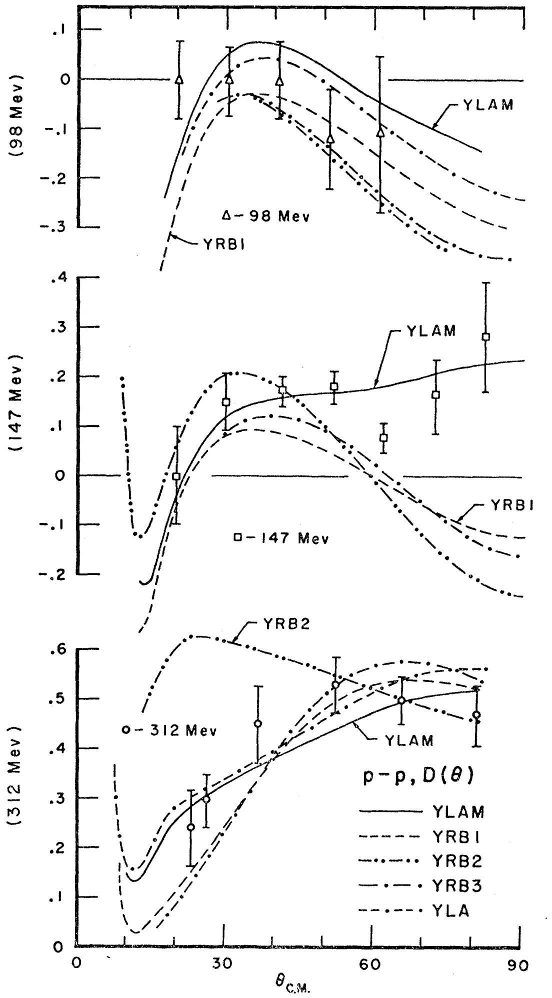

FIG. 17. Representations of the proton-proton depolarization parameter, $D(\theta)$, at 98, 147, and 312 Mev compared with experiment. The recent data at 98 Mev have been used in only a few searches for YLAM and not at all in the other searches. The theoretical curves with other search designations used at this energy have been calculated at 95 Mev in machine runs preceding availability of 98-Mev data. Fits of the YRB series disagree with the 147-Mev Harvard data in the same general manner as the SM potential. An important contributor to the difference between the YLAM and YRB1 results at 147 Mev is the difference between the values of $\delta^{P}{}_{0}$ for the two fits. Changing this phase shift in the YLAM phase-parameter set to its YRB1 value gives a value of D roughly half way at 90° and a crossing with YRB1 at about 55°. The K_0 has a relatively small effect. The Harwell values are below the YRB1 curve at larger angles: −0.19 at 50°, −0.18 at 60°, −0.39 at 70°. The course of the searches giving the $\delta^{P}{}_{0}$, K_0 for YLAM versus YRB1 appears to be sensitive to assignments of other parameters such as those of odd parity, $J=2$.

In a recent publication[12] the applicability of OPEP phase parameters for $L\geq 3$ at 275 Mev inferred essentially in the above manner was interpreted as an indication of the dominance of the OPEP generally denoted as $V^{(2)}$ at $r>1.6(5)\times10^{-13}$ cm or $x>1.1(7)$.

Although low-energy (a few Mev) p-p and n-p data can be well represented by means of s waves alone with due account of vacuum polarization, it has been shown by Hull and Shapiro[25] and confirmed by MacGregor[26] that it is possible to represent the data by admitting p waves in the analysis and that appreciable differences between the three ^3P phase shifts are admissible resulting in appreciable polarization. The fits YRB1, YLA, and the others mentioned in this report give very small polarizations. Thus, at 18.2 Mev and $\theta=50°$ the calculated values are 0.06% and 0.08%, respectively, for a modification of YRB1 and another version of the YRB search procedure. These may be compared with the Blanpied[27] measured value of (0.6±0.5)% at 16.0 Mev at $\theta=12.5°$ and the 3.3-Mev values of Alexeff and Haeberli[28] of (0.08±0.16)% at $\theta=30°$, (0.25±0.16)% at 45°, (0.59±0.24)% at 53°. These values have not been included in the searches for phenomenologic fits reported on. There is likely to be difficulty in reconciling the larger values in these difficult experiments with potentials currently in vogue.

