The conversion process then goes on in this imaginary potential ($V_M - V_{M^*}$). Again the conversion comes as a result of a small longitudinal kick in the pseudopotential field. We assume that π mesons are coupled to bosons of mass M^* which represents a 3-meson intermediate state through a term in the Hamiltonian of the form,

$$\int \lambda^2(M^*)\phi_{M^*}{}^*\phi_\pi dM^*,$$

$$\phi_{M^*}{}^* = \phi_\pi{}^3 \quad (\text{i.e., } M^* \rightarrow 3\pi).$$

Using this interaction, we compute the following cross section for π+nucleus$\rightarrow M^*$+nucleus.

$$\frac{d\sigma}{dM^*} = \frac{\lambda^4(p^2+M^{*2})^{\frac{1}{2}}}{4(M^{*2}-M^2)^2}\left|1-\left(\frac{\sigma_{M^*}}{\sigma_\pi}\right)^{\frac{1}{2}}\right|^2 \frac{q_0{}^2\rho(M^*)\sigma_\pi}{(q_{||}{}^2+q_0{}^2)(2\pi)^9},$$

where σ_π = the cross section for diffraction scattering of a "bare" π meson by the nucleus (assuming $\sigma_\pi > \sigma_{M^*}$), σ_{M^*} = the cross section for diffraction scattering of the "bare" state of mass M^*, $q_{||} = (M^{*2}-M^2)/2p$, and $\rho(M^*)$ = covariant density of states in the M^* center-of-mass system (i.e., between M^* and M^*+dM^*).

In order to get an estimate of a cross section, one must make an assumption at this point. We assume that the cross section for "bare" and "dressed" states are approximately the same. One sees that the process is fairly likely until $q_{||} \gtrsim q_0$. Again this means that the process is likely only as long as the intermediate state of mass M^* can live a distance the order of the radius of the nucleus, $A^{\frac{1}{3}}/m_\pi$. When $q_{||} \gg m_\pi/A^{\frac{1}{3}}$, then so much momentum must be transferred that the collision point is localized well inside the nucleus and consequently will very likely disrupt the nucleus.

Note added in proof. R. F. Sawyer has pointed out an exception to our argument concerning quantum numbers. Nuclei are not in eigenstates of G conjugation, since charge conjugation (which produces anti-nuclei) is a part of the G operation. This means that the G quantum number of the beam particle need not be conserved in a diffraction production process.

In a similar way, in the diffraction production of θ_1's from a beam of θ_2's, the PC quantum number of the beam particle does change (from -1 to $+1$), as a consequence of the fact that the nucleus is not in an eigenstate of PC.

Then the diffraction production $\pi \rightarrow n\pi$ is allowed regardless of whether n is even or odd (i.e., regardless of G conjugation), with the single exception that $\pi \rightarrow 2\pi$ is forbidden by angular momentum and parity considerations.

ACKNOWLEDGMENTS

We feel that these ideas are worthy of more detailed theoretical treatment, as well as search for experimental verification. We again acknowledge our indebtedness to the many Russian authors who started these considerations some years ago. We acknowledge numerous conversations with our colleagues here at Wisconsin, in particular Dr. Fry, Dr. Ebel, Dr. Holladay, Dr. Lewis, Dr. Sachs, Dr. Sakita, and Dr. Sawyer.

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Particle Creation in Electron-Electron Collisions

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Pair production in high-energy electron-electron collisions is studied with special attention given to pion pair production. A method of calculation is formulated which yields results with reasonable directness in the relativistic limit. The orders of magnitude of counting rates for various experimental settings are ascertained. A complete result is obtained for the case in which two pions emerge with equal energies and opposite momenta.

I. INTRODUCTION

Experiments in which oppositely directed beams of electrons clash and interact over long periods of time are now in preparation.[1] These beams will permit measurements of electron-electron (Møller) scattering at center-of-mass energies of 500 Mev or more. Cross sections for pion and muon pair production are of the order of $(\alpha/\pi)^2$ relative to the Møller cross section, though they may be greatly enhanced in certain cases. Such processes furnish the opportunity—albeit, a re-

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[1] G. K. O'Neill and E. J. Woods, Phys. Rev. **115**, 659 (1959); Barber, Richter, Panofsky, O'Neill, and Gittelman, High-Energy Physics Laboratory, Stanford University Report, June, 1959 (unpublished). W. K. Panofsky, Fourth Annual International Conference on High-Energy Nuclear Physics, 1959 (unpublished). Similar projects have been undertaken at MURA (Midwestern Universities Research Association).

mote one at the present time—to study electromagnetic form factors and Compton (two-photon) interactions of the produced particles uncomplicated by the presence of nucleons. This would be particularly interesting in the case of pions, inasmuch as it would provide the most direct means of studying some effects of the pion-pion strong interaction recently proposed and discussed by many authors.[2]

In this paper, we study the characteristics of pair productions which accompany highly relativistic electron collisions with the following aims: (a) to formulate a method of calculation which yields results with reasonable directness in the relativistic limit; (b) to appreciate the orders of magnitude of counting rates for various experimental arrangements and so to determine which lines of investigation are most feasible.

As a result of such analysis, we can point to two types of experiments which appear to be more practicable than others. One case is treated semiquantitatively; in the other, which corresponds to the most obvious and simple experimental setting, the complete result is obtained. The discussion cites only pion production explicitly, but the qualitative considerations apply equally well to muon or, for that matter, electron production.

