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Nuclear $E1$ Peak Energies

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The peak energies E_g of electric dipole giant resonances in photonuclear reactions considerably exceed the shell-model spacings E_s in the same nuclei: The discrepancy $\delta=(E_g-E_s)/E_s$ is of order unity, although the giant resonances are supposed to arise mainly from transitions of nucleons between successive shells. The present note attempts to understand δ by noting that $|E_g-E_h|\ll E_g$ and considering sum rule expressions for the harmonic energy E_h. These sum rule expressions are well known to yield $E_h=E_d+E_x$, where E_x is special to the $E1$ operator and the charge exchange component of nuclear forces. Theoretical and empirical arguments are adduced that (E_g-E_s) comprises mainly E_x, plus two secondary corrections which happen practically to cancel. Comparison with experiment gives a constant $E_x\approx 8$ Mev and thence a satisfactory account of δ. Previous discussions of δ appear to have neglected E_x and hence have failed to obtain even the right order of magnitude of δ for real nuclei. The use of a constant E_x also allows an improved fit to the curve of $\int\sigma dE$ as a function of A. It is pointed out that the value of E_x is comparable in significance with the average nuclear potential V: By virtue of the $E1$ excitation mode, E_x represents the *difference* of even-parity and odd-parity two-nucleon interactions, while V represents a *sum* of even and odd interactions. The effective mass for the model ground-state wave function is treated as a derived quantity and turns out to be $M^*/M\gtrsim 1$; this large value is attributed to a Thomas shift associated with finite nuclear boundaries.

I. INTRODUCTION AND SUMMARY

Electric dipole "giant resonances" in photonuclear reactions have been interpreted on an independent-particle nuclear shell model as reflecting promotion of single nucleons between successive major shells.[1] A difficulty with this interpretation has always been the large relative discrepancy,

$$\delta=(E_g-E_s)/E_s\sim 1, \quad (1)$$

between the giant resonance peak energy E_g and the single-particle shell spacing E_s in the same nucleus—as measured, for example, by (d,p) reactions.[2,3] At least three suggestions have been made regarding this discrepancy; they are not necessarily consistent, and each appears to have some deficiencies. We summarize them as follows: (*i*) Insertion of an effective mass $M^*=\frac{1}{2}M$ in the single-particle shell model[4] yields E_g in fair accord with observation, except for the A dependence; but E_s is correspondingly increased, so that one still has $\delta\sim 0$. The only solution at this level is to assert that $M^*=\frac{1}{2}M$ for photonuclear excitation, $M^*=M$ for other means of measuring E_s; this does not solve the problem of $\delta\sim 1$ but merely recasts it in terms of M^*, which may not be the most perspicuous approach. (ii) The basic single-particle picture is maintained, but the $E1$ excited state is recognized as a coherent sum over many single-particle excitations or particle-hole pairs, where the hole is left in the Fermi sea of the ground state by removal of the excited particle.[5] These particle-hole pairs will, of course, interact with each other; but such interaction energies will sum incoherently to a relatively small total for most states of the nucleus. Just the coherence of the $E1$ excited state, however, gives rise to a coherent sum over these interactions, which may amount to a substantial shift of E_g. Arguments are given to show that this shift will have the observed sign (increase), but quantitative estimates are neither attempted nor shown to be feasible. Moreover, consideration of large scale oscillations in extended nuclear matter seems to suggest that this shift is negligibly small.[6] (iii) Classical, plasma-type oscillations of infinite nuclear matter are discussed in which neutrons and protons execute opposite motions.[6] The plasma frequency is as usual determined by kinetic factors relating to the nucleon mass and the density of particles in the Fermi sea. Use of an effective mass M^* for the nucleons takes account of the *average* effect of internucleon forces. Because, in this case, neutrons and protons move oppositely, there should be added a special, nonaverage contribution related to the nuclear symmetry energy, which raises the plasma frequency to ω_p'. This shift is implicitly related to δ for infinite nuclear matter and evaluated[6] as $\delta=(\omega_p'-\omega_p)/\omega_p\approx 15\%$. No reason is given to anticipate that application to finite nuclei will increase this estimate by almost one order of magnitude, as required by experiment.

The present note attempts to shed some light on this situation by exploiting a relation suggested by the giant resonance experimental data themselves; namely,

$$E_g\approx E_h\gg|E_g-E_h|, \quad (2)$$

where the harmonic mean energy is $E_h=I/I_h$, $I=\int\sigma dE$, $I_h=\int\sigma dE/E$. Relation (2) implies that to first approximation δ can be discussed in terms of E_h rather than E_g. This is desirable because E_g depends strongly on the nuclear model invoked, while E_h can be expressed in terms of sum rules and is considerably less model-dependent. Starting with assumption (2),

[1] D. H. Wilkinson, Physica **22**, 1039 (1956).
[2] J. P. Schiffer, L. L. Lee, Jr., and B. Zeidman, Phys. Rev. **115**, 427 (1959).
[3] B. L. Cohen and R. E. Price, Nuclear Phys. **17**, 129 (1960).
[4] D. H. Wilkinson, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, California, 1959), Vol. 9, p. 1.
[5] G. E. Brown and M. Bolsterli, Phys. Rev. Letters **3**, 472 (1959).
[6] K. A. Brueckner and R. Thieberger, Phys. Rev. Letters **4**, 466 (1960).