TABLE V. Measurements of the differential cross section shown in Fig. 18 in addition to those used in gradient sources.

Energy (Mev)	Angle	Quantity plotted	Value (σ in mb/sr)	Source of data
18.8	90°	$k^2\sigma$	0.616±0.016	a
21.9	90°	$k^2\sigma$	0.602±0.014	a
25.2	90°	$k^2\sigma$	0.569±0.010	a
28.16	90°	$k^2\sigma$	0.552±0.008	b
29.4	90°	$k^2\sigma$	0.577±0.011	c
31.15	90°	$k^2\sigma$	0.551±0.006	b
31.8	90°	$k^2\sigma$	0.552±0.006	a
34.20	90°	$k^2\sigma$	0.550±0.006	b
36.9	90°	$k^2\sigma$	0.540±0.006	b
39.6	90°	$k^2\sigma$	0.534±0.006	b
41	90°	$k^2\sigma$	0.564±0.040	d
50.15	90°	$k^2\sigma$	0.508±0.006	b
52	90°	$k^2\sigma$	0.553±0.039	d
61.92	90°	$k^2\sigma$	0.504±0.006	b
70	90°	$k^2\sigma$	0.503±0.030	d
134	90°	σ	3.80±0.13	e
160	90°	σ	4.16±0.19	f
164	90°	σ	3.60±0.17	g
230	90°	σ	3.58±0.19	f
86	45°	$k^2\sigma$	0.521±0.026	h
86	45°	σ	5.03±0.27	h
230	45°	σ	3.58±0.19	f

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[d] U. E. Kruse [private communication to Hess, reference 14].
[e] Reference P13 of Hess, reference 14, based on T. G. Pickavance (private communication to W. N. Hess), and J. M. Cassels, Proc. Phys. Soc. (London) **A69**, 495 (1956).
[f] Reference C13 of Hess, reference 14, based on O. Chamberlain, G. Pettengill, E. Segrè, and C. Wiegand, Phys. Rev. **95**, 1348 (1954), G. H. Pettengill, thesis, University of California Radiation Laboratory Report UCRL-2808 (unpublished), and private communication to W. N. Hess.
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The polarization correlation coefficient C_{nn} according

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[27] W. A. Blanpied, Phys. Rev. **116**, 738 (1959).
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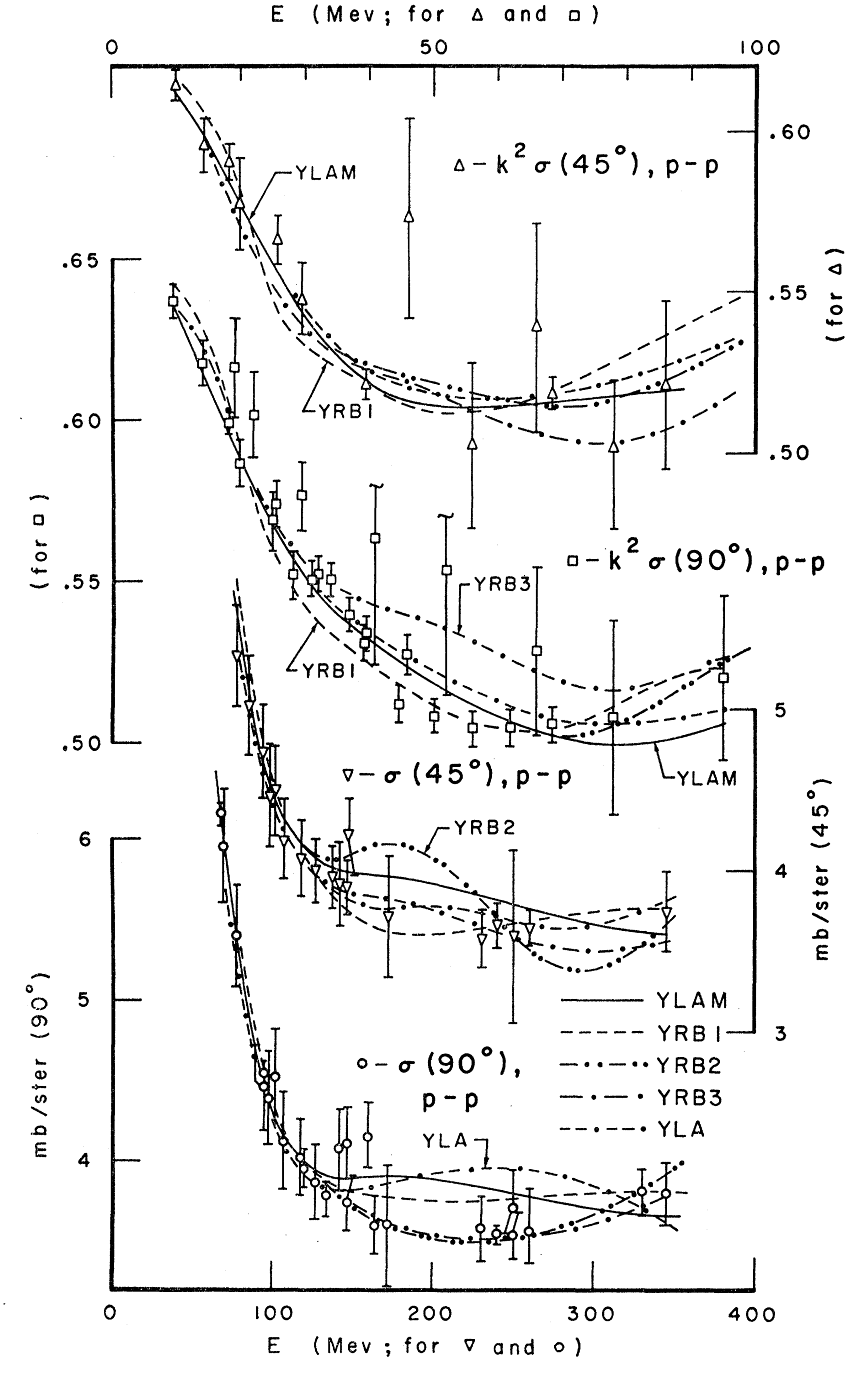

