Electron Density Fluctuations in a Plasma*

E. E. SALPETER
Newman Laboratory of Nuclear Studies and Center for Radiophysics and Space Research, Cornell University, Ithaca, New York

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We consider the spatial Fourier transform ρ_{ke} for wave vector $\mathbf{k}$ of the charge distribution of the electrons in a plasma with particle density n, electron and ion temperatures T and T_i and Debye length D. We assume the absence of a magnetic field, neglect collisions and assume $nD^3\gg1$. The statistical average of $|\rho_{ke}|^2$ is calculated as a function of $\alpha=1/kD$ assuming complete thermodynamic equilibrium; that component of $|\rho_{ke}|^2$ which keeps in phase with the ion charge density fluctuations is also calculated.

The frequency spectrum of the time-varying function ρ_{ke} is obtained at thermal equilibrium and simplified, assuming the ion mass to be much larger than the electron mass, for general values of α and T/T_i. For small α the main component of the spectrum has the characteristic Doppler broadening shape corresponding to the electron's thermal velocity. For large α we have a component with narrow width corresponding roughly to the ion velocity Doppler spread and very narrow side bands at plus and minus the frequency of electrostatic plasma oscillations.

1. INTRODUCTION

IN the last decade or two many calculations have been carried out on the time development of fluctuations of charge density in an ionized gas under a variety of conditions. For a given volume V containing N electrons and N/Z positive ions of atomic charge Z quantities ρ_{ke} and ρ_{ki} have been introduced, mainly for mathematical convenience, which are essentially the spatial Fourier transforms for wave vector $\mathbf{k}$ of the electron and ion charge densities. More specifically

$$\rho_{ke}(t)=-e\sum_{j=1}^{N}e^{-i\mathbf{k}\cdot\mathbf{r}_j},\quad \rho_{ki}(t)=Ze\sum_{j=1}^{N/Z}e^{-i\mathbf{k}\cdot\mathbf{R}_j},$$
$$\rho_{kt}=\rho_{ke}+\rho_{ki}, \quad (1)$$

where $\mathbf{r}(t)$ and $\mathbf{R}(t)$ are the positions, as a function of time t, of the jth electron and ion, respectively, and $\mathbf{k}$ is a constant wave vector.

The use of backscattering of a radar beam from the ionosphere at great heights or from the exosphere has been proposed recently by Gordon[1] for measuring electron density and temperature at various heights. Radar frequencies of 50 to 1000 Mc/sec are used which are very large compared with the electron plasma frequency ω_p (of the order of 0.1 to 10 Mc/sec),

$$\omega_p=(4\pi ne^2/m)^{\frac{1}{2}}, \quad (2)$$

where $n\equiv N/V$ is the particle density of the electrons alone, e is the charge (in cgs units) and m the mass of the electron. Standard magneto-ionic theory replaces the electrons by a continuous medium, whose refractive index is close to unity at these high frequencies, and would not lead to reflection or attenuation if the density of the medium is assumed to be smooth and varying slowly. The actual amplitude of radiation scattered through a scattering angle θ from a volume V of ionized gas (dimensions of the order of 1 km) is then simply the sum of the Thomson scattering amplitudes from each of the N electrons in the volume (Thomson scattering from the positive ions is negligible because of their large mass). Since the electrons are highly nonrelativistic, we can neglect retardation effects and the scattering amplitude from each electron contains a phase-factor like those appearing in Eq. (1) with the wave number k given by $k=4\pi\sin(\frac{1}{2}\theta)/\lambda$, where λ is the wavelength of the electromagnetic radiation. The total amplitude of backscatter as a function of time is thus proportional to the quantity $\rho_{ke}(t)$, defined in Eq. (1), for a fixed value of k. For low enough intensity of the radar beam and for radar frequency large compared with ω_p we can neglect altogether the effect of the electromagnetic radiation on the quantity ρ_{ke}.