the outline of the argument is as follows, with details to be supplied in the next section: It has long been known that the sum rule expression for I is of the form $I=I_d+I_x$, where I_x is related to nuclear exchange forces and depends on the specific form of the electric dipole operator. The corresponding $E_h=(I_d/I_h)+(I_x/I_h)=E_d+E_x$ thus contains a term E_x entirely peculiar to the $E1$ mode of excitation; if this term is large, it can be the dominant contribution to δ. Evaluation in the next section indicates, in contrast to previous calculation, that E_d and E_x differ in dependence on nuclear mass number A; they can hence be distinguished by comparison with experiment, and E_x indeed turns out to be large, $E_x\sim 8$ Mev. If one also subtracts the 15% correction[6] from E_d and makes a very crude estimate of (E_g-E_h), a fair account of δ can be given over the range of real nuclei, in which E_x is much the largest component contributing to δ. This provides some *a posteriori* justification of assumption (2). The final discussion concludes that none of the previous discussions of δ have accounted for the exchange contribution E_x and have thus been inadequate.

The final formulas obtained by fitting to the data have generous uncertainties, which do not obscure the qualitative significance of the various terms:

$$E_g=[(40\pm6)A^{-\frac{1}{3}}+(7.5\mp1.5)]\text{ Mev}, \tag{26}$$

$$E_s=(38\pm8)A^{-\frac{1}{3}}\text{ Mev}, \tag{29}$$

whence

$$\delta\approx0.2[A^{\frac{1}{3}}+0.25](\pm20\%). \tag{30}$$

An improved fit to the curve of $I=\int\sigma dE$ versus A is obtained, corresponding to a nuclear radius parameter of $r_0{}^2=(1.1\pm0.2)$ f^2. A relatively sensitive exchange parameter is obtained that can be compared with various nuclear force prescriptions from nuclear spectroscopy: The quite uncertain numerical value seems to favor effective two-body potentials with substantial attraction in the odd-parity states. A simple estimate is derived for the effective nuclear mass in the ground-state trial function:

$$M^*/M\approx1.2\pm0.2. \tag{37}$$

This large value is interpreted as reflecting a Thomas shift characteristic of finite nuclei.

II. FORMULATION

The sum-rule formula for E_h is

$$E_h=\tfrac{1}{2}\langle[D,[H,D]]\rangle_{00}/\langle D^2\rangle_{00}, \tag{3}$$

where the subscripts indicate evaluation in the ground state Ψ_0 of the nucleus, H is the nuclear Hamiltonian, and D the dipole operator,

$$D=\tfrac{1}{2}\sum_{j=1}^{A}(\tau_z{}^j+\mathcal{V})z^j,\quad \mathcal{V}=\left(\frac{N-Z}{A}\right). \tag{4}$$

The quantity $\mathcal{V}$ is an insignificant correction to the present considerations and will be dropped. Equation (3) is valid only in case the model functions Ψ_n are exact eigenfunctions of the Hamiltonian H that is employed, $H\Psi_n\equiv E_n\Psi_n$; otherwise, there is an uncertainty in Eq. (3) of order[7]

$$\epsilon=\sum_{q\neq0}\langle D^2\rangle_{0q}\langle H\rangle_{q0}/\langle D^2\rangle_{00}, \tag{5}$$

where the sum is over all excited model function states. Since we shall choose both H and Ψ_0 for their simplicity and conventionality, ϵ will not vanish identically; it will be necessary to give an explicit argument below that ϵ is small.

For the Hamiltonian we take

$$H=\sum_i\frac{p_j{}^2}{2M}+\sum_{i<j}\mathcal{W}+\mathcal{B}P_{ij}{}^\sigma-\mathcal{H}P_{ij}{}^\tau-\mathcal{M}P_{ij}{}^\sigma P_{ij}{}^\tau, \tag{6}$$

where P^σ, P^τ are real and isotopic spin exchange operators and $\mathcal{W}$, $\mathcal{B}$, $\mathcal{H}$, $\mathcal{M}$ are functions of r_{ij}. Explicit momentum dependence is not included in these functions, for it should be possible to make their dependence on r_{ij} sufficiently complicated to give a good representation of nuclear forces in the limited momentum range involved. The present calculation goes only as far as taking simple averages over these radial functions, however. The use of the effective central potentials in place of tensor or spin-orbit forces is sufficient for $E1$ interactions, which are not primarily connected with nucleon spins. The choice of the free nucleon mass in Eq. (6) is essentially a matter of convention; a different choice $M^*\neq M$ would entail a corresponding redefinition of the potential terms in such a way that the net effect of H remains unchanged.[8] The convention of Eq. (6) is still mainly used for calculations of nuclear spectroscopy, with which we wish to compare the exchange parameters indicated by the present analysis.

For the model function Ψ_0 we take the ideal harmonic oscillator without spin-orbit coupling (i.h.o.). This appears to provide a quite good approximation for the $E1$ giant resonance,[4] and has the virtue of reducing the complicated sums in Eqs. (3)–(5) to simple forms. Moreover, it seems plausible that in this case ϵ provides a first estimate of (E_g-E_h),

$$\epsilon\sim E_g-E_h, \tag{7}$$

since both quantities vanish identically if the i.h.o. model is correct and the giant resonance becomes a single sharp line.[9]

The i.h.o. functions are completely specified in terms of a single parameter a with the dimensions of a

[7] F. C. Barker, Phil. Mag. 2, 780 (1957). The authors thank Dr. Barker for constructive suggestions about the present discussion.