FIG. 18. Representations of $k^2\sigma(\theta)$ and $\sigma(\theta)$ at $\theta=45°$ and 90° as functions of energy compared with experiment. The energy scale for $k^2\sigma(\theta)$ is at the top of the figure, while that for $\sigma(\theta)$ is at the bottom. Below 70 Mev the curve $k^2\sigma(90°)$ for YRB2 follows closely that for YLAM and is not shown to avoid confusion.

to Allaby, Ashmore, Diddens, and Eades[29] at $\theta=90°$ and $E=320$ is 0.75 ± 0.11. The lower limit of their standard error belt, i.e., 0.64 is in agreement with the calculated value of 0.63(6) for fit YRB1 but is appreciably higher than the expected value ~0.52 for fit YLAM. The latter fit is on the whole, however, the better of the two. The value of C_{nn} (90°) obtained by Ashmore, Diddens, and Huxtable[30] at 382 Mev is 0.42 ± 0.085 and agrees better with YLAM calculation at 320 Mev but this agreement is at the wrong energy. Since C_{nn} measurements are available only in a few cases, the disagreements just mentioned are not definite enough to give preference to YRB1 over YLAM. Additional measurements of this parameter at

[29] J. V. Allaby, A. Ashmore, A. N. Diddens, and J. Eades, Proc. Phys. Soc. (London) **74**, 482 (1959).

[30] A. Ashmore, A. N. Diddens, and G. B. Huxtable, Proc. Phys. Soc. (London) **73**, 957 (1959).

