TABLE III. Momentum cutoff values and estimated energies at which the Born and Bethe cross sections differ by 1% and 10%.

	$\kappa_c a_0$	$E(1\%)$ (ev)	$E(10\%)$ (ev)
$5s$-$6p$	0.024	19	3
$5p$-$6d$	0.027	13	2.3
$5d$-$6f$	0.030	9	1.8
$5f$-$6g$	0.0360	5	1.2
$5g$-$6h$	0.04611	2.5	0.7

spectively. It will be noted that the Bethe formula reproduces the Born results to relatively low energies. It should be emphasized that $E(1\%)$ and $E(10\%)$ in this paper and in Papers I and II are rough estimates only.

Current work at St. John's University is aimed at extending these cross section results to large values of n by more approximate methods, and also to testing the range of validity of Born's approximation for the calculations to date.

ACKNOWLEDGMENTS

Thanks are due to Gerard C. McCoyd and Carl C. Krolik for checking calculations. We are indebted to the Fairchild Astrionics Division, Wyandanch, New York, for generously permitting our use of the digital computer facility. We are grateful to Miss Maureen Loughran, Mr. Sam Singer, and Mrs. Agnes Kadar for assistance with the figures and manuscript of the papers in this series.

PHYSICAL REVIEW VOLUME 120, NUMBER 5 DECEMBER 1, 1960

High-Frequency Region of the Bremsstrahlung Spectrum*

R. H. PRATT

Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California

(Received July 25, 1960)

The McVoy-Fano theory of the connection between the atomic photoelectric effect and the high-frequency region of the bremsstrahlung spectrum has been extended to next order in αZ. The contribution from p states is determined and is important in heavy elements. Predictions for the high-frequency limit are in reasonable agreement with experiment. Information is also obtained concerning angular distributions and polarization correlations.

I. INTRODUCTION

In the high-frequency region of the bremsstrahlung spectrum almost all the energy of the incident electron is radiated. Since the outgoing electron is of low energy, the process cannot be treated with the Bethe-Maximon[1] methods. In contrast to the Born-approximation prediction of the Bethe-Heitler formula,[2] the bremsstrahlung cross section remains finite in the high-frequency limit, when the photon energy achieves its maximum value and the electron velocity $\beta=0$. Fano, Koch, and Motz[3] have noted that in this limit bremsstrahlung is an approximate inverse of the atomic photoelectric effect, and that a prediction for its cross section follows from the theoretical work on the photoeffect. Using Nagasaka's results[4] for the photoeffect, they obtain fairly good agreement with experiments on the bremsstrahlung "tip." A more rigorous, but limited, derivation of the relationship has been given by McVoy and Fano[5]: To lowest order in $a\equiv Ze^2$ the matrix elements for inverse photoeffect from the K shell and for the high-frequency limit of bremsstrahlung are identical, apart from normalization factors.[6]

It is now known that Nagasaka's expression for the high-energy limit of the photoeffect, which corrected the Hall formula[7] in order a, is itself incorrect in order a^2.[8–10] For all a the total cross section for the K shell in the high-energy limit is fairly well represented by[8]

$$\sigma=(4\pi e^2a^5/k)(1-4\pi a/15)\times\exp\{2[-1+(1-a^2)^{\frac{1}{2}}]\ln a-2a\cos^{-1}a\}, \quad (1)$$

* Supported in part by the U. S. Air Force through the Air Force Office of Scientific Research.

[1] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954); H. Olsen, L. C. Maximon, and H. Wergeland, Phys. Rev. **106**, 27 (1957).

[2] W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, New York, 1954), 3rd ed.; further terms in the Born expansion have been calculated by C. Kacser, Proc. Roy. Soc. (London) **A253**, 103 (1959).

[3] U. Fano, H. W. Koch, and J. W. Motz, Phys. Rev. **112**, 1679 (1958), hereafter referred to as FKM, which see for references to previous work.

[4] F. G. Nagasaka, Ph.D. thesis, University of Notre Dame, 1955 (unpublished). See also G. W. Grodstein, National Bureau of Standards Circular No. 583 (U. S. Government Printing Office, Washington, D. C., 1957); and R. T. McGinnies, NBS Supplement to Circular 583 (1959).