We remark first that the Møller cross section is infinite in the forward direction because of the infinite range of the Coulomb force, or equivalently stated, because of the singularity in the propagator for the exchanged photon. An echo of this feature is found in pair creations where many of the beam electrons are deviated less than about 10^{-3} radian from their original directions, at the relevant incident energies, owing to the "nearly singular" nature of certain photon propagators. These electrons move with considerably reduced energy and increased path curvature in the magnetic field that guides the beam and hence might be experimentally observable. In virtue of this forward concentration of electrons, a measurement which counts one meson of a produced pair and one forwardly scattered electron in coincidence may be of special interest. The counting rate is then proportional to the meson phase-space factor, but is independent of the electron detector's angular resolution provided the latter is somewhat larger than 10^{-3} radian and much less than one radian. If all electron energies were accepted in this hypothetical measurement,[3] the principal contribution to the cross section would differ from the Møller (wide-angle) cross section by a factor proportional to the fractional energy resolution[4] of the pion counter, a (small) factor of $(\alpha/\pi)^2$, and a (large) factor which is the logarithm of a dimensionless ratio of the order of 10^6. Besides, the cross section is multiplied by a (possibly large) factor due to strong interaction between the produced pions. The basis of this estimate is discussed more fully in Sec. 3.

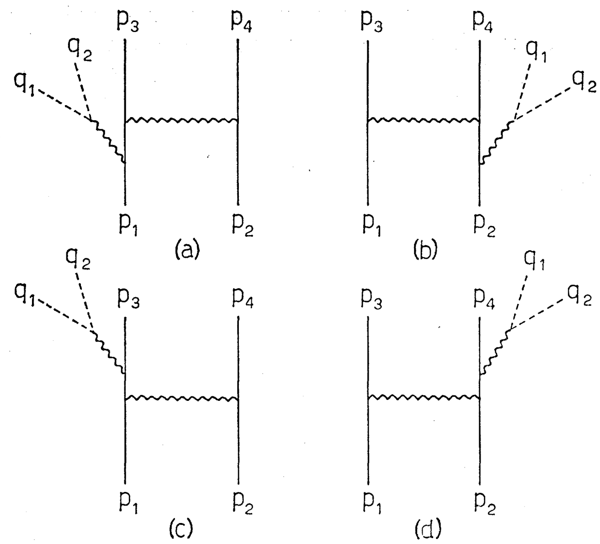

FIG. 1. One-photon ("odd") interactions for pair production in electron-electron collisions.

Pair production in electron-electron collisions may be considered, in the first perturbative approximation, to take place through two distinct processes, namely through one-photon interactions, Fig. 1, and two-photon interactions, Fig. 2. These two modes of production are physically distinguishable: In the first case the two produced pions are in a pure $J=1$, $T=1$ state; in the second case the state of the produced pions is a superposition of even angular momentum states. The effect of strong pion-pion interaction will also intervene differently in the two cases. In the first case it is completely described by the electromagnetic pion form factor, which is supposed, according to recent suggestions,[2] to have a resonant behavior just in the energy region of present interest. In the second case the effect of strong interactions intervenes through the modification of the Compton (two-photon) pion process.

The experiment described above would count pions arising from both modes of production, and would also "integrate" over one pion. Therefore, one would measure an admixture of both strong-interaction effects, integrated over a whole energy region.

An experiment which is instead sensitive only to the two-photon interaction and capable of giving informa-

[2] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2**, 365 (1959), and Phys. Rev. **117**, 1609 (1960); F. Bonsignori and F. Selleri, Nuovo cimento **15**, 465 (1960); I. Derado, Nuovo cimento **15**, 853 (1960); M. Gourdin and A. Martin, Nuovo cimento **16**, 78 (1960); F. Cerulus, Nuovo cimento **14**, 827 (1959). For investigation of pion-pion interaction through high-energy ("clashing beams") electron-electron and electron-positron collision experiments: L. M. Brown and F. Calogero, Phys. Rev. Letters **4**, 315 (1960), and Phys. Rev. (to be published); N. Cabibbo and R. Gatto, Phys. Rev. Letters **4**, 313 (1960).

[3] Otherwise, in estimating the order of magnitude of the counting rate, one should include another factor which accounts for the fraction of electrons actually counted.

[4] The cross section is also proportional to the angular resolution of the pion counter but a corresponding factor appears in the Møller cross section.

tion on strong-interaction effects not integrated over energy is achieved by counting in coincidence a produced pair with zero total momentum. Such mesons are mostly produced in collisions which rescatter both of the electrons into their forward directions. We find that the counting rate, although somewhat less than that described above for the first experiment (because of the smaller fraction of pion phase space accepted) is much greater than the rate for a two-pion coincidence experiment recording mesons at arbitrary angles. The calculation for this case is given in full in Sec. 4.

2. BASIC FORMULAS AND DISCUSSION OF THE METHOD

We consider the creation of a pion pair with four-momenta q_1, q_2 in a collision between electrons with initial momenta p_1, p_2 and final momenta p_3, p_4, respectively. In the clashing beam experiment, the center-of-mass and laboratory frames are the same. The energies and three-momenta are labeled as follows:

$$p_1=(E;\mathbf{p}),\ p_2=(E;-\mathbf{p}),\ p_3=(E';\mathbf{p}'),\ p_4=(E'';-\mathbf{p}''),$$
$$q_1=(\omega';\mathbf{q}'),\quad q_2=(\omega'';-\mathbf{q}''),$$
$$p_1^2=p_2^2=p_3^2=p_4^2=m^2,\quad q_1^2=q_2^2=\mu^2.$$

Let $\mathbf{n}$, $\mathbf{n}'$, $\mathbf{n}''$ be unit vectors in the directions of $\mathbf{p}$, $\mathbf{p}'$, $\mathbf{p}''$. The components of any vector $\mathbf{a}$ parallel to and transverse to $\mathbf{n}$, the beam direction, will be called a_z and $\mathbf{a}_\perp$, respectively. We use θ', φ', and $x'=\cos\theta'$ to denote the spherical coordinates of $\mathbf{n}'$ relative to $\mathbf{n}$, and θ'', φ'', $x''=\cos\theta''$ for the coordinates of $\mathbf{n}''$ relative to $\mathbf{n}$. When $\mathbf{p}$, $\mathbf{q}'$, $\mathbf{q}''$, $\mathbf{n}'$ are specified, all other variables are determined by conservation laws and mass shell relations. Production probabilities are therefore proportional to $d\mathbf{q}'d\mathbf{q}''\sin\theta' d\theta' d\varphi'$.