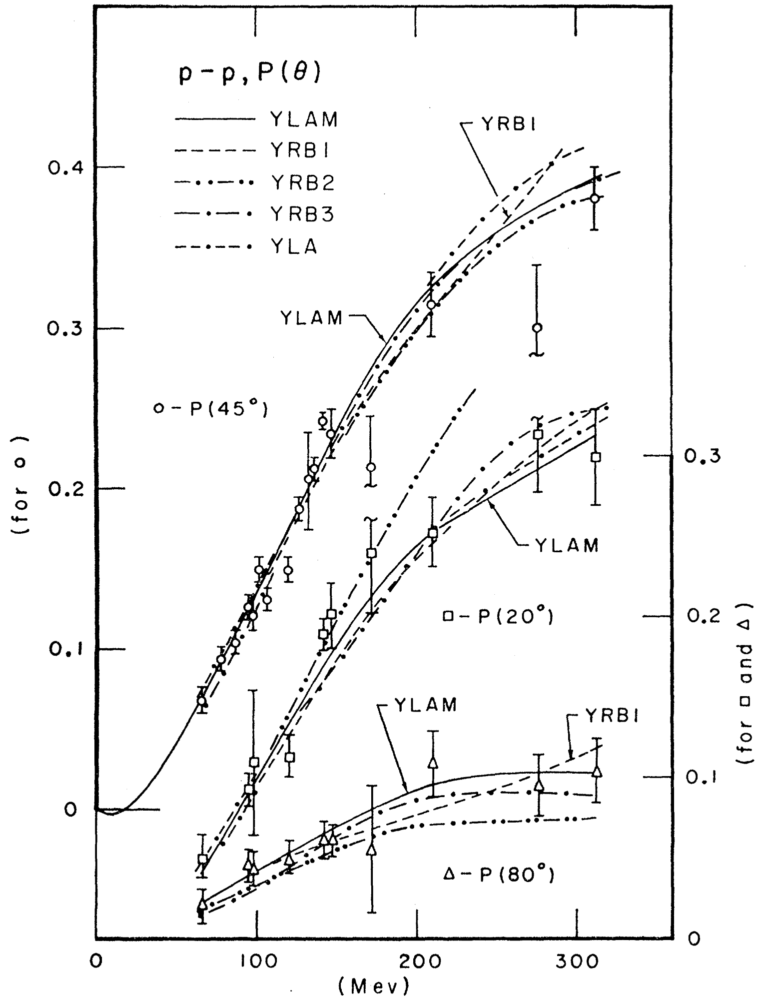

FIG. 19. Representations of the proton-proton polarization, $P(\theta)$, at $\theta=20°$, 45° and 80° as a function of the energy compared with experiment. Only the 45° datum at 86 Mev among those shown has not been used in the search.

more energies and angles would be helpful. According to Ashmore, Diddens, Huxtable, and Skarsveg,[31] denoting triplets and singlets by t and s,

$$C_{nn}=(\sigma_t-\sigma_s)/(\sigma_t+\sigma_s),$$

an exact relation neglecting the relatively small Coulomb scattering. It should accordingly be possible to resolve the usual σ into σ_s and σ_t. The calculated $C_{KP}(90°)$ changes from 0.44±0.05 for Set 1 of Cziffra, MacGregor, Moravcsik, and Stapp[11] to 0.49±0.09 for Set 2 at 310 Mev and appears to be not sensitive to the choice of phase parameter. The experimentally available value[30] of 0.83±0.10 at 382 Mev is not truly comparable being at an appreciably different energy.

A connection between the energy rate of change of a phase shift to the time delay in scattering from a system has been pointed out by Eisenbud.[32] This semiclassical connection has been more generally related to *causality* by Wigner[33] who worked out a rigorous lower limit for $d\delta/dk$ and studied it in special cases. It is also apparent[34] that the inequality is equivalent to the condition that the energy rate of change of the radial logarithmic derivative be negative. Wigner's discussion is based on properties of $\mathcal{R}$, the derivative matrix, and his result applies, therefore, under more general circumstances than those covered by a static potential description of the interaction between the two parts of the system whose relative motion is described by the phase shift δ. The presentation[34] of the related inequality for the energy rate of change of the radial logarithmic derivative has the same degree of generality. The connection with causality makes it perhaps of special interest to test the phase parameters by means of the inequalities mentioned.

[31] A. Ashmore, A. N. Diddens, G. B. Huxtable, and K. Skarsveg, Proc. Phys. Soc. (London) **72**, 289 (1958); see H. P. Stapp, University of California Radiation Laboratory Report UCRL-3098, 1955 (unpublished).

[32] L. Eisenbud, dissertation, Princeton, 1948 (unpublished).

[33] E. P. Wigner, Phys. Rev. **98**, 145 (1955).

[34] G. Breit, *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1959), Vol. **41**, Part 1, Sec. 47(β).