[5] K. W. McVoy and U. Fano, Phys. Rev. **116**, 1168 (1959), hereafter referred to as MF, also U. Fano, Phys. Rev. **116**, 1156 (1959).

[6] We use unrationalized units and set $\hbar=c=m_e=1$; $O(x)$ shall mean "of order x" and $y=O(x)$ shall mean "y is of order x."

[7] H. Hall, Revs. Modern Phys. **8**, 358 (1936).

[8] R. H. Pratt, Phys. Rev. **117**, 1017 (1960), hereafter referred to as I.

[9] E. Guth (private communication).

[10] The results obtained in I have now been verified by H. Hall (private communication). For lead a similar result was obtained earlier by R. H. Boyer, Phys. Rev. **117**, 475 (1960).

where k is the energy of the incident photon, and hence the cross section is not well represented by the lowest order in a. If the connection between the bremsstrahlung tip and the photoeffect is to be of practical use, as the work of FKM suggests, the relationships between the processes must exist to a higher order in a and be valid for a finite portion of the high-frequency spectrum. Further, FKM and MF assume that in both processes only the s states for the low-energy bound or continuum electron need be considered. However, recent work on the L-shell photoeffect[11] has shown that in heavy elements the p states are not negligible even in the high-energy limit (as contrasted with the nonrelativistic prediction).

The present paper hence has three objectives: (1) to extend the connection of the photoeffect and the high-frequency region of bremsstrahlung beyond lowest order in a and to finite β, (2) to determine the importance of the contribution of higher angular momentum states to the bremsstrahlung process, and (3) to use the more recent results on the photoeffect as a basis for bremsstrahlung predictions. We will begin by establishing the important regions in r space for the integrals over electron wave functions which determine the matrix elements. The relationship of the two processes results from an equivalence of bound and continuum electron wave functions in these regions. After considering screening and energy extrapolations, we can make predictions for bremsstrahlung based on the photoeffect work of I and II and compare with experiment.

The main ideas of the analysis to follow can be displayed in the nonrelativistic problem for s states at the high-frequency limit. In this case, when initial and final wave functions are expanded in a, the matrix element for bremsstrahlung and the complex conjugate of the matrix element for the photoeffect both have the form

$$\int d^3r[f_0(r)+af_1(r)+a^2f_2(r)] \times O[g_0(r)+ag_1(r)+a^2g_2(r)],$$

through order a^2 in the wave functions. The g's represent the high-energy electron wave function and are the same in the two processes. One goes from bremsstrahlung to photoeffect by changing the f's. What MF noted was that the lowest order in a of the *matrix element* is contributed to both by f_0 and by f_1, and that, apart from normalization, these f's are the same for bremsstrahlung and photoeffect. What we now note is that f_2, which is different for the two processes, does not contribute to the matrix element in relative $O(a)$, but in relative $O(a^2)$. A similar argument can be made for any angular momentum state and for the entire high-frequency region of bremsstrahlung. In the relativistic problem the large and small components of the electron spinors must be expanded separately, after which a similar analysis can be made.

[11] R. H. Pratt, Phys. Rev. **119**, 1619 (1960), hereafter referred to as II. The notation of this paper will in general be followed in the present work.

II. BREMSSTRAHLUNG AND PHOTOEFFECT

The comparison of bremsstrahlung and photoeffect is a comparison of matrix elements. It is assumed that the outgoing low-energy electron in the bremsstrahlung process is not observed, for when correlations with this electron are detected the correspondence with the photoeffect does not apply. The bremsstrahlung cross section can hence be represented as a single sum over cross sections for the various angular momentum states of the outgoing electron, without interference terms. The differential cross section for an electron $\mathbf{p}$ (of total energy E) to radiate a photon $\mathbf{k}$ and then be in a state of definite angular momentum (j,l,m) is