Let D be the Debye length, defined for the electrons alone,

$$D=(\kappa T/4\pi ne^2)^{\frac{1}{2}}, \quad (3)$$

where n is the electron particle density and T the electron temperature, and let Λ be the dimensionless ratio

$$\Lambda=nD^3\sim(e^2n^{\frac{1}{3}}/\kappa T)^{-\frac{3}{2}}\sim D\kappa T/e^2. \quad (4)$$

We shall only consider cases throughout this paper where $\Lambda\gg1$, i.e., where a sphere of radius equal to the Debye length contains very many electrons and where the Coulomb interaction energy between "nearby" electrons (separations $\sim n^{-\frac{1}{3}}$) is small compared with the thermal energy κT. This inequality certainly holds for densities and temperatures encountered in the ionosphere and exosphere, as well as for many laboratory experiments. We further assume throughout that the gas is ionized enough so that collisions with neutral gas atoms or molecules can be neglected. This assumption is invalid for the lower ionosphere but holds for heights of about 300 km. and higher. For a highly ionized gas with $\Lambda\gg1$ the main collision process for electrons and ions is multiple Coulomb scattering through small angles and the effective mean free path l for appreciable deflections

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[1] W. E. Gordon, Proc. Inst. Radio Engrs. **46**, 1824 (1958).

is[2] of the order of $D\Lambda/\log\Lambda \gg D$. We shall assume that Λ is sufficiently large for the mean free path l to be large compared also with our effective scalelength k^{-1} and we shall neglect collisions altogether. We shall calculate ρ_{ke} in this paper only in the absence of any magnetic field. For the ionospheric applications the neglect of the earth's magnetic field is not justified at the lower frequencies of about 50 Mc/sec and is expected to give a moderately good approximation at radar frequencies of about 400 Mc/sec or higher. In many calculations in this paper we assume complete thermodynamic equilibrium but we shall also discuss some limited derivations from such equilibrium.

In all our calculations the following dimensionless parameter α will be of importance:

$$\alpha = 1/kD = (4\pi ne^2/k^2\kappa D)^{\frac{1}{2}} = \lambda(ne^2/4\pi\kappa T)^{\frac{1}{2}}(\sin\tfrac{1}{2}\theta)^{-1}. \quad (5)$$

In the limit of $\alpha \to 0$ the collective effects of the Coulomb interactions, which become important only over distances as large as the Debye length D or larger, are negligible over distances as small as the scale length k^{-1} and the electrons are randomly distributed in space. In this case we have completely incoherent scattering from each of the electrons, at least at thermodynamic equilibrium, i.e., in evaluating $|\rho_{ke}|^2$ all the cross-terms average to zero and we simply obtain Ne^2 for this quantity. In the ionospheric applications for a radar frequency of 450 Mc/sec, for instance, $\alpha < 1$ at heights of 1000 or 2000 km and higher and this case of $\alpha \to 0$ has been treated in detail by Gordon.[1] We shall carry out calculations in this paper for arbitrary values of α for which the spatial correlations between all the electrons and ions have to be taken into account.

In Sec. 2 we evaluate the time average of the intensity of the electron density fluctuation, $\langle|\rho_{ke}|^2\rangle_{\rm av}$, for arbitrary values of the parameter α and the atomic charge Z of the positive ions at complete thermodynamic equilibrium. We shall calculate these averages from first principles, although they could be obtained more easily using results from the Debye-Hückel theory. Such a calculation does not give the time development of $\rho_{ke}(t)$ or its frequency Fourier transform, but the intensity can be divided into two parts with different characteristic frequency spreads if the ion mass M is very large compared with the electron mass m. The ions move very slowly compared with the electrons and, if we consider the ions fixed, we can evaluate the average correlation of the electron density distribution with that of the ions. Such a calculation will give that part of ρ_{ke} which varies very slowly with time (characteristic of ion thermal velocities). The remaining part of ρ_{ke} varies rapidly with time (characteristic of electron thermal velocities).

In Sec. 3 we derive formulas for $|Q_{ke}(\omega)|^2$ where $Q_{ke}(\omega)$ is the frequency Fourier transform of $\rho_{ke}(t)e^{-\gamma t}$ in the limit of $\gamma \to 0$. This quantity is relevant if a frequency spectrum is observed over a long but finite time period. The calculations are carried out for complete thermodynamic equilibrium except that the electron and ion temperatures T and T_i need not be equal. In Sec. 4 the general results are simplified and approximations evaluated, using the fact that the ion mass M is much larger than the electron mass m. In Sec. 5 a special kind of deviation from equilibrium is discussed where we assume that an external agent suddenly alters the degree of ionization in a nonuniform manner at some time but the medium is allowed to relax to equilibrium after this time.

[2] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, New York, 1956), Chap. 5.