Figures 1 and 2 picture the processes of interest. In the rest system of the meson pair, reversal of the sign of meson charge means interchange of mesons. Matrix elements for Fig. 1 are linear in meson charge while those for Fig. 2 are quadratic. Thus, the mesons of Fig. 1 emerge in a relative odd angular momentum state; we term these processes "odd." The processes of Fig. 2, for the analogous reason, will be called "even." In odd processes, pion-pion interactions are summarized by a pion electromagnetic form factor $eF(q_1+q_2)$ at the creation vertex. We shall not attempt to indicate explicitly the role played by strong pion interactions in even processes, though the possibility of large effects should be kept in mind.

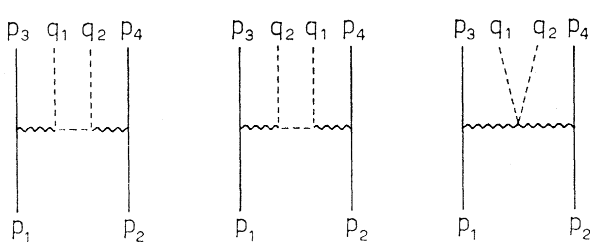

FIG. 2. Two-photon ("even") interactions for pair production in electron-electron collisions. The third diagram is present only in boson production.

Interference between even and odd processes adds to the cross section a term which is odd under change of sign of meson charge. Hence, an experiment which does not distinguish between positive and negative mesons encounters no interference of this type.

Set $k_1=p_1-p_3$, $k_2=p_2-p_4$, $q_{12}=q_1+q_2$, for the momentum transfers of the "first" electron, "second" electron, and meson line, respectively, and define "polarization" vectors

$$e_\mu'=k_1^{-2}\langle p_3|\gamma_\mu|p_1\rangle,\quad e_\mu''=k_2^{-2}\langle p_4|\gamma_\mu|p_2\rangle,\quad f_\mu=q_{12}^{-2}(q_1-q_2)_\mu, \tag{2.1}$$

which are true polarization vectors supplemented by photon propagator factors for notational convenience. The matrix elements of γ_μ between positive-energy states of unspecified spin are two-dimensional spin operators. With all four-spinors normalized to $u^*u=1$, we have, for the first electron ($\boldsymbol{\sigma}_1$ is the Pauli spin operator),

$$\langle p_3|\gamma_0|p_1\rangle=\{1+(E+m)^{-1}(E'+m)^{-1}\mathbf{p}\cdot\mathbf{p}'+i(E+m)^{-1}(E'+m)^{-1}(\boldsymbol{\sigma}_1\cdot\mathbf{p}'\times\mathbf{p})\}\times[4EE'(E+m)^{-1}(E'+m)^{-1}]^{-\frac{1}{2}}, \tag{2.2a}$$

$$\langle p_3|\boldsymbol{\gamma}|p_1\rangle=\{(E'+m)^{-1}\mathbf{p}'+(E+m)^{-1}\mathbf{p}+i\boldsymbol{\sigma}_1\times[(E'+m)^{-1}\mathbf{p}'-(E+m)^{-1}\mathbf{p}]\}\times[4EE'(E+m)^{-1}(E'+m)^{-1}]^{-\frac{1}{2}}. \tag{2.2b}$$

The formulas for $\langle p_4|\gamma_\mu|p_2\rangle$ are inferred from (2.2) by replacing $\mathbf{p}$, $\mathbf{p}'$, E', $\boldsymbol{\sigma}_1$ with $-\mathbf{p}$, $-\mathbf{p}''$, E'', and $\boldsymbol{\sigma}_2$.

The matrix element M_0 for odd processes, exclusive of factors of charge, 2π, i, is the sum of four parts as illustrated in Fig. 1:

$$M_0=(M_a+M_b+M_c+M_d)F(q_1+q_2), \tag{2.3}$$

where

$$M_a=\langle p_3|\gamma e''[\gamma(p_3-k_2)-m]^{-1}\gamma f|p_1\rangle, \tag{2.4a}$$

$$M_c=\langle p_3|\gamma f[\gamma(p_1-k_2)-m]^{-1}\gamma e''|p_1\rangle; \tag{2.4b}$$

M_b, $M_d=M_a$, M_c, respectively, but with $p_1\leftrightarrow p_2$, $p_3\leftrightarrow p_4$, $k_1\leftrightarrow k_2$, $e'\leftrightarrow e''$.

The matrix element M_e for even processes is

$$\begin{aligned}M_e&=e_\mu'e_\nu''\{2\delta_{\mu\nu}+[(q_1-k_1)^2-\mu^2]^{-1}\times(2q_1+k_1)_\mu(2q_2+k_2)_\nu+[(q_2-k_1)^2-\mu^2]^{-1}\times(2q_2+k_2)_\mu(2q_1+k_1)_\nu\}\\&=\{e'e''+4(e'q_1)(e''q_2)[(q_1-k_1)^2-\mu^2]^{-1}\}+\{q_1\leftrightarrow q_2\},\end{aligned} \tag{2.5}$$

since $e'k_1=e''k_2=0$.

The cross section $\sigma(q',q'')$ for pair production, summed over final electron states, is determined by the usual rules[5] (S denotes average over initial and sum

[5] See, for example, J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1955), Chap. 8–6.

over final electron spins):

$$d\sigma(q',q'')=(\alpha/2\pi)^4(E/p)d\mathbf{q}'d\mathbf{q}''\int(p'^2/\omega'\omega'')\frac{dp'}{dE}\times S|M_0+M_e|^2d\mathbf{n}'. \quad (2.6)$$

In this description of the cross section, we have not included any exchange terms. Later, we shall observe that exchange terms do not contribute in the relativistic limit to the two types of experiments described in the introduction.