$$d\sigma=(2\pi)^{-2}p^{-1}E|H|^2d^3k\delta(E-k-\epsilon), \tag{2}$$

where

$$H_{\text{brem}}=-e(2\pi/k)^{\frac{1}{2}}\int d^3r\,\psi_{\text{fin}}^*(\epsilon,j,l,m)\boldsymbol{\alpha}\cdot\mathbf{e}^*e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{\text{in}}, \tag{3}$$

and $\epsilon>1$ is the *total* energy of the outgoing electron. The final electron has the usual normalization to an energy δ function and the initial electron is normalized to a unit volume. This is to be compared with the differential cross section

$$d\sigma=(2\pi)^{-2}|H|^2d^3p\delta(E-k-\epsilon), \tag{4}$$

for a photon $\mathbf{k}$ to eject an electron $\mathbf{p}$ from the (n,j,l,m) bound state of an atom. To make the comparison clearer we give the complex conjugate of the matrix element:

$$H_{\text{photo}}^*=-e(2\pi/k)^{\frac{1}{2}} \times\int d^3r\,\psi_{\text{in}}^*(n,j,l,m)\boldsymbol{\alpha}\cdot\mathbf{e}^*e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{\text{fin}}, \tag{5}$$

where the initial state has its conventional normalization and the final state is normalized to a unit volume; $\epsilon<1$ is the total energy of the bound electron.

The main difference between the matrix elements (3) and (5) is the replacement of a continuum wave function by a bound-state wave function with the same (j,l,m). Now for small r it is possible to show that these wave functions are similar, whereas for large r their behaviors are of course very different. Thus the first question is to determine which regions give a significant contribution to the integrals. For the photoeffect this problem was investigated in II. The bound-state wave function at the origin is proportional to $r^{\gamma-1}$, where γ is determined by the angular momentum of the state. The remaining r dependence (including the characteristic bound-state exponential) can simply be written as a power series in (ar) and the matrix element obtained as a power series in a by integrating term by term. Bremsstrahlung has similar properties whenever the final-state momentum $q \ll 1$: The low-energy continuum wave function is proportional to $r^{\gamma-1}$ at the origin and otherwise consists of a power series in (ar) and (qr); on integrating term by term the matrix element is obtained as a power series in a and q. A detailed proof of these statements (and particularly a justification of the term-by-term integration) has been given by MF. In both processes the argument will fail if $p \ll 1$; the series will not converge, indicating the importance of larger regions in r. Hence it is not permissible to compare photoeffect near threshold with bremsstrahlung from nonrelativistic electrons.

We will soon show that the first few terms of the series in r for bound and continuum wave functions differ only in normalization. In order to compare cross sections, we must know what order in a is contributed to the matrix element by each term of the series. For the photoeffect this was determined in II, and similar conclusions are obtained for bremsstrahlung. We summarize the results, using a to represent either a or q, and removing all normalization factors from the matrix element. There are two cases. (1) $j=l+\frac{1}{2}$. The leading order in the matrix element is $O(a)$. Terms in the matrix element of this order are contributed by the first two terms of the power series for the "large component" g and the first term of the "small component" f. (2) $j=l-\frac{1}{2}$. The leading order in the matrix element is $O(a^2)$. Terms of this order are contributed by the first three terms of g and the first two terms of f. In both cases the next two terms of the power series for g and f contribute $O(a^2)$ relative to the leading order, and further terms to even higher orders. Terms of relative $O(a)$ are contributed by the factor $r^{\gamma-1}$, which however is the same for the two processes. Hence if we show that the terms in the wave function power series which contribute in lowest order are the same (apart from normalization) for the two processes, we have automatically obtained a relationship which holds neglecting only relative $O(a^2)$ in the matrix elements. This is a generalization of the MF result, which was stated to hold for s states neglecting relative $O(a)$.

It is simple to show that the terms of bound and continuum series which contribute to lowest order matrix elements are the same. In II it was demonstrated that for the photoeffect such terms were, except for normalization, independent of principal quantum number, i.e., independent of $|\epsilon-1|=O(a^2)$. The proof could be made directly from the coupled differential equations for the radial wave functions, and did not require use of the knowledge that one was dealing with a bound state. Hence it continues to apply for $\epsilon-1=O(q^2)>0$, and says that neglecting $O(q^2)$ and $O(a^2)$ these terms of the continuum and bound wave functions are the same, except for normalization. This implies a further generalization of MF, as we do not require $q\equiv 0$.