2. SOME TIME AVERAGED INTENSITIES

We consider a volume V containing N electrons and N/Z positive ions of atomic charge Z with N and V extremely large but with electron particle density $n = N/V$ fixed and finite. In this section we assume complete thermodynamic equilibrium at temperature T but, unlike the remaining sections, need to make no assumptions about the collision mean free path being large and evaluate only statistical averages of various quantities. We do have to assume, however, that the dimensionless parameter Λ defined in Eq. (4) is large compared with unity; in an expansion in inverse powers of Λ we shall calculate explicitly only the leading term and give only qualitative estimates of higher order corrections.

Let $\mathbf{k}$ be any wave vector which satisfies periodic boundary conditions in the volume V and, for any given spatial distribution of all the electrons and ions, define complex quantities ρ_{ke}, ρ_{ki} and ρ_{kt} according to Eq. (1). Except for the relation $\rho_{-\mathbf{k}} = \rho_{\mathbf{k}}^*$, different values of $\mathbf{k}$ represent independent modes. We shall write

$$\rho_{ke} = \mu_{ke}e^{-i\delta_{ke}}, \quad \rho_{ki} = \mu_{ki}e^{-i\delta_{ki}}, \quad \rho_{kt} \equiv \rho_{ke} + \rho_{ki} = \mu_{kt}e^{-i\delta_{kt}}, \quad (6)$$

where each μ and δ is real and positive. We shall need the electrostatic potential $\phi(\mathbf{r})$ and electric field $\mathbf{E}(\mathbf{r})$ due to the given distribution of electrons and ions. After carrying out a Fourier transformation of the Coulomb potential $-e/|\mathbf{r}-\mathbf{r}_j|$ due to the jth electron (and of its gradient) and summing over all electrons and ions, we find

$$\begin{aligned} \phi(\mathbf{r}) &= \textstyle\sum_{\mathbf{k}}(8\pi/Vk)\mu_{kt}\cos(\mathbf{k}\cdot\mathbf{r}-\delta_{kt}), \\ \mathbf{E}(\mathbf{r}) &= \textstyle\sum_{\mathbf{k}}\mathbf{k}(8\pi/Vk^2)\mu_{kt}\sin(\mathbf{k}\cdot\mathbf{r}-\delta_{kt}). \end{aligned} \quad (7)$$

where the $\mathbf{k}$ summation is carried only over half of all the possible $\mathbf{k}$ vectors (those with positive z component, say).

We now consider $N-1$ of the electrons and all the ions as fixed, introduce an Nth electron and ask for the statistical ensemble average of $\cos(\mathbf{k}\cdot\mathbf{r}-\delta)$ over all positions $\mathbf{r}$ of this extra electron, $\mathbf{k}$ and δ being fixed. This average is

$$\langle\cos(\mathbf{k}\cdot\mathbf{r}-\delta)\rangle = \int d^3r\cos(\mathbf{k}\cdot\mathbf{r}-\delta)e^{e\phi(\mathbf{r})/\kappa T} \Big/ \int d^3r\, e^{e\phi(\mathbf{r})/\kappa T}, \quad (8)$$

where κ is Boltzmann's constant. As $V\rightarrow\infty$ we could write, no matter what the value of Λ,

$$e^{e\phi(\mathbf{r})/\kappa T}=\prod_{\mathbf{k}}[1+(8\pi e/Vk^2\kappa T)\mu_{kt}\cos(\mathbf{k}\cdot\mathbf{r}-\delta_{kt})], \quad (9)$$

and rewrite this infinite product as an infinite series of terms with successive positive powers of $1/\kappa T$. However, this series will converge rapidly if, and only if, $|e\phi(\mathbf{r})/\kappa T|\ll 1$. If $\Lambda\gg 1$ the Coulomb interaction between "neighboring" particles is weak compared with κT and this inequality is satisfied for all values of $\mathbf{r}$ except those very close to one of the fixed charges. In the integral in Eq. (8) distances away from a fixed charge which are small compared with both the scale-length k^{-1} and the Debye length D are unimportant. At a distance k^{-1} the potential energy of an electron is of order e^2k and the use of an expansion in powers of $1/\kappa T$ will give rapid convergence as long as $(\Lambda\alpha)^{-1}\sim e^2k/\kappa T\ll 1$, as well as $\Lambda\gg 1$. We shall assume that this inequality also holds (it breaks down only for $\alpha\ll\Lambda^{-1}\ll 1$ and for such very small values of α the Coulomb correlations are negligible and the problem trivial in any case). Using the expansion of Eq. (9) in the integrands of Eq. (8) we keep only the zero order term (unity) in the denominator. In the numerator the zero order term gives no contribution, we keep only the terms of first order in $1/\kappa T$ and, since the cosine terms for different values of $\mathbf{k}$ are orthogonal, we obtain