[8] J. S. Levinger, N. Austern, and P. Morrison, Nuclear Phys. 3, 456 (1957).

[9] The relative signs of ϵ and (E_g-E_h) in Eq. (7) follow from the discussion of reference 7.

length: If the axial quantum numbers are the nonnegative integers ξ, η, ζ, a shell of total quantum number m contains a total number of nucleons

$$A_m = 2(m+1)(m+2), \quad (8)$$

with mean square radius and momentum

$$\langle r^2\rangle_m = (m+3/2)a^2,$$
$$\langle p^2\rangle_m = (m+3/2)(\hbar/a)^2. \quad (9)$$

For the entire nucleus,

$$A = \sum_{m=0}^{n} A_m = \tfrac{2}{3}(n+1)(n+2)(n+3),$$
$$\langle r^2\rangle = \tfrac{3}{4}(n+2)a^2 \approx 0.86 A^{\frac{1}{3}} a^2. \quad (10)$$

One can get an order-of-magnitude of a^2 by comparing the mean squared radius of the outer shell $\langle r^2\rangle_n$ with $(r_0 A^{\frac{1}{3}})^2$, or the mean squared radius over the nucleus with $\frac{3}{5}(r_0 A^{\frac{1}{3}})^2$, where $r_0 \approx (1.1\pm 0.1)$ f (1 f$=10^{-13}$ cm). The result is

$$a^2 \approx (0.8\pm 0.1) r_0^2 A^{\frac{1}{3}} \approx (1.0\pm 0.2)\ \text{f}^2 A^{\frac{1}{3}} = a_0^2 A^{\frac{1}{3}}. \quad (11)$$

The i.h.o. estimate of $\langle D^2\rangle_{00}$ is[10]

$$\langle D^2\rangle_{00} = \tfrac{1}{4}\sum_j \langle (z^j)^2\rangle_{00} + \tfrac{1}{4}\sum_{i\neq j} \langle \tau_z^i \tau_z^j z^i z^j\rangle_{00}$$
$$= Aa^2/8 = A^{\frac{4}{3}} a_0^2/8. \quad (12)$$

In the i.h.o., z^j can induce only transitions $\zeta^j \leftrightarrow \zeta^j \pm 1$; in conjunction with the exclusion principle this implies that the second term in Eq. (12) vanishes entirely, while the first extends only over the last shell. Even for the last shell only the transition sequence $\zeta^j \to \zeta^j + 1 \to \zeta^j$ contributes, and $\zeta^j \to \zeta^j - 1 \to \zeta^j$ is forbidden; hence, an extra factor of $\frac{1}{2}$ in the result. Allowance for exclusion is equivalent to antisymmetrization of Ψ_0, which thus need not be explicit.

Similarly,

$$\tfrac{1}{2}\langle [D,[H,D]]\rangle_{00} = A\hbar^2/8M$$
$$- \tfrac{1}{2}\langle \textstyle\sum_{n,p} (z^n - z^p)^2 \mathcal{Y}(r_{np}) P_{np}\rangle_{00}, \quad (13)$$
$$\mathcal{Y} = \mathfrak{M} + \tfrac{1}{2}\mathcal{H}, \quad P_{np} = -P_{np}{}^{\sigma} P_{np}{}^{\tau}.$$

Here $\sum_{n,p}$ indicates summation over neutron and proton states separately, and P_{np} is the space exchange operator. As a preliminary to evaluating the exchange term in Eq. (13), consider

$$\langle \sum_{n,p} (z^n - z^p)^2 P_{np}\rangle_{00} = 4\int\int d^3r_1 d^3r_2 |Q(12)|^2,$$
$$Q(12) = \sum_{m=0}^{n} \psi_m(1)\psi_m{}^*(2)(z^1 - z^2) \quad (14)$$
$$= \frac{a}{\sqrt{2}} \sum_{m=0}^{n} [(\zeta+1)^{\frac{1}{2}} \psi_{\zeta+1}(1)\psi_\zeta{}^*(2) - (\zeta)^{\frac{1}{2}} \psi_\zeta(1)\psi_{\zeta-1}{}^*(2) - (1\leftrightarrow 2)].$$

[10] J. S. Levinger, Phys. Rev. **97**, 122 (1955).

Here $\sum_m = \sum_{\xi+\eta+\zeta=m}$; the ψ are i.h.o. orbitals with only the quantum number ζ given explicitly as a subscript, but ξ and η are to be understood as well. Because of exclusion the $\sum_m$ collapses as in Eq. (12) to $\sum_n$, the last filled shell only. In this last shell approximate ζ by $\zeta_n = n/3$ throughout; then

$$Q(12) \approx a\left(\frac{n+3}{6}\right)^{\frac{1}{2}} \sum_n \begin{vmatrix} \psi_{\zeta+1}(1) & \psi_{\zeta+1}{}^*(2) \\ \psi_\zeta(1) & \psi_\zeta{}^*(2) \end{vmatrix}. \quad (15)$$