Thus the difference between continuum and bound wave functions in the matrix elements (3) and (5) is small, once the normalizations are taken into account. There are two other differences between these matrix elements: The photon polarization vector $\mathbf{e}$ is replaced by $\mathbf{e}^*$, and an incoming continuum electron wave function ψ_{in} is replaced by an outgoing function ψ_{out}. These latter differences would not occur if comparison were made with the inverse photoeffect. Since the reciprocity theorem connects photoeffect and its inverse (matrix elements can be identified if all spins are reversed), photoeffect and bremsstrahlung can be connected subject to the same reversal of spins.

The differential cross sections (2) and (4) may hence be connected with the relation

$$|H|_{\text{brem}}{}^2/|H|_{\text{photo}}{}^2=(N_{\text{cont}}/N_{\text{bound}})^2\equiv R^2, \tag{6}$$

where N_{cont} and N_{bound} are the normalizations of continuum and bound electron wave functions of the same (j,l,m). The restrictions on the validity of Eq. (6) have previously been outlined. N_{bound} is easily obtained from II. For consistency, terms of relative $O(a^2)$ should be neglected, and then for $j=l\pm\frac{1}{2}$ the large components of the bound-state wave function near the origin behave as

$$C(n,l)(2ar)^{\gamma-1}, \tag{7a}$$

$$C(n,l)(2ar)^{\gamma-1}a^2(2l+1)/(2l), \tag{7b}$$

respectively, where

$$C(n,l)=\frac{1}{(2l+1)!}\left[\frac{(n+l)!}{2n(n-l-1)!}\right]^{\frac{1}{2}}(2a)^{\frac{3}{2}}n^{-(\frac{3}{2}+l)},\qquad \gamma=(\kappa^2-a^2)^{\frac{1}{2}}. \tag{8}$$

A similar expansion of the continuum wave functions, neglecting relative $O(a^2)$ and $O(q^2)$, yields (apart from a phase factor)

$$D(l)(2qr)^{\gamma-1}, \tag{9a}$$

$$D(l)(2qr)^{\gamma-1}aq(2l+1)/(2l), \tag{9b}$$

where

$$D(l)=\frac{|\Gamma(l+1+i\nu)|}{(2l+1)!}e^{\frac{1}{2}\pi\nu}(2q/\pi)^{\frac{1}{2}},\qquad \nu=a/q. \tag{10}$$

From the preceeding we see that for both $j=l+\frac{1}{2}$ and for $j=l-\frac{1}{2}$

$$R=[D(l)/C(n,l)]\nu^{-l}. \tag{11}$$

Specializing to the s and p states, for which photoeffect information is available,

$$\begin{aligned}R(s)&=a^{-1}n^{\frac{3}{2}}(1-e^{-2\pi\nu})^{-\frac{1}{2}}\\ R(p)&=a^{-1}n^{\frac{5}{2}}(n^2-1)^{-\frac{1}{2}}(1+\nu^{-2})^{\frac{1}{2}}(1-e^{-2\pi\nu})^{-\frac{1}{2}}.\end{aligned} \tag{12}$$

With (11) or (12) it is possible to discuss the entire high-frequency region of bremsstrahlung. For the high-frequency limit $q=0$ one may use the relation

$$\lim_{q\to 0} |\Gamma(\gamma+ia\epsilon/q)| e^{\frac{1}{2}\pi a\epsilon/q} q^{\gamma-\frac{1}{2}} = (2\pi)^{\frac{1}{2}} a^{\gamma-\frac{1}{2}}, \quad (13)$$

noted by MF to derive

$$R(q=0) = a^{-1} n^{2+l} \left[\frac{(n+l)!}{(n-l-1)!} \right]^{-\frac{1}{2}}. \quad (14)$$

In agreement with (12)

$$R(s,\, q=0) = a^{-1} n^{\frac{3}{2}}, \quad R(p,\, q=0) = a^{-1} n^{\frac{3}{2}} (n^2-1)^{-\frac{1}{2}}. \quad (15)$$