The computational labor required to evaluate cross sections such as (2.6) is often prohibitively great, unless the work is arranged wisely. If, for example, the sums over the spins described by e', e'' were performed by the trace method, the calculator would be disconcerted to find most of his labor spent on terms which mutually cancel at a late stage of the calculation. In fact, when the contributions are ordered with respect to their magnitude in the relativistic limit (all energies$\gg m$), $S|M_0|^2$ vanishes to order m^2 and $S|M_e|^2$ vanishes to order m^4.

These cancellations occur because, in the relativistic limit, major parts of the polarizations $\langle p_3|\gamma_\mu|p_1\rangle$, $\langle p_4|\gamma_\mu|p_2\rangle$ are proportional to $(p_3-p_1)_\mu$, $(p_4-p_2)_\mu$, respectively. In other words, the photons exchanged by the electrons are largely "longitudinal." We are then forewarned of the cancellations by the principle of gauge invariance which tells us that completely "longitudinal" photons produce no physical effect.

In our approach the "longitudinal" parts of the polarizations are identified and treated separately. The leading contributions to the matrix elements and spin sums in the relativistic limit may then be ascertained with comparative ease. The "longitudinal" contributions, though of lesser magnitude than might have been originally anticipated, are not ignorable and must be taken into account in the calculation.

In the relativistic limit, the denominators of e', e'' are

$$k_1^2=(p_1-p_3)^2=-2EE'(1+\epsilon'-x'), \quad (2.9a)$$

$$k_2^2=(p_2-p_4)^2=-2EE''(1+\epsilon''-x''), \quad (2.9b)$$

where

$$\epsilon'=\frac{1}{2}\frac{m^2(E-E')^2}{E^2E'^2},\quad \epsilon''=\frac{1}{2}\frac{m^2(E-E'')^2}{E^2E''^2}. \quad (2.10)$$

For $E\gtrsim 500$ Mev, ϵ' and ϵ'' are typically of the order of 10^{-6} and are of interest only for $x'\approx 1$, $x''\approx 1$.

We define auxiliary symbols β', β'':

$$\beta'=(E+E')/(E-E'),\quad \beta''=(E+E'')/(E-E''). \quad (2.11)$$

Let (2.2a), (2.2b), and k_1 be expanded in powers of m and compared. Note that we can ignore m relative to E, E', but cannot ignore m relative to $E\sin\theta'$, $E(1-x')$, etc., since the denominator dictates $x'\approx 1$. We find

$$e'=\mathbf{e}_\perp'+\mathbf{L}'+C'k_1, \quad (2.12)$$

where

$$\mathbf{e}_\perp'=-[4EE']^{-1}(1+\epsilon'-x')^{-1}\times\{\beta'\mathbf{n}'+i\boldsymbol{\sigma}_1\times[\mathbf{n}_\perp'-\mathbf{n}(2\epsilon')^{\frac{1}{2}}]\}_\perp, \quad (2.13a)$$

$$L_z=-[4E(E-E')]^{-1}\{1+x'+i(\boldsymbol{\sigma}_1\cdot\mathbf{n}'\times\mathbf{n})\}, \quad (2.13b)$$

$$\mathbf{L}_\perp=[4E(E-E')]^{-1}\{1+i(1+x')(\boldsymbol{\sigma}_1\cdot\mathbf{n}'\times\mathbf{n})|\mathbf{n}_\perp'|^{-2}\}\mathbf{n}_\perp'-[4EE']^{-1}i\boldsymbol{\sigma}_1\times\mathbf{n}, \quad (2.13c)$$

$$C'=[2k_1^2(E-E')]^{-1}\{1+x'+i(\boldsymbol{\sigma}_1\cdot\mathbf{n}'\times\mathbf{n})\}. \quad (2.13d)$$

The formula for e'' is obtained by replacing $\mathbf{n}$, $\mathbf{n}'$, β', C', $\boldsymbol{\sigma}_1$ with $-\mathbf{n}$, $-\mathbf{n}''$, β'', C'', $\boldsymbol{\sigma}_2$, etc. The vector $\mathbf{n}_\perp'$ contains a factor of $\sin\theta'$. Notice that $\sin\theta'\times(1+\epsilon'-x')^{-1}$ and $(\epsilon')^{\frac{1}{2}}(1+\epsilon'-x')^{-1}$ both attain maximum values of the order of $(\epsilon')^{-\frac{1}{2}}$. Thus, whenever divided by powers of $(1+\epsilon'-x')$, a factor of $\sin\theta'$ is essentially equivalent to a factor of $(\epsilon')^{\frac{1}{2}}$ for purposes of estimating magnitude.

Equation (2.12) expresses e' as the sum of a "concentrated" part $\mathbf{e}_\perp'$ containing a denominator nearly singular at $x'=1$, and a "diffuse" part without a denominator. The concentrated part describes processes in which the first electron is rescattered predominantly without change of direction. For most of these electrons, the angle of deviation θ' from their forward direction is restricted by $1-\cos\theta'\lesssim\epsilon'$; that is,

$$\theta'\lesssim(\epsilon')^{\frac{1}{2}}\approx 10^{-3}. \quad (2.14)$$

The "longitudinal" term in (2.12), although it has a denominator of k_1^2 buried in the definition of C', must be classified as diffuse because in working out the full matrix element, a compensating factor of k_1^2 emerges in the numerator. Thus, if (2.12) and its counterpart for e'' are used in (2.4), (2.5), the portion of M_e dependent on C' and C'' is

$$(k_1^2-k_1q_1)^{-1}\{-2C'k_1^2(\mathbf{q}''\cdot\mathbf{e}_\perp'')+2C''k_2^2(\mathbf{q}'\cdot\mathbf{e}_\perp')+C'C''k_1^2k_2^2\}+(q_1\leftrightarrow q_2), \quad (2.15)$$

and the part of M_a+M_c dependent on C'' is

$$(k_2^2+2p_1k_2)^{-1}(k_2^2-2p_3k_2)^{-1}\tfrac{1}{2}C''k_2^2(\gamma f)\times(p_1+p_3)_\mu(q_1+q_2)_\mu. \quad (2.16)$$

3. ORDERS OF MAGNITUDE OF COUNTING RATES FOR VARIOUS EXPERIMENTAL ARRANGEMENTS

In view of the foregoing, it is natural to set

$$\sigma(\mathbf{q}',\mathbf{q}'')=\sigma_D(\mathbf{q}',\mathbf{q}'')+\sigma_C(\mathbf{q}',\mathbf{q}'')+\sigma_{CC}(\mathbf{q}',\mathbf{q}''), \quad (2.17)$$

thus distinguishing between parts of $\sigma(\mathbf{q}',\mathbf{q}'')$ of type D (diffuse), type C (one electron concentrated in forward direction), and type CC (both electrons concentrated).