Converting to momentum space and recalling that i.h.o. functions are self-conjugate under this transformation (except for a factor i^n), we write the exchange term in Eq. (13) as

$$-a^2\left(\frac{n+3}{6}\right)\int d^3\kappa\, Y(\kappa) J(\kappa^2),$$

$$Y(\kappa) = (2\pi)^{-3}\int d^3r\, \exp(i\boldsymbol{\kappa}\mathbf{r})\mathcal{Y}(r), \quad (16)$$

$$J(\kappa^2) = 4\sum_{n,n'} d^3k d^3k' [\psi_{\zeta+1}(\mathbf{k}+\boldsymbol{\kappa})\psi_\zeta(\mathbf{k}')]$$
$$\times [\psi_{\zeta'+1}(\mathbf{k})\psi_{\zeta'}(\mathbf{k}'-\boldsymbol{\kappa}) + (\zeta' \leftrightarrow \zeta'+1)].$$

The cutoff function $J(\kappa^2)$ has the properties that $J(0) = A_n$, $J(\kappa^2) \to 0$ as $\kappa^2 \gg \langle p^2/\hbar^2\rangle_n \approx 1.4 r_0^{-2}$. Since this limit is independent of A, we take

$$J(\kappa^2) \approx A_n \exp[-(\kappa r_0)^2],$$

$$\int d^3\kappa\, Y(\kappa) J(\kappa^2) = bY(0)\int d^3\kappa\, J(\kappa^2)$$
$$\approx \frac{2}{3}\left(\frac{\sqrt{\pi}}{r_0}\right)^3 A_n Y(0). \quad (17)$$

Because of the relatively short cutoff in $J(\kappa^2)$ we take $Y(\kappa) \approx Y(0)$ and allow for a slight reduction in $Y(\kappa)$ over the range of nonvanishing $J(\kappa^2)$ by introducing a factor b. The value $b \approx \frac{2}{3}$ comes from the specific example

$$b = \left[\int_0^\infty (1+t^2)^{-1} \exp(-t^2) t^2 dt\right] \Big/ \left[\int_0^\infty \exp(-t^2) t^2 dt\right].$$

In the optical approximation of infinite nuclear matter with an equilibrium volume of $(4\pi/3) r_0^3$ per nucleon,

$$(2\pi)^3 Y(0) = \int d^3r\, \mathcal{Y}(r) = \int d^3r [\mathfrak{M} + \tfrac{1}{2}\mathcal{H}]$$
$$= (4\pi/3) r_0^3 Y, \quad (18)$$

where Y is the average value over the equilibrium volume. Correspondingly one can define an average total potential V for nuclear matter,

$$\int d^3r [\mathcal{W} + \tfrac{1}{2}\mathcal{B} - \tfrac{1}{2}\mathcal{H} - \tfrac{1}{4}\mathfrak{M}] = (4\pi/3) r_0^3 V. \quad (19)$$

The ratio,

$$y=(Y/V), \tag{20}$$

is thus a measure of the exchange character of nuclear forces in nuclei.

Collecting Eqs. (13)–(20), we have

$$\tfrac{1}{2}\langle [D,[H,D]]\rangle_{00}=(A/8)[\hbar^2/M-(4/9\pi^{\frac{1}{2}})a^2(yV)]. \tag{21}$$

Insertion of Eqs. (21) and (12) in Eq. (3) yields

$$E_h=\hbar^2/Ma^2-0.25(yV). \tag{22}$$

Comparison with Eq. (11) shows that the direct term in Eq. (22) should be of order $E_d=(40\pm8)A^{-\frac{1}{3}}$ Mev, while the exchange term $E_x=-0.25(yV)$ is to first order independent of A. This suggests analysis of the experimental E_g with a similar form, $(cA^{-\frac{1}{3}}+c')$, which appears in the next section to provide an excellent fit.

This section concludes with a discussion of the quantity ϵ in Eq. (5). For the i.h.o. it is

$$\epsilon=\left(\frac{8}{Aa^2}\right)^{\frac{1}{4}}\sum_{\sigma_z,\tau_z}\sum_{\xi+\eta+\zeta=m}\sum_{\xi'+\eta'+\zeta'=m'} \times\Big[\sum_{m=n,n-1}\sum_{m'=m+2}(-1)^{m-n}\langle z^2\rangle_{mm'}\langle p^2/2M+\bar{V}\rangle_{m'm} +2\sum_{m_1\neq m_2=n}\sum_{m_1'\neq m_2'=n+1}\langle \tau_z{}^1\tau_z{}^2z^1z^2\rangle_{m_1m_2;m_1'm_2'} \times\langle V(12)\rangle_{m_1'm_2';m_1m_2}\Big], \tag{23}$$

where exclusion has been taken into account. In the first term the quantity $\bar{V}$ is the average potential seen by a single nucleon, due to all other nucleons. Now the optimum i.h.o. model is defined by some criterion that makes $\bar{V}$ as close as possible to a quadratic potential: if the equivalence were perfect, then $\langle p^2/2m+\bar{V}\rangle_{mm'}=0$ for $m\neq m'$. Of course, this condition fails in practice but the corresponding term in Eq. (23) should still be small. The last sum in Eq. (23) is convertible to an expression in momentum space like Eq. (16), except that terms with $(\xi,\eta,\zeta)=(\xi',\eta',\zeta')$ are excluded. The corresponding function $K(\kappa^2)$ therefore has $K(0)=0$; since also $K(\kappa^2)\rightarrow 0$ for large κ^2 in much the same way as $J(\kappa^2)$, it is plausible that $K(\kappa^2)$ is always small. To see what these quantities are small with respect to, consider the kinetic term, which can be evaluated directly