Making use of these R's, we may write the cross section for bremsstrahlung, which is a sum over partial cross sections of different (j,l,m), as a weighted sum over photoeffect cross sections:

$$d\sigma_{\mathrm{brem}} = p^{-1} E \sum_{j,l,m} R^2 d\sigma_{\mathrm{photo}}, \quad (16)$$

subject to energy conservation. In view of our normalization to an energy δ function, the total cross section is simply related to the cross sections for the photoeffect by

$$\sigma_B = (d\sigma/dk)_{\mathrm{brem}} = [(E-1)/(E+1)] \sum_{j,l,m} R^2 \sigma_{\mathrm{photo}}, \quad (17)$$

in agreement with FKM.

To this point the discussion has assumed hydrogen-like wave functions for the electrons and has neglected the screening effect of the electron cloud. It has been argued in II that for the regions in r space near the origin which are important in the photoeffect the *shape* of the wave functions will be largely unaffected, but the change in *normalization* must be taken into account. The same arguments apply to the high-frequency region of bremsstrahlung; for their validity we must still require that we are not dealing with bremsstrahlung from nonrelativistic electrons or with photoeffect near threshold. For relativistic electrons the change in normalization is small. Hence the concern is for the normalization of the wave functions ψ_{cont} and ψ_{bound} of given (j,l,m) previously discussed. Equation (6) may still be considered valid, and from an experimental point of view a measurement of the cross sections for bremsstrahlung in the high-frequency region and for the photoeffect may be viewed as providing information concerning the normalization of electron wave functions at the origin. A theoretical estimate of the change in normalization is obtained by comparing the hydrogen-like wave functions near the origin with wave functions computed numerically from more accurate potentials. For the K and L shells this has been done by Brysk and Rose,[12] who give the correction factors in graphical form. The K-shell corrections to the wave function normalization are generally less than 5%, but the L-shell corrections are large except in the heaviest elements. Corrections to continuum wave functions may be obtained from the work of Reitz.[13] Except within perhaps 10 kev of the tip they are small, and may generally be neglected in view of the other approximations which have been made. Hence we may use the unscreened R's (11)–(15) in the expressions (16) and (17) for bremsstrahlung if we also use unscreened predictions for the photoeffect cross sections.

III. APPLICATIONS

Cross sections for the high-frequency region of bremsstrahlung have been related to cross sections for the photoeffect. This is of practical use only if photoeffect results, either experimental or theoretical, are available. Our purpose in the present section will be to use the photoeffect data which now exists as a basis for specific predictions regarding bremsstrahlung, and to compare these predictions with experiment. The greatest amount of information concerns total cross sections, but some results for angular distributions and polarization correlations may also be obtained. When further results for the photoeffect become available[14] the conclusions of this section may be extended in an evident manner. In the process of obtaining predictions for bremsstrahlung we shall also see the relative importance of s, p, etc., states, and otherwise investigate the validity of the approximations made. Additional related processes will be noted.

The K-shell photoeffect total cross sections at relativistic energies are fairly well known. The high-energy limit was established in I; when combined with Gavrila's[15] work on the energy dependence of the cross section, it gives an extrapolation formula useful down to the energy (1.1 Mev) at which Hulme's[16] numerical values are available. For lower energies theory and experiment are in reasonable accord with the values tabulated by Grodstein.[4] Information on L-shell cross sections is not yet as satisfactory. The high-energy limits were obtained in II; the energy dependence of the $2s$ cross section is the same as for $1s$, whereas for the $2p$ cases the dependence is not completely known. It was assumed in II that all these cross sections had the same energy dependence in the relativistic region;

[12] H. Brysk and M. E. Rose, Revs. Modern Phys. **30**, 1169 (1958).

[13] J. R. Reitz, Phys. Rev. **77**, 10 (1950), and J. R. Reitz, Department of Physics, University of Chicago, 1949 (unpublished). In the first of these references the screening correction to the sum of the squares of large components for $s_{\frac{1}{2}}$ and small components for $p_{\frac{1}{2}}$ is tabulated. For a more complete analysis one must examine the wave functions as tabulated in the second reference. The potentials used have been tabulated by N. Metropolis and J. R. Reitz, J. Chem. Phys. **19**, 555 (1951).