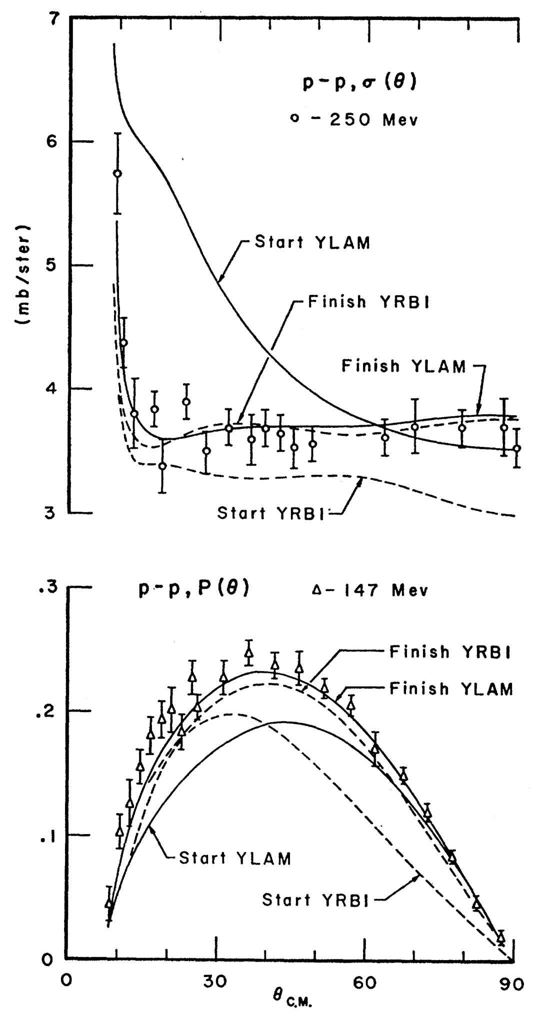

FIG. 20. Illustrations of improvements in fits produced through gradient searches in search series YRB1 and YLAM. Comparisons made for $\sigma(\theta)$ at 250 Mev and $P(\theta)$ at 147 Mev.

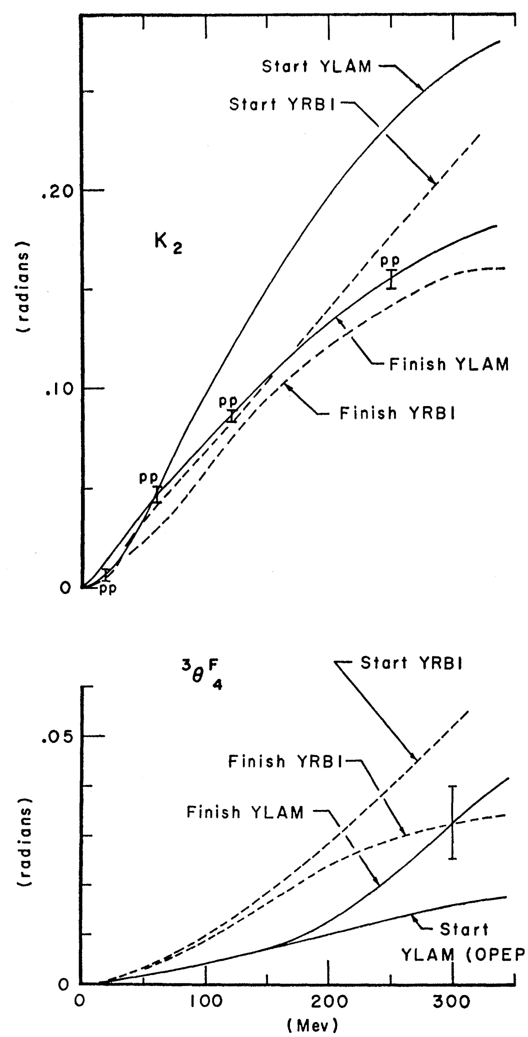

FIG. 21. Examples of the changes produced in phase parameters by the searches which give the improvement in fit to data illustrated in Fig. 20. It should be noted that while the fit YRB1 is based on the SM potential for $E<150$ Mev, the singlet range has been modified, as decribed in the text, and K_2 is not exactly the same as that of SM. In all cases, the phase parameters used for the YRB1 start were computed before published values were available and therefore differ slightly in some cases from those of SM.