$$\epsilon_d=\left(\frac{2}{Aa^2}\right)\sum_{\xi+\eta+\zeta=m}\sum_{\xi'+\eta'+\zeta'=m'}\sum_{m=n,n-1} \times\sum_{m'=m+2}(-1)^{m-n}\langle z^2\rangle_{mm'}\langle p^2/2M\rangle_{m'm} =-\tfrac{1}{2}(\hbar^2/Ma^2)=-\tfrac{1}{2}E_d\approx-20A^{-\frac{1}{3}}\text{ Mev}. \tag{24}$$

according to the discussion following Eq. (22). The arguments immediately above then suggest that

$$|\epsilon|\ll|\epsilon_d|, \tag{25}$$

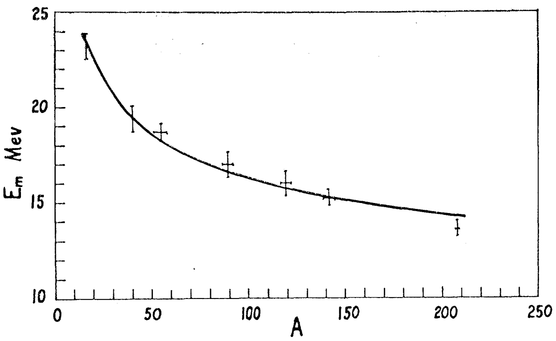

FIG. 1. Nuclear $E1$ peak energies, E_g, for closed shell nuclei. The experimental data[11] have been averaged over a range of A values, shown by the horizontal bars, having the same magic numbers of neutrons or protons. The smooth line is the curve $E_g=(7.5+40A^{-\frac{1}{3}})$ Mev. The ordinate should read E_g instead of E_m.

which will be a relatively small uncertainty in the treatment of the experimental data.

III. COMPARISON WITH EXPERIMENT

A least-squares fit of the measured E_g [11] to the form $(cA^{-\frac{1}{3}}+c')$ yields

$$E_g=[(40\pm6)A^{-\frac{1}{3}}+(7.5\mp1.5)]\text{ Mev}. \tag{26}$$

The optimum fit is shown in Fig. 1, where the data are chosen from groups of nuclei averaged around closed shells of neutrons or protons at average $\bar{A}=16$, 40, 54, 89, 119, 141, 208. These are the nuclei in which the observed giant resonances are sharpest, and for which the i.h.o. might be expected to apply best. Actual trial indicates, however, that Eq. (26) would be essentially unchanged by including data from all nuclei.

An effort was made to obtain an estimate of (E_g-E_h) for the groups of nuclei shown in Fig. 1, except that $\bar{A}=12$ was substituted for $\bar{A}=16$, 40. The results are extremely crude but suggest an average $(\bar{E}_g-\bar{E}_h)\approx-1.5$ Mev, which would justify the assumption of Eq. (2). If one pursues the details further and tries to obtain a fit of the form $(cA^{-\frac{1}{3}}+c')$,

$$(E_g-E_h)\sim-(4A^{-\frac{1}{3}}+0.5)\text{ Mev}. \tag{27}$$

The uncertainties are so large as to render this detailed form almost meaningless; but since the magnitude is small, no great error should ensue from using it—in

[11] R. Montalbetti *et al.*, Phys. Rev. **91**, 659 (1953); J. Goldemburg and L. Katz, Can. J. Phys. **32**, 49 (1954); R. Nathans and J. Halpern, Phys. Rev. **93**, 437 (1954); R. Nathans and P. F. Yergin, Phys. Rev. **98**, 1296 (1955); P. F. Yergin and B. P. Fabricand, Phys. Rev. **104**, 1334 (1956); G. A. Ferguson *et al.*, Phys. Rev. **95**, 776 (1954); E. G. Fuller *et al.*, Phys. Rev. **112**, 554 (1958); J. H. Carver and K. H. Lokan, Australian J. Phys. **10**, 312 (1957); J. H. Carver and W. Turchinetz, Proc. Phys. Soc. (London) **A73**, 69, 110 and 589 (1959); P. Axel and J. D. Fox, Phys. Rev. **102**, 400 (1956); M. B. Scott *et al.*, Phys. Rev. **100**, 209 (1955). The first of these references gives a fit to the data of $E_g=37A^{-0.186}$ Mev; over the limited range $12\lesssim A\lesssim 210$ the function $E_g=(40A^{-\frac{1}{3}}+7.5)$ Mev is very well approximated by $E_g=40A^{-0.195}$ Mev.

lieu of anything better—as an explicit connection between E_g and E_h. Accordingly,

$$E_h = E_d + E_x,$$
$$E_d \approx (44\pm8)A^{-\frac{1}{3}}\text{ Mev},\quad E_x \approx (8\mp2)\text{ Mev}, \tag{28}$$

with an extra allowance for uncertainty contributed by Eq. (27).