[14] M. Gavrila (to be published), S. Hultberg (private communication), H. Hall and R. H. Pratt (work in progress).

[15] M. Gavrila, Phys. Rev. **113**, 514 (1959). *Note added in proof.* The work of B. Nagel, Arkiv. Fysik **18**, 1 (1960), is now also available.

[16] H. R. Hulme, J. McDougall, R. A. Buckingham, and R. H. Fowler, Proc. Roy. Soc. (London) **A149**, 131 (1935).

reasonable agreement with experiment was obtained after screening effects were taken into account. The important result was that in heavy elements $2p$ cross sections were comparable to $2s$ cross sections, contrary to the nonrelativistic result. For bremsstrahlung this means that p state contributions are large, and that it may be necessary to consider d states and higher.

It is convenient to write the photoeffect cross sections as

$$\sigma_P{}^J \equiv \sigma_0 P^J(E) = \sigma_0[P^J(E)/P^J(\infty)]P^J(\infty), \quad \sigma_0 = 4\pi e^2 a^5/k, \tag{18}$$

where σ_0 is the high-energy small-Z limit of the K-shell cross section, and $P^J(\infty)$ gives the Z dependence of the high-energy limit, J indexing the quantum numbers of the bound state. In a similar manner we write the bremsstrahlung cross section at the tip as

$$\begin{aligned}\sigma_B{}^J &\equiv \sigma_1 B^J(E) = \sigma_1[B^J(E)/B^J(\infty)]B^J(\infty) \\ &\approx \sigma_1[P^J(E)/P^J(\infty)]B^J(\infty), \\ \sigma_1 &= (4\pi e^2 a^3/k)[(E-1)/(E+1)], \\ \sigma_B &\equiv \sum \sigma_B{}^J \approx \sigma_1[P(E)/P(\infty)]B(\infty), \\ B(\infty) &= \sum B^J(\infty),\end{aligned} \tag{19}$$

where σ_1 is the high-energy small-Z limit and we have used (1) the fact that the constant of proportionality which approximately connects photoeffect and bremsstrahlung is independent of energy, and (2) the assumption that all shells have the same energy dependence. Then, using only s and p states, we may use the results of the previous section to write

$$\begin{aligned}B(\infty) &\approx P^{1s}(\infty) + (32/3)[P^{2p\frac{1}{2}}(\infty) + P^{2p\frac{3}{2}}(\infty)] \\ &\approx 8P^{2s}(\infty) + (32/3)[P^{2p\frac{1}{2}}(\infty) + P^{2p\frac{3}{2}}(\infty)].\end{aligned} \tag{20}$$

It should be realized that (20) does *not* represent a consistent expansion in $a \equiv Ze^2$: It is correct to use P^{1s} or P^{2s} only neglecting $O(a^2)$, whereas the P^{2p} terms are $O(a^2)$. For the photoeffect it was established in I and II that the neglected terms of $O(a^2)$ are small once the cross section is written in the form (1), and we are assuming the same will be true for bremsstrahlung. It is of course possible to use the methods of I to calculate the $O(a^2)$ terms in bremsstrahlung, and see whether they tend to cancel the P^{2p} terms. However this does not appear too useful, since, as in the photoeffect, one does not know how good a representation various forms of power series give until an exact numerical calculation is available.