The inequalities have been used employing a generalization of the form involving the energy rate of change of the radial logarithmic derivative.

Employing radial functions normalized in the convention of Eq. (6) the inequality is

$$\partial[\partial\mathcal{F}/\mathcal{F}\partial r]/\partial E<0. \tag{7}$$

According to the derivations previously mentioned, the left-hand side is supposed to be evaluated outside the region within which nucleon-nucleon interactions take place. The derivations[33,34] are not directly applicable to the nucleon-nucleon problem, however. One reason for the inapplicability is the existence of coupled states having the same J but different L. The description of such states by means of a single radial function $\mathcal{F}/r$ is inadequate, two radial functions being required. Accordingly, an inequality involving at least these two functions will replace (7). Another reason for the inapplicability of the derivations is that they deal with a collection of a fixed number of particles. In the two-nucleon problem it is necessary to deal with the production of real and virtual mesons as well and the potential energy of nuclear reaction theory has to be replaced by an expression involving meson creation and destruction operators. Working in the system of zero total momentum in Fock space, a generalization of the logarithmic derivative inequality assumes a simple form if the nucleon motion is treated nonrelativistically and the recoil of the nucleons caused by meson emission is neglected. The inequality then takes the form

$$\sum_{\alpha(n),n} u_{\alpha(n),n}{}^2\partial(\partial u_{\alpha(n),n}/u_{\alpha(n),n}\partial r)/\partial E = -(M/\hbar^2)\int_0^r \sum_{\alpha(n),n} u_{\alpha(n),n}{}^2 dr<0. \quad (7.1)$$

Here n is the number of mesons and $u_{\alpha(n),n}(r)/r$ is the radial function for relative motion of the two nucleons for a state the condition of which is described by $\alpha(n)$. The latter quantity specifies the way in which the total angular momentum $\mathbf{J}$, in units $\hbar$, is compounded from the relative orbital angular momentum of nucleon motion $\mathbf{L}$, the total nucleon spin $\mathbf{S}$, the total pion angular momentum $\mathbf{l}_\pi$ as well as the way the latter is compounded from orbital angular momenta of individual pions. The parity of all states in (7.1) is taken to be the same. In the derivation of Eq. (7.1) the spin angular factors were used such that u^2dr is the number of systems at r in dr. Each $\alpha(n)$, n plays the rôle of a channel in nuclear reaction theory with the difference that its definition does not depend on the separation of configuration space into interior and exterior regions. Referring to this kind of channel as a "Fock channel," there is seen to be a complication in the entrance of a number of Fock channels in the inequality. If r is sufficiently large, then for energies below the threshold of meson production only the closed channels need be considered. Simplifying the problem still further by the assumption that only one n is of importance for the open channels, so as to reproduce the condition of free nucleons, there are left at most two functions u in Eq. (7.1). If there is only one L of the two-nucleon system giving the desired J and parity, the result reduces to Eq. (7). If there are two, then

$$u_1{}^2\partial(\partial u_1/u_1\partial r)/\partial E+u_2{}^2\partial(\partial u_2/u_2\partial r)/\partial E<0 \quad (7.2)$$

follows. Both (7) and (7.2) are seen to follow only if the additional assumptions are made. The assumption regarding the possibility of neglecting all closed Fock channels cannot be accurately satisfied unless r is sufficiently large. The condition that the closed channel terms on the left-hand side of (7.1) be negligible in comparison with open channel terms is, however, not a necessary but only a sufficient one. It is only necessary that these terms, if negative, should not overbalance the right-hand side of the first part of Eq. (7.1) and when the radial functions in these channels are only partially attenuated this condition may still be satisfied. It is to be expected, therefore, that (7) and (7.2) will be satisfactory criteria for values of r not too small in comparison with the range of nuclear forces. As a rough criterion one would expect the inequalities to be true whenever the OPEP is the main part of the interaction and possibly down to distances at which the next term, $V^{(4)}$, is comparable with the OPEP term $V^{(2)}$. This requirement is approximately satisfied for $x=rm_\pi c/\hbar$ having value 1.4, as estimated on the basis of the potential recently proposed by Bryan[23] and is in approximate agreement with the calculations of Gupta.[35]