An effort to estimate E_s in accordance with Eq. (28) is as follows: The term E_x is entirely associated with the peculiarities of $E1$ excitation and should be dropped. For an i.h.o., E_d would equal E_s, except for the possibility of special corrections to E_d because of the opposite motion of neutrons and protons. The next section argues that corrections are exactly those computed in reference 6; accepting the value of 15% found there, one has the prescription for E_s,

$$E_s = E_g - [(E_g - E_h) + 0.15E_d + E_x] \approx (38\pm8)A^{-\frac{1}{3}}\text{ Mev}. \tag{29}$$

Comparison with Eqs. (27), (28) shows that the first two of the bracketed terms practically cancel; because of this apparent accident, the discrepancy δ arises almost entirely from E_x. Equations (1), (26)–(29) indicate

$$\delta \approx 0.2[A^{\frac{1}{3}} + 0.25](\pm 20\%). \tag{30}$$

The observed values[11,12] at $\bar{A}\approx 90$, 208 are, respectively, $E_g = 17.0\pm0.7$, 13.5 ± 0.5 Mev, $E_s = 8.3$, 5.4 Mev (assumed error ±1 Mev), or hence $\delta = 1.0\pm0.2$, 1.5 ± 0.2. The values calculated from Eq. (30) for these cases are $\delta = 0.9\pm0.2$, 1.2 ± 0.2.

Although Eq. (30) and its interpretation were the primary goals of the present calculation, certain other features appear to merit discussion. The fact that the exchange terms computed here do not vary as $A^{-\frac{1}{3}}$, as does E_d, allows an improved fit to the A dependence of $\int \sigma dE$. Using the sum rule expression

$$\int \sigma dE/E = (2\pi)^2\alpha \langle D^2\rangle_{00}, \tag{31}$$

where $\alpha = e^2/\hbar c = 1/137$, one has by Eqs. (12) and (28)

$$\int \sigma dE = E_h \int \sigma dE/E \approx 1.6 a_0^2 A[1 + 0.18A^{\frac{1}{3}}]. \tag{32}$$

Figure 2 compares Eq. (32) with those nuclei used to obtain Eq. (26), with the solid line corresponding to $a_0^2 = 0.8$ f^2. Another measure of a_0^2 comes from Eqs. (22) and (28).

$$E_d = (\hbar^2/Ma_0^2)A^{-\frac{1}{3}} = (44\pm8)A^{-\frac{1}{3}}\text{ Mev}, \tag{33}$$

whence $a_0^2 = (0.9\pm0.2)$ f^2. Allowing a similar uncertainty in the fit of Fig. 2, one has as an average

$$a_0^2 = (0.85\pm0.15)\text{ f}^2, \tag{34}$$

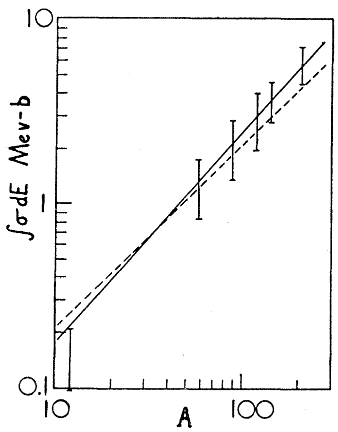

FIG. 2. Integrated cross sections $\int\sigma dE$ for closed shell nuclei. The lower limits of the cross sections are based on data obtained in the giant resonance region[11] ($\bar{E}\leqslant 20$–30 Mev); the upper limits include the additional contributions from high-energy ($E\leqslant 150$ Mev) gamma rays [L. W. Jones and K. M. Terwilliger, Phys. Rev. **91**, 699 (1953)]. Photoproton emission [J. Halpern and A. K. Mann, Phys. Rev. **83**, 370 (1951); E. V. Weinstock and J. Halpern, Phys. Rev. **94**, 1209 (1954); L. Cohen *et al.*, Phys. Rev. **104**, 108 (1956)] has been included and makes a substantial contribution to the cross sections for the two points of lowest A. The value for middle-weight nuclei relies on Ni^{58} $(\gamma,n)+(\gamma,p)+(\gamma,2n)+(\gamma,np)$ measurements [J. H. Carver and W. Turchinetz, Proc. Phys. Soc. (London) **A73**, 585 (1959)]. Total cross sections for the light nuclei are very difficult to estimate though somewhat more certain for the "near-magic" nucleus C^{12} than for O^{16}. We have used the results of W. C. Barber *et al.* [Phys. Rev. **98**, 73 (1955)] for the $C^{12}(\gamma,n)$ cross sections since these extend to 260 Mev, The $C^{12}(\gamma,p)$ cross section is predominant in the giant resonance region, and it has been assumed that the ratio of (γ,p) to (γ,n) cross sections is the same at high energies as in the giant resonance region. The full line corresponds to $1.3A(1+0.18A^{\frac{1}{3}})$ f^2 Mev; the dotted line corresponds to the Levinger-Bethe expression [J. S. Levinger and H. A. Bethe, Phys. Rev. **78**, 115 (1950)] $\int\sigma dE = 15A(1+0.8x)$ Mev-mb, with $x=\frac{1}{2}$.

which corresponds by Eq. (11) to $r_0^2 = (1.1\pm0.2)$ f^2. The harmonic integral evaluated by Eq. (34) is

$$\int \sigma dE/E = \tfrac{1}{2}\pi^2\alpha a_0^2 A^{\frac{4}{3}} \approx (0.31\pm0.05)A^{\frac{4}{3}}\text{ mb}, \tag{35}$$

and is displayed in Fig. 3.