The predictions for $B(\infty)$ which follow from Eq. (20) are given in Table I. The proper choice of P's is somewhat ambiguous, and results are given for two cases. (1) Exact numerical photoeffect results are used. (2) Cross sections are taken from the analytic expressions, such as Eq. (1) for s states. The difference between these two cases are terms of $O(a^2)$, which it perhaps may be argued are particular to the photoeffect and not to be included for bremsstrahlung. In any event, a comparison of the two gives some idea as to the order of magnitude to be expected from such terms; in the remaining discussion the predictions of case (2) will be used. Table I also gives the separate contributions of s and p states. The p states are a 5% effect in Fe, and in Pb they are more than a third of the total. Hence in heavy elements the contribution from d states and higher is not necessarily negligible. If we estimate their magnitude by assuming the contributions from $(s_{\frac{1}{2}}, p_{\frac{1}{2}}, d_{\frac{3}{2}}, \cdots)$ and from $(s_{\frac{1}{2}}, p_{\frac{3}{2}}, d_{\frac{5}{2}}, \cdots)$ form geometrical series (as is true for the Z dependence for small Z), then for $a<0.4$ the higher states are unimportant, while for Pb they may contribute 20% of the total. Such effects are of the same order of magnitude as the $O(a^2)$ effects which we are also neglecting. It is clear that we have no reason to expect an accuracy of better than 20–30% in bremsstrahlung predictions for heavy elements.

Experiments on the high-frequency limit of bremsstrahlung were performed by FKM, and analyzed with the theory they developed. The results of the present paper modify that theory in two ways, both only for heavy elements. (1) Nagasaka's cross section,[4] which they used for photoeffect above 1 Mev, is replaced by the extrapolation of I, somewhat reducing the result at 4.5 Mev and appreciably decreasing it at 15.1 Mev. (2) The inclusion of p-state bremsstrahlung increases the prediction at all energies—again using the assumption that the s and p energy-dependences are similar. Hence FKM's comparison of theory and experiment for Al, which gave good agreement, is not affected. In heavy elements we must ask to how low an energy we are willing to apply our formalism. Of course our proofs are not valid for energies of the order of the binding energies.[17] At somewhat higher energies it is necessary to specify that we are comparing bremsstrahlung and photoeffect at the same *photon* energy and assume the difference in electron energies is small. This is important, since for low energies the photoeffect varies rapidly in

TABLE I. High-frequency limit of bremsstrahlung. Contributions to $B(\infty)$ from the s and p states are given and summed. When there are two lines for a given value of a, the first uses numerical photoeffect results and the second analytic forms neglecting $O(a^2)$.

a	B^s	$B^{p\frac{1}{2}}$	$B^{p\frac{3}{2}}$	B^{s+p}
0.1	0.699	0.002	0.006	0.707
0.2	0.514	0.008	0.017	0.539
0.3	0.394	0.018	0.027	0.439
	0.391	0.017	0.029	0.437
0.4	0.314	0.033	0.036	0.383
	0.306	0.030	0.039	0.375
0.5	0.260	0.056	0.042	0.358
	0.246	0.047	0.045	0.338
0.6	0.222	0.090	0.044	0.356
	0.200	0.073	0.048	0.321
0.7	0.196	0.143	0.046	0.385
	0.166	0.110	0.050	0.326

[17] However, if we simply correct the FKM value for Au at 0.05 Mev by including the p wave contribution it is brought into good agreement with experiment.

TABLE II. High-frequency limit of bremsstrahlung. Comparison with theory and experiment of FKM for Au. $kZ^{-2}\sigma_B$ is plotted in mb.

Electron kinetic energy (Mev)	Theory FKM	Theory present work	Experiment FKM
0.50	3.4	5.0	5.2 $\pm$2.0
1.0	1.8	3.0	1.7 $\pm$0.7
4.5	2.0	1.9	1.8 $\pm$0.3
15.1	1.77	1.6	1.47$\pm$0.44

the photon energy range appropriate to the tip region of bremsstrahlung (<200 kev). Results for Au are compared with experiment in Table II. At 4.5 Mev and 15.1 Mev our two modifications of the FKM theory mainly cancel; the agreement with experiment is slightly improved, but both sets of predictions are well within experimental errors. At 1.0 Mev our value is higher, since we include p states and the FKM value did not depend on Nagasaka's work. The agreement with experiment is worsened, but is still within the combined experimental and theoretical uncertainties. For 0.5 Mev the inclusion of p states gives better agreement with experiment, although the quoted experimental errors are very large.