The D part of $\sigma(\mathbf{q}',\mathbf{q}'')$ is obtained by using only the diffuse terms of e', e''. The final electrons have no strongly preferred directions. This part is the most laborious to calculate; exchange terms contribute and the dependence of the matrix elements on θ', φ' renders the integration over $d\mathbf{n}'$ in (2.6) complex. The D part is also comparatively difficult to observe. The counting rate depends on the amount of phase space—dimensionlessly expressed as $(dq'/\omega')(dq''/\omega'')(d\Omega'/4\pi)(d\Omega''/4\pi)$—that the counters envelop. The rate is then much less than the rate for measurements alluded to in the introduction and discussed below which depend on fewer differential factors.

When calculating concentrated parts of $\sigma(\mathbf{q}',\mathbf{q}'')$, we observe that the integral over $d\mathbf{n}'$ of $(1+\epsilon'-x')^{-2}$ is much larger than the integral of $(1+\epsilon'-x')^{-1}$ because $\epsilon'\ll 1$. Therefore, we select only the terms with the most factors of the "nearly" singular denominator. One consequence is that exchange terms are ignorable.

The C part of $\sigma(\mathbf{q}',\mathbf{q}'')$ has the same order of magnitude as $\sigma_D(\mathbf{q}',\mathbf{q}'')$ from which it may be distinguished, however, by using a counter to catch the forward electron, in coincidence with the meson counters. If the aperture of the electron counter is larger than $\theta'=(\epsilon')^{\frac{1}{2}}\approx 10^{-3}$ radian and much less than one radian, a negligible amount of diffuse production is counted and a negligible amount of concentrated production is omitted by this triple coincidence experiment. If we omit the second meson counter, thus integrating over $(dq''/\omega'')\times(d\Omega''/4\pi)$ without losing control of the experiment, a physically interesting quantity is obtained at a much higher counting rate. The estimate given in the introduction is easily obtained from the above formulas, keeping also in mind the considerations given after formula (2.13).

Finally, the CC part of $\sigma(\mathbf{q}',\mathbf{q}'')$ is obtained by using the concentrated parts $\mathbf{e}_\perp'$ and $\mathbf{e}_\perp''$, and corresponds to both electrons being rescattered in their forward directions. This also forces the meson pair to appear with zero transverse momentum. The special case in which the meson pair has zero total momentum is considered in detail in the following section.

4. PRODUCTION OF PIONS WITH OPPOSITE MOMENTA

Setting $\mathbf{q}'=\mathbf{q}''$ we obtain the special case in which the pions possess equal energy and emerge in opposite directions. The final electrons likewise have opposite momenta, $\mathbf{p}'=\mathbf{p}''$. The notation can be simplified somewhat. Because doubly primed variables are now equal to their singly primed counterparts, we put $\bar{E}=E'=E''$, $\mathbf{n}'=\mathbf{n}''$, and drop primes on other variables. (Exception: the space parts of e', e'' are opposite. Put $\mathbf{e}_\perp=\mathbf{e}_\perp'=-\mathbf{e}_\perp''$.)

The total matrix element for odd processes vanishes identically (not merely in the relativistic limit) in this case. This can be proved as follows[6]: M_a is a Lorentz invariant function of p_1, p_2, p_3, p_4, and f which we write $M_a=M(1234f)$. Then $M_b=M(2143f)$. Now, $f_\mu=(q_1-q_2)_\mu/(q_1+q_2)^2$ is a vector without a time component, and M_a depends linearly on f. Under a spatial reflection, M_a is invariant, but p_1 and p_2 are interchanged, p_3 and p_4 are interchanged, and f changes sign. Therefore,

$$M_a=-M(2143f)=-M_b.$$

Similarly, $M_c=-M_d$. Hence, $M_0=M_a+M_b+M_c+M_d=0$, as asserted.

Therefore the production in this case arises only from even processes. The CC part of the cross section, in virtue of its greater number of nearly singular denominators dominates the C and D parts. Thus, we replace e', e'' with $\mathbf{e}_\perp, -\mathbf{e}_\perp$, respectively, in (2.5). Furthermore, the meson propagator may be evaluated at $\theta=0$:

$$[(k_1-q_1)^2-\mu^2]|_{\theta=0}=-2\omega(\omega-q_z);$$

then

$$\begin{aligned}M_e&=\{\mathbf{e}_\perp^2-[\omega(\omega-q_z)]^{-1}2(\mathbf{q}\cdot\mathbf{e}_\perp)^2\}+(\mathbf{q}\leftrightarrow-\mathbf{q})\\&=2\{\mathbf{e}_\perp^2-[\omega^2-q_z^2]^{-1}2(\mathbf{q}\cdot\mathbf{e}_\perp)^2\}.\end{aligned}\tag{4.1}$$