Comparison of Eqs. (22) and (28) indicates that

$$y \approx -(32/V) \sim 0.4, \tag{36}$$

if we take the average nuclear potential near the top of the Fermi sea to be $V\approx -90$ Mev, as suggested with fair uncertainty by elementary considerations of nuclear saturation.[13] The values of y predicted by various choices of nuclear exchange parameters used in nuclear spectroscopy are shown in Table I. It is seen that a small value of y or hence of E_x corresponds to relative equality of even-parity and odd-parity terms in the two-body potential. This is quite simply understood from $E_x \sim \langle Q|V|Q\rangle$ and Eq. (15) for Q, where because of the $E1$ mode of excitation the orbitals ψ_ζ, $\psi_{\zeta+1}$ have opposite parity. Therefore, in the matrix element

[12] B. L. Cohen, J. B. Mead, R. E. Price, K. S. Quisenberry, and C. Martz, Phys. Rev. **118**, 499 (1960).

[13] R. Karplus and K. M. Watson, Am. J. Phys. **25**, 641 (1957).

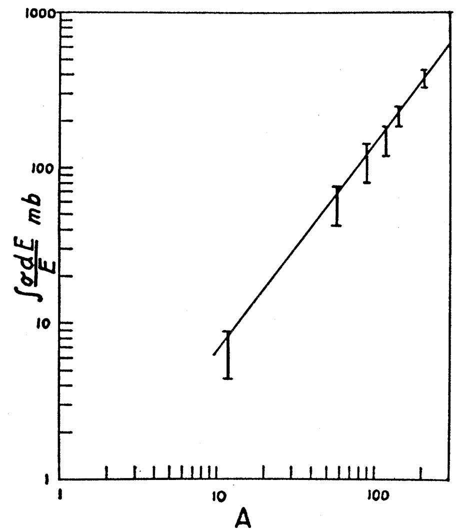

FIG. 3. Harmonic cross sections $\int\sigma dE/E$ for the same cases as Fig. 2. The solid line corresponds to $\int\sigma dE/E=0.31A^{4/3}$ mb.

$\langle Q|V|Q\rangle$ the direct terms involve only the even-parity parts of V, the exchange terms only the odd-parity parts, and the total matrix element is the difference of these. One may thus compare E_x in significance with the average (optical) potential in a nucleus, which is a sum over even-parity and odd-parity terms. Unfortunately, the numerical work leading to the value $y\sim 0.4$ is not very reliable; one can safely say only that Eq. (36) indicates effective forces between nucleons that are attractive in odd- as well as even-parity states. Since the "effective forces" are supposed to represent by central interactions all of the true interaction effects, this is perhaps not unreasonable.

The i.h.o. parameter a^2 can be expressed in terms of the oscillator frequency and effective mass[14] $a^2=(\hbar/M^*\omega)$. The i.h.o. relation $E_s=\hbar\omega$ then implies, by Eqs. (29) and (34)

$$M^*/M=\hbar^2/Ma^2E_s\approx 1.2\pm 0.2, \qquad (37)$$

which exceeds by about a factor 2 the estimate[15] for infinite nuclear matter. Although the $E1$ giant resonance involves mainly nucleons in the last shell, this circumstance can be estimated to increase the effective M^*/M only slightly, say by about 10%. On the other hand, (d,p) measurements of the type emphasizing the large size of δ have also given rise to direct estimates[12] of M^*/M exceeding even Eq. (37). It seems possible to imagine that Eq. (37) may reflect the boundary-dependent ("Thomas") shift of levels in finite nuclei.[16] For a square well there is a difference of $\pi/2$ between the boundary conditions at $r=R$ for an infinitely bound state ($\psi=0$) and a completely free one ($\psi'=0$). In a finite well the transition from one boundary condition to the other occurs mainly in the region of zero binding: If the "transition region" has an effective width of N level spacings, the change in radial momentum across the region will be $\Delta p=(N-\frac{1}{2})\pi\hbar/R$; while for a potential of infinite depth the corresponding momentum change across the region would be $\Delta p_\infty=N\pi\hbar/R$ because of the constant boundary condition. The relative increments in kinetic energy are then

$$\Delta T/\Delta T_\infty\approx\Delta p/\Delta p_\infty=(N-\tfrac{1}{2})/N. \qquad (38)$$

Thus to represent by an infinite well (as the i.h.o.) the succession of levels in a finite well in the region near zero binding requires the substitution

$$M^*\rightarrow M^*N/(N-\tfrac{1}{2}), \qquad (39)$$

in the corresponding infinite well. This condition makes Eq. (37) compatible with M^*/M for infinite nuclear matter if the effective $N\sim 1$ to 2. This is corroborated by explicit calculations with a truncated oscillator potential,[17] which show $\Delta T/\Delta T_\infty\sim\frac{1}{2}$.

IV. DISCUSSION

The considerations above provide a basis for examining previous discussions of the discrepancy δ. Association[4] of δ with $M^*=\frac{1}{2}M$ is completely contradicted by Eq. (37), which latter is compatible with an empirically satisfactory treatment of δ. Although Eq. (37) is a crude, secondary relation and possibly more model-dependent than some of our other estimates, it seems clear that δ is quite unrelated to any assumption like $M^*/M=\frac{1}{2}$.