By integrating the differential equation including the OPEP towards the small r, $d\mathfrak{F}/\mathfrak{F}dr$ has been calculated at $x=0.678$, 0.846, 1.00, 1.406 for K_0, $\delta^P{}_0$, $\delta^P{}_1$, $\delta^F{}_3$, and K_2. For K_0 the test gave a slight violation of the inequality for $x=0.678$, the logarithmic derivative being practically constant throughout the complete energy range. For the other x the inequality is satisfied. The slight deviation for $x=0.678$ which takes place at the smaller E does not require for its explanation the closed channels of Eq. (7.1) but follows naturally if an attractive potential is assumed to exist in addition to the OPEP. For $\delta^P{}_0$ the deviations are present for $x=0.678$, 0.846, 1.00, the region of violation moving to larger E as x increases. For $x=1.406$ there is no violation. The same explanation as for K_0 applies in this case. For $\delta^P{}_1$ no violations were found for $x=0.846$, 1.000, 1.406 and a slight one for $x=0.678$. For $\delta^F{}_3$ definite violations were found at all x, the least marked being for $x=1.406$. In this case, Eq. (7) is satisfied from $E=40$ Mev to 300 Mev but not for $E<40$ Mev. This again is understandable in terms of an additional potential which is relatively more important at small E. For K_2 the situation is similar to that for $\delta^F{}_3$. From these tests there appears to be no reason for suspecting the fits.

Some similar tests have been made with (7.2) for the coupled cases, with similar results. In employing (7.2) the work is more laborious because one has to consider not only the two eigenstates but also linear combinations of them and to use the strongest conditions thus obtained. The work is straightforward but laborious. Since, on the other hand, there are deviations in the uncoupled cases which are explicable only if one postulates a potential acting in addition to the OPEP or which have possibly to do with omitted effects of the closed channels in (7.1), it proved simpler to look for potentials applicable to states of sharp J and parity and capable of representing the phase parameters as functions of E. If a potential is found, the inequalities (7), (7.2) are satisfied automatically. At this point an element of personal judgement unfortunately has to enter because a potential may be found which is unreasonable.

Starting with the mathematical form used by Bryan[23] but regarding the coefficients of the x^{-n} as adjustable parameters, gradient searches have been made for adjustment of these coefficients to give a representation of the phase-parameter graphs. No difficulty has been experienced in obtaining such a representation of YLAM within the error limits of the YLAM search with separate potentials for different J and parity. The potentials are not qualitatively

[35] S. N. Gupta, Phys. Rev. **117**, 1146 (1960).

different in character from other potentials in current literature if the latter are calculated for individual J and parity combinations. They may, therfore, be considered reasonable by ordinary criteria and the tests do not invalidate the results presented. The search for potentials has not progressed far enough to give an exact representation of the most probable YLAM phase parameters although even this requirement has been practically accomplished for the more important cases except for odd $J=2$ states. There has been no indication so far that this will not prove possible even in this test which may be surmised to be unnecessarily stringent. The publication of the phenomenological potentials just referred to is being postponed.

After this manuscript was completed there appeared data[36] giving measurements of $D(\theta)$ at 210 Mev. These favor YLAM which gives values of 0.175 at 30° and 0.330 at 60° to be compared with 0.19±0.02 and 0.33±0.03, respectively. Fit YRB2 agrees well at the larger angle but gives about 0.59 at the smaller and appears to be excluded as at 312 Mev. Fit YRB1 gives 0.115 and 0.167 at the smaller and larger angles and is definitely not favored by the data. On the other hand, YRB3 gives 0.168 and 0.285 at the smaller and larger angles and is conceivably admissible so far as these measurements go. It is seen that all in all YLAM is the best over-all fit so far.

Note added in proof. According to a communication by Dr. B. Rose at the Tenth International Conference on High Energy Physics at Rochester, August 25–September 1, 1960, and a letter from Dr. A. E. Taylor, a remeasurement of the Harwell D discussed in connection with Fig. 17 improves agreement with Harvard 147 Mev data. The new Harwell D together with cross sections of Caverzasio and Michaloanvicz have been included in new searches in progress which modify YLAM in the general direction of YRB1 in the 150 Mev region. With reference to Fig. 12 a letter from Professor R. Wilson indicates the presence of a special source of error in the measurement of polarization at the lowest Harvard angle.

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