When $\mathbf{e}_\perp$, taken from (2.13a), is inserted in (4.1), we obtain an equation of the form

$$M_e=\tfrac{1}{8}(E\bar{E})^{-2}(1+\epsilon-x)^{-2}\times\{X+i\boldsymbol{\sigma}_1\cdot\mathbf{X}_1+i\boldsymbol{\sigma}_2\cdot\mathbf{X}_2-\boldsymbol{\sigma}_1\cdot\mathbf{X}\cdot\boldsymbol{\sigma}_2\},\tag{4.2}$$

where $\mathbf{X}$ is a dyadic in three dimensions. Then

$$\mathcal{S}|M_e|^2=\tfrac{1}{64}(E\bar{E})^{-4}(1+\epsilon-x)^{-4}\times\{X^2+\mathbf{X}_1^2+\mathbf{X}_2^2+\mathrm{trace}\,\mathbf{X}\mathbf{X}^T\},\tag{4.3}$$

where the superscript T denotes transpose. To calculate the "X's," choose the azimuth $\varphi(=\varphi'=\varphi'')$ so that $\mathbf{n}_\perp'\cdot\mathbf{q}=|\mathbf{q}_\perp|\sin\theta\cos\varphi$. Use the abbreviations

$$s=2\mathbf{q}_\perp^2(\omega^2-q_z^2)^{-1}=2\mathbf{q}_\perp^2(\mu^2+\mathbf{q}_\perp^2)^{-1},\tag{4.4}$$

$$\mathbf{a}=\mathbf{n}'\times\mathbf{n},\tag{4.5a}$$

$$\mathbf{b}=\mathbf{n}_\perp'-(2\epsilon)^{\frac{1}{2}}\mathbf{n},\tag{4.5b}$$

$$\mathbf{c}=\mathbf{b}\times(\mathbf{q}_\perp/|\mathbf{q}_\perp|).\tag{4.5c}$$

Therefore, by (2.13a), (4.1), and (4.2),

$$X=\beta^2\sin^2\theta(1-s\cos^2\varphi),\tag{4.6a}$$

$$\mathbf{X}_1=\mathbf{X}_2=\beta[(2\epsilon)^{\frac{1}{2}}\mathbf{a}-s\sin\theta\cos\varphi\,\mathbf{c}],\tag{4.6b}$$

$$\mathbf{X}=\mathbf{I}\mathbf{b}^2-\mathbf{aa}-\mathbf{bb}-s\mathbf{cc}\quad(\mathbf{I}\text{ is the unit dyadic}).\tag{4.6c}$$

Noting that $\mathbf{a}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{b}=0$, we have

$$\mathrm{trace}\,\mathbf{X}\mathbf{X}^T=\mathbf{a}^4-2\mathbf{a}^2\mathbf{b}^2+2\mathbf{b}^4+2s[(\mathbf{a}\cdot\mathbf{c})^2-\mathbf{b}^2\mathbf{c}^2]+s^2\mathbf{c}^4.\tag{4.7}$$

After $\mathbf{a}^2$, $\mathbf{b}^2$, etc. are expressed in terms of θ, φ, ϵ and the integration over φ is carried out, we have

$$(2\pi)^{-1}\int X^2d\varphi=\beta^4\sin^4\theta(1-s+\tfrac{3}{8}s^2),\tag{4.8a}$$

$$(2\pi)^{-1}\int(\mathbf{X}_1^2+\mathbf{X}_2^2)d\varphi=\beta^2\sin^2\theta\times[4\epsilon(1-s+\tfrac{1}{2}s^2)+\tfrac{1}{4}s^2\sin^2\theta],\tag{4.8b}$$

$$(2\pi)^{-1}\int\mathrm{tr}\,\mathbf{X}\mathbf{X}^Td\varphi=(\sin^4\theta+4\epsilon\sin^2\theta+8\epsilon^2)\times(1-s+\tfrac{1}{2}s^2)-\tfrac{1}{8}s^2\sin^4\theta.\tag{4.8c}$$

[6] This proof is due to L. M. Brown.

When (4.8) and (4.3) are used in (2.6) and the integration is performed, we obtain the final result, which can be expressed in the form

$$d\sigma(\mathbf{q},-\mathbf{q})=(\alpha^4/2\pi m^2)(d\mathbf{q}'/4\pi\omega^3)(d\mathbf{q}''/4\pi\omega^3)Q, \quad (4.9)$$

where

$$Q=[E\omega(\mu^2+\mathbf{q}_\perp^2)]^{-2}\{\omega^2(E^2+\bar{E}^2)(\mu^4+\mathbf{q}_\perp^4)+\tfrac{4}{3}E^2\bar{E}^2(2\mu^4+\mathbf{q}_\perp^4)\}. \quad (4.10)$$

Consider now the cross section $\sigma_{CC}(\mathbf{q}',\mathbf{q}'')$ as a function of $(\mathbf{q}'-\mathbf{q}'')_\perp$ with $q_z'=q_z''$. Equation (4.9) is its value for $(\mathbf{q}'-\mathbf{q}'')_\perp=0$. The cross section is strongly peaked about this value and if $(\mathbf{q}'-\mathbf{q}'')_\perp$ differs only slightly from zero, drops to a negligible value compared with its peak value. This is because, when both electrons are well concentrated in their forward directions, the transverse component of the meson-pair momentum must be concentrated about zero.

We are therefore led to consider the following type of experiment. Let one counter record all mesons with momentum between $\mathbf{q}$ and $\mathbf{q}+d\mathbf{q}'$. Let a second counter, placed in coincidence, record mesons of momentum $\mathbf{q}''$, where q_z'' is between $-q_z$ and $-q_z+dq_z''$, and with $\mathbf{q}_\perp''$ such that

$$0\leq|(\mathbf{q}-\mathbf{q}'')_\perp|/2\bar{E}\leq\rho. \quad (4.11)$$

Here, ρ should be large compared with 10^{-3} and small compared with unity—say, $\rho\approx10^{-2}$. The characteristics (energy and angular resolution) of the counters will imply some relations between $d\mathbf{q}'$, dq_z'', and ρ, but for simplicity in this discussion, we assume they can be chosen independently. In any case (see below) the cross section depends weakly on the resolution factor ρ.