The theory as developed in this paper also gives predictions for the shape of the spectrum in the high-frequency region, perhaps for some 200 kev below the tip. If $kZ^{-2}(d\sigma/dk)$ is plotted in this region (as done by FKM) we expect two effects to modify the value which is predicted for the limit. (1) The photoeffect cross section is to be taken for a smaller k and hence is larger. (2) From Eq. (12) the whole result will be increased by $(1-e^{-2\pi\nu})^{-1}$, and the p wave part of it further by $(1-\nu^{-2})$. Both effects cause a plot of $k\sigma_B$ to increase as one goes away from the tip. At high energies the energy-dependence of the photoeffect is k^{-1}, while for low energies more powers of k^{-1} are appropriate. Hence $k\sigma_{\text{photo}}$ is insensitive to small change in k for large k, but increases rapidly with decreasing k for small k. Since the second effect which has been listed is independent of photon energy we expect that the slope of the plots will decrease as incident electron energy increases, and this is indeed the case in the plots of FKM. However more quantitative predictions are not very successful, the agreement with experiment being only fair in Au and poor in Al.

Angular distributions for the photoeffect are not yet well known—the experimental work of Hultberg[18] should be noted. In I the high-energy limit was combined with Gavrila's results to provide an extrapolation formula for the total cross section; Gavrila's angular distribution may be treated in the same fashion and an extrapolation formula obtained for the differential cross section. At the same time this gives the correlations with linearly polarized photons (and unpolarized electrons). In a qualitative way we may apply this result directly to the bremsstrahlung case. The formula will predict that the average angle between photon and electron approaches 0° as the energy increases, but it also predicts that the emission at 0° is zero,[19] apparently contrary to the photoeffect experiments. It is also easy to see that as the energy increases the correlations with photon linear polarization disappear. At much lower energies the comparison of linear polarization in bremsstrahlung and photoeffect has been used successfully by Motz and Placious.[20] Finally, the correlations between longitudinally polarized electrons and circularly polarized photons recently discussed by several authors[21] for photoeffect and one photon pair annihilation will also apply to the high-frequency limit of bremsstahlung, subject to the reversal of both spins.[22]

In summary: By extending the McVoy-Fano theory of the connection between the photoeffect and the high-frequency region of bremsstrahlung, comparison of the two processes has been placed on firmer ground. This is not sufficient for a completely quantitative discussion of bremsstrahlung near the tip, but does provide several predictions which can be checked against experiment. For a better discussion, numerical evaluation of the proper matrix elements is probably required; the magnitudes we have found for s- and p-state contributions provide information for such an endeavor. Finally, just as the photoeffect is one of several processes with identical matrix elements at high energies (I and II), bremsstrahlung is related to other processes, for example, pair creation with a low-energy electron, and a similar analysis may be made for them.

Note added in proof. In recent work at Illinois, Hall, Hanson and Jamnick[23] find for Th $kZ^{-2}\sigma_B=1.71\pm0.30$, in satisfactory agreement with our theory, which predicts about 1.75. Also, Johnson and Mullin[24] have used a modified *SM* function (compare with Nagel, reference 14) and by explicit calculation find that in the high-energy limit the second term of the series in a is $-19\pi a/15$. This is the same as the similar photoeffect result from Eq. (1), as our discussion has led us to predict.

[18] S. Hultberg, Arkiv Fysik **15**, 307 (1959).

[19] M. Gavrila, Nuovo cimento **15**, 691 (1960). *Note added in proof.* It has now been shown by Nagel (reference 14) and by Kolbenstvedt and Olsen (to be published) that there are nonvanishing terms in the forward direction of relative $O(a^2)$.

[20] J. W. Motz and R. C. Placious, Phys. Rev. **112**, 1039 (1958).

[21] H. Banerjee, Nuovo cimento **11**, 220 (1959); U. Fano, K. W. McVoy, and J. R. Albers, Phys. Rev. **116**, 1147 (1959); H. Olsen, Kgl. Norske Videnskab Selskabs Forh. **31**, 11 and 11a (1958).

[22] U. Fano, K. W. McVoy, and J. R. Albers, Phys. Rev. **116**, 1159 (1959).

[23] H. E. Hall, A. O. Hanson, and D. Jamnick (private communications).

[24] W. R. Johnson and C. J. Mullin, Phys. Rev. **119**, 1270 (1960).