The qualitative ascription of δ to coherent interactions between particle-hole pairs excited by $E1$ radiation[5] might appear at first to find support in a suitable verbal interpretation of Eq. (15) in which one regards the nucleon orbital ψ as destroying a particle or creating a hole in a filled shell, and conversely for ψ^*. Then $Q(12)$ represents the creation (destruction) of a coherent set of hole-particle pairs; but closer inspection of its derivation shows that in $Q(12)$ a neutron combines with a proton hole or vice versa. The interaction energy $\langle Q|V|Q\rangle_{00}$ can thus depend only on the charge exchange part of the nuclear potential as is of

[14] Because the notion of effective mass has occasioned some confusion in connection with $E1$ nuclear excitation, it may be worth while to repeat in outline the argument used here. The insensitivity of Eq. (3) to inconsistency between H and Ψ_0 is an important feature of all sum-rule calculations, since one would otherwise require a self-consistent solution of the nuclear A-body problem, as yet unavailable. This inconsistency can extend to the choice of M^*, which must therefore be considered separately for H and Ψ_0. For H the choice $M=M^*$ is equivalent to adoption of a certain, still widespread convention for describing effective two-body nuclear potentials. For Ψ_0 we originally specify only a length parameter a; when subsequent analysis and comparison with experiment indicate that $E_s\approx 38A^{-\frac{1}{3}}$ Mev, we can attempt to make our i.h.o. model Ψ_0 fit not only a but also $\hbar\omega=E_s$. This then implies $M^*/M\approx 1.2$ for the model Ψ_0, a value that seems not unreasonable on consideration of the Thomas shift.

[15] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

[16] E.g., A. M. Lane and R. G. Thomas, Revs. Modern Phys. **30**, 257 (1958).

[17] F. C. Barker (private communication).

TABLE I. Exchange factor y.

Potential	A_{01}	A_{10}	A_{00}	A_{11}	y	"x"
Rosenfeld[a]	1	0.6	−1.8	−0.33	∞	0.9
Elliott and Flowers[b]	1	0.7	0.5	−0.26	2.4	0.5
Serber	1	1	0	0	1.3	0.5
Barker[c]	1	(0.6±0.2)	(0.9±0.2)	(0.6±0.4)	0.3±0.3	$\approx\frac{1}{2}y$

[a] L. Rosenfeld, *Nuclear Forces* (North-Holland Publishing Company, Amsterdam, 1948).
[b] J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **242**, 57 (1957).
[c] See reference 17.

The potentials here are specified in terms of average values A_{TS} for the isotopic and real spin states of two nucleons, in these terms:

$$y=\frac{2(3A_{01}+A_{10}-3A_{11}-A_{00})}{9A_{11}+3A_{10}+3A_{01}+A_{00}}.$$

Note that since y is normalized to the effective nuclear potential in the nucleus, it can differ substantially from the conventional fraction x of exchange forces [J. S. Levinger and H. A. Bethe, Phys. Rev. **78**, 115 (1950)]. For a pure Wigner plus Majorana potential, $y=x(1-5/4x)^{-1}$; in a more general case we define

$$``x" = (\mathfrak{M}+\tfrac{1}{2}\mathcal{H})/(\mathcal{W}+\tfrac{1}{2}\mathcal{B}+\mathfrak{M}+\tfrac{1}{2}\mathcal{H}) = \tfrac{1}{2}[1-(3A_{11}+A_{00})/(3A_{01}+A_{10})]$$

to give a simple basis of comparison with the conventional x. The values of "x" are also given in the table: It is seen that y is a more sensitive measure of the forces' exchange character.

course true of E_x. The hole-particle pairs of reference 5 are entirely charge-conservative ($n-\bar{n}$ and $p-\bar{p}$); the corresponding $\langle Q'|V|Q'\rangle$ cannot contribute to E_x, and arguments have been presented[6] that its contribution to E_d is very small. Our empirical treatment does not yield any estimate of $\langle Q'|V|Q'\rangle$ but indicates that it is of minor significance relative to $\langle Q|V|Q\rangle$.

Unfortunately, even the collective treatment of $E1$ oscillations does not at present[6] account for the exchange term E_x. This arises from what has been called an inherent difficulty in defining classical, collective coordinates in the presence of exchange potentials.[18] Such a difficulty can be foreseen on the most elementary picture of neutron and proton fluids, undergoing opposite oscillations: The tacit assumption is always made that each fluid preserves its identity throughout; but if elements from the two fluids can exchange roles during the oscillation, this simple picture breaks down. In classical terms such exchange of fluid elements should have the qualitative effect of continuously advancing (or retarding, if the sign of the exchange effect is reversed) the phase of the oscillation: i.e., of increasing or decreasing its frequency. Quantum mechanically, because of exchange the occupation number n_j in Eq. (4) of reference 6 does not commute with the potential element $(K_{ijij}-K_{ijji})$, nor is the resulting V_τ diagonal in the occupation number n_j. In taking averages over the ground state of nuclear matter, all such questions are resolved by calculating only terms diagonal in occupation numbers; but the $E1$ oscillations consist just of the fluctuations in occupation numbers, and nondiagonal terms may be important—and are, in fact, according to the previous sections. The corrections computed by the collective approach with neglect of exchange effects must be attributed to E_d in the present treatment.

One is thus in the position of concluding that all previous discussions of δ have omitted the main contribution, which is that of E_x. This conclusion is supported by definitely improved fits to the giant resonance data over the range of real nuclei. It is interesting to note that the gross effect of E_x has been known in principle for as many years as the sum rule, and that modern refinements seem to change the picture very little.

[18] J. Fujita, Progr. Theoret. Phys. (Kyoto) **16**, 112 (1956).