Define

$$d\sigma_{CC}(\mathbf{q})=\int_{|\mathbf{q}_\perp-\mathbf{q}_\perp''|\leq2\bar{E}\rho}\frac{d\sigma_{CC}(\mathbf{q},\mathbf{q}'')}{d\mathbf{q}_\perp''}d\mathbf{q}_\perp''\bigg|_{q_z''=q_z}. \quad (4.12)$$

With $\rho\geq10^{-3}$, essentially all pairs produced (with $q_z'-q_z''\approx0$) in CC processes will be counted with cross section $d\sigma_{CC}(\mathbf{q})$. The counting rate for C or D processes will be supplied with an additional (very small) factor of ρ. Thus the experiment really measures only $d\sigma_{CC}(\mathbf{q})$. The evaluation of (4.12) is examined in the Appendix. The result is

$$d\sigma_{CC}(\mathbf{q})=(\alpha^4/2\pi E^2)(d\mathbf{q}/4\pi\omega^3)(dq_z''/\omega)f(\mathbf{q},E), \quad (4.13)$$

where

$$f(\mathbf{q},E)=(\mu^2+\mathbf{q}_\perp^2)^{-2}[A\mu^4+B(\mu^4+\mathbf{q}_\perp^4)], \quad (4.14)$$

and

$$A=\left(\frac{E-\omega}{\omega}\right)^2\left[1-\frac{\pi^2}{3}-2\ln\frac{2\rho^2}{\epsilon}+\left(\ln\frac{2\rho^2}{\epsilon}\right)^2\right],$$

$$B=A-\left(\frac{E-\omega}{E}\right)\ln\frac{2\rho^2}{\epsilon}+\left(\frac{2E-\omega}{2E}\right)^2\left[\left(\ln\frac{2\rho^2}{\epsilon}\right)^2-\frac{\pi^2}{3}\right]. \quad (4.15)$$

Note that the cross section depends weakly on the angle at which the pions are produced relative to the beam direction.

The ratio R of the counting rate of this experiment to that of a 90° Møller scattering experiment,[7] for equal angular apertures of the counters, is given by

$$R=\frac{\alpha^2}{18\pi^2}\left(\frac{dq}{\omega}\right)\left(\frac{dq_z''}{\omega}\right)\left(\frac{q}{\omega}\right)^2f(\mathbf{q},E). \quad (4.16)$$

For $\rho=3\times10^{-2}$, $E=500$ Mev, $\omega=300$ and 400 Mev we get $R=5\times10^{-6}(dq/\omega)(dq_z''/\omega)$ and $R=2\times10^{-6}(dq/\omega)\times(dq_z/\omega)$, respectively, for mesons emitted transverse to the beam direction. The true rate of reaction might turn out to be much larger, thereby revealing the effect of strong pion-pion interactions.

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APPENDIX

The calculation of $d\sigma_{CC}(\mathbf{q}',\mathbf{q}'')$ in the neighborhood of $\mathbf{q}'=\mathbf{q}''=\mathbf{q}$ follows the line laid out in Sec. 4, but with some attention to the distinction between singly primed and doubly primed variables. To obtain (4.12), we may, however, set $q_z'=q_z''$. We define $\bar{E}=E-\frac{1}{2}(\omega'+\omega'')$ which coincides, in the zero-momentum case, with the definition of Sec. 4. Further, let $\mathbf{\Delta}=(2\bar{E})^{-1}(\mathbf{q}'-\mathbf{q}'')_\perp$. The conservation laws applied to m^2 tell us that

$$\mathbf{p}\cdot(\mathbf{p}'-\mathbf{p}'')=0, \quad (A.1a)$$

$$E''-E'=2(\bar{E}-E'), \quad (A.1b)$$

$$E'=(1+\mathbf{n}'\cdot\mathbf{\Delta})^{-1}\bar{E}(1-\mathbf{\Delta}^2)-\tfrac{1}{2}[\bar{E}(1-\mathbf{\Delta}^2)]^{-1}m^2. \quad (A.1c)$$

The only point of difficulty in the calculation, and the only one we discuss in detail, relates to the angular integrals which are of the general type

$$\int\sin^n\theta'\cos^p\varphi'\sin^m\theta''\cos^q\varphi''k_1^{-4}k_2^{-4}d\mathbf{n}'. \quad (A.2)$$

The coefficient of a factor like (A.2) may be evaluated at $\mathbf{q}'=\mathbf{q}''=\mathbf{q}$, i.e., at the peak value of the cross section, but the integrals (A.2), since they define the shape of the peak, must be treated more carefully. We consider, first of all, those integrals without a dependence on φ', φ'' in the numerator. These may be reduced to the

[7] The comparison is here made with Born approximation Møller scattering, without taking into account radiative corrections.

basic forms

$$I_{n,m}=\int k_1^{-2n}k_2^{-2m}d\mathbf{n}'; \quad n=1,2; \quad m=1,2, \tag{A.3a}$$

and

$$I=\int \sin\theta' \sin\theta'' k_1^{-4}k_2^{-4}d\mathbf{n}'. \tag{A.3b}$$

To arrive at (4.13), one must then compute the two-dimensional integrals [$\mathbf{\Delta}$ has only transverse components and $d\mathbf{\Delta}=(2\bar{E})^{-2}d\mathbf{q}_\perp''$],

$$J_{n,m}=\int_{|\mathbf{\Delta}|\leq\rho} I_{n,m}d\mathbf{\Delta}, \quad J=\int_{|\mathbf{\Delta}|\leq\rho} Id\mathbf{\Delta}. \tag{A.4}$$

We make use of the well-known formula

$$(ab)^{-1}=\int_0^1 [a+z(b-a)]^{-2}dz, \tag{A.5a}$$

and the related formulas

$$(ab^2)^{-1}=\int_0^1 [a+z(b-a)]^{-3}2zdz, \tag{A.5b}$$

$$(a^2b^2)^{-1}=\int_0^1 [a+z(b-a)]^{-4}6z(1-z)dz, \tag{A.5c}$$

obtainable from (A.5a) by differentiation with respect to a, b.

With the substitutions $a=k_1^2$, $b=k_2^2$ we infer, by (A.1),

$$\begin{aligned} a+z(b-a) &= 2\{EE'-\mathbf{p}\cdot\mathbf{p}'-m^2+z[E(E''-E')+\mathbf{p}\cdot(\mathbf{p}'-\mathbf{p}'')]\} \\ &= 2E\bar{E}(1+\mathbf{n}'\cdot\mathbf{\Delta})^{-1}[B-\mathbf{n}'\cdot\mathbf{A}]+2m^2C, \end{aligned} \tag{A.6}$$

where

$$\mathbf{A}=(1-\mathbf{\Delta}^2)\mathbf{n}-2z\mathbf{\Delta}, \tag{A.7a}$$

$$B=(1-\mathbf{\Delta}^2)+2z\mathbf{\Delta}^2, \tag{A.7b}$$

$$\begin{aligned} C=&[2\bar{E}(1-\mathbf{\Delta}^2)]^{-1}E\mathbf{n}'\cdot(\mathbf{\Delta}-2z\mathbf{\Delta}+\mathbf{n}) \\ &+[2E(1+\mathbf{n}'\cdot\mathbf{\Delta})]^{-1}\bar{E}(1-\mathbf{\Delta}^2)-1. \end{aligned} \tag{A.7c}$$

Furthermore,

$$B^2=\mathbf{A}^2+4z(1-z)\mathbf{\Delta}^2(1-\mathbf{\Delta}^2). \tag{A.8}$$

Because of (A.6), (A.7), the J's are nearly singular when $\mathbf{n}'$ is in the direction $\mathbf{A}$ and $\mathbf{\Delta}^2$ is very small. As the C term is significant only in this case, we evaluate it for $\mathbf{\Delta}=0$, $\mathbf{n}'=\mathbf{n}$, obtaining

$$m^2C=E\bar{E}\epsilon.$$

Moreover, $|\mathbf{\Delta}|\leq\rho\ll 1$ in (A.4), so $\mathbf{\Delta}^2$ is ignorable relative to unity. We may also replace the factor $(1+\mathbf{n}'\cdot\mathbf{\Delta})$ by unity in (A.6) as it ultimately yields $\mathbf{\Delta}^2$ corrections only. Therefore, by (A.8),

$$B=[\mathbf{A}^2+4z(1-z)\mathbf{\Delta}^2]^{\frac{1}{2}}=A+2z(1-z)\mathbf{\Delta}^2. \tag{A.9}$$

Therefore, by (A.5c)

$$\begin{aligned} J_{2,2}&=\int_{|\mathbf{\Delta}|\leq\rho} d\mathbf{\Delta}\int_0^1 dz\int d\mathbf{n}' \\ &\quad\times[A-\mathbf{n}'\cdot\mathbf{A}+2z(1-z)\mathbf{\Delta}^2+\epsilon]^{-4} \\ &\qquad\times(2E\bar{E})^{-4}6z(1-z) \\ &=\int_0^1 dz\int_{|\mathbf{\Delta}|\leq\rho} d\mathbf{\Delta}[2z(1-z)\mathbf{\Delta}^2+\epsilon]^{-3} \\ &\qquad\times(E\bar{E})^{-4}\tfrac{1}{4}\pi z(1-z) \\ &=\pi^2\epsilon^{-2}(2E\bar{E})^{-4}. \end{aligned} \tag{A.10}$$

Similarly, (A.5b) and (A.5a) lead to

$$\begin{aligned} J_{2,1}=J_{1,2}&=\frac{-\pi^2}{4(E\bar{E})^3\epsilon}\int_0^1 [2z(1-z)-\epsilon\rho^{-2}]^{-1}dz \\ &=\frac{-\pi^2}{8(E\bar{E})^3\epsilon}\ln\left(\frac{2\rho^2}{\epsilon}\right), \\ J_{1,1}&=\frac{\pi^2}{4(E\bar{E})^2}\int_0^1 dz\left(\frac{1}{z}+\frac{1}{1-z}\right)\ln[1+2z(1-z)\rho^2/\epsilon] \\ &=\frac{\pi^2}{2(E\bar{E})^2}\int_0^1 \frac{dz}{z}\ln[1+2z(1-z)\rho^2/\epsilon]. \end{aligned} \tag{A.11}$$

Since ϵ/ρ^2 is small,

$$1+2z(1-z)\rho^2/\epsilon=(1+b_1z)(1+b_2z),$$

where

$$b_1=(\rho^2/\epsilon)[1+(1+2\epsilon/\rho^2)^{\frac{1}{2}}]\approx 2\rho^2/\epsilon,$$

$$b_2=(\rho^2/\epsilon)[1-(1+2\epsilon/\rho^2)^{\frac{1}{2}}]\approx -1.$$

Consequently[8]

$$\begin{aligned} J_{1,1}&=\frac{\pi^2}{2(E\bar{E})^2}\int_0^1 \frac{dz}{z}\left[\ln(1-z)+\ln\left(1+\frac{2\rho^2}{\epsilon}z\right)\right] \\ &=\frac{\pi^2}{2(E\bar{E})^2}\left[-\frac{\pi^2}{6}+\frac{1}{2}\left(\ln\frac{2\rho^2}{\epsilon}\right)^2\right]. \end{aligned} \tag{A.12}$$

Using these methods, one may also demonstrate that $\sin\theta' \sin\theta''$ in (A.3b) can be approximated by $\sin^2\theta$, reducing (A.3b) to integrals of type (A.3a), and that occurrences of φ', φ'' in (A.2) may be averaged (with $\varphi'=\varphi''$) as in Sec. 4, Eq. (4.8).

The only serious obstacles to the calculation of $d\sigma_{CC}(\mathbf{q})$ are now overcome. The result is given in (4.13) of the text.

[8] See, for example, H. B. Dwight, *Tables of Integrals and Other Mathematical Data* (The Macmillan Company, New York, 1949), pp. 137 and 201.