$$\langle\cos(\mathbf{k}\cdot\mathbf{r}-\delta)\rangle=(4\pi e/Vk^2\kappa T)\mu_{kt}\cos(\delta-\delta_{kt}), \quad (10)$$

where the μ_{kt} and δ_{kt} refer to $N-1$ electrons and N/Z ions. A similar calculation for $\langle\cos(\mathbf{k}\cdot\mathbf{R}-\delta)\rangle$, where $\mathbf{R}$ is the position of an additional positive ion, simply gives $(-Z)$ times the expression in Eq. (10).

The use of Eq. (1) gives a double sum over indices j, l for a quantity like $\mu_{ke}{}^2=\rho_{ke}{}^*\rho_{ke}$ for all N electrons. In this double sum we separate out the terms with $j=l$ for which the phase factors cancel. On taking a statistical ensemble average the various terms in the remaining double summations give identical results and, replacing $N(N-1)$ by N^2 and dropping the subscript k, we obtain

$$\begin{aligned}\langle\mu_e{}^2\rangle&=Ne^2[1+N\langle\cos\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_l)\rangle],\\ \langle\mu_i{}^2\rangle&=Ne^2[Z+N\langle\cos\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_l)\rangle], \quad (11)\\ \langle\mu_t{}^2\rangle&=\langle\mu_e{}^2\rangle+\langle\mu_i{}^2\rangle-2N^2e^2\langle\cos\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{R}_l)\rangle,\end{aligned}$$

where $\mathbf{r}_j$ and $\mathbf{R}_j$ are the positions of the jth electron and ion, respectively.

The cosine expectation values are the Fourier transforms of the two particle correlation functions, which would vanish for randomly distributed electrons and ions, and we evaluate them as follows. We keep ρ_t for all particles except electrons j and l fixed, keep $\mathbf{r}_j$ fixed at first and average over $\mathbf{r}_l$. This average can be obtained in analogy with the derivation of Eq. (10) but in Eq. (9) we have to add the term $[1-(8\pi e^2/Vk^2\kappa T)\cos\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_l)]$, which represents the Coulomb interactions between the two singled out electrons. We have

$$\langle\cos\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_l)\rangle_{\mathbf{r}_j}=(4\pi e/Vk^2\kappa T)[-e+\mu_t\cos(\mathbf{k}\cdot\mathbf{r}_j-\delta_t)],$$

with $\mathbf{r}_j$ fixed. We average next over $\mathbf{r}_j$, still keeping ρ_t fixed, use Eq. (10) and finally average over the remaining particles as well. Using the same procedure on the other cosine terms we find

$$\begin{aligned}\langle\cos\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_l)\rangle&=Z^{-2}\langle\cos\mathbf{k}\cdot(\mathbf{R}_j-\mathbf{R}_l)\rangle\\ &=-Z^{-1}\langle\cos\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{R}_l)\rangle\\ &=(\alpha^2/N)[\alpha^2\langle\mu_t{}^2\rangle(Ne^2)^{-1}-1], \quad (12)\end{aligned}$$

where the dimensionless parameter α is defined in Eq. (5). $\langle\mu_t{}^2\rangle$ refers to $N-2$ electrons but $N-2$ may be replaced by N, since we expect no cancellation of large terms.

Substituting Eq. (12) into Eq. (11) to eliminate $\langle\mu_e{}^2\rangle$ and $\langle\mu_i{}^2\rangle$ yields an explicit expression for $\langle\mu_t{}^2\rangle$ and substituting this expression back into Eq. (12) gives explicit expressions for the cosine expectation values. Using again Eq. (11) we finally obtain the desired expressions

$$\begin{aligned}\langle\mu_e{}^2\rangle&=Ne^2(1+Z\alpha^2)[1+(Z+1)\alpha^2]^{-1},\\ \langle\mu_i{}^2\rangle&=ZNe^2(1+\alpha^2)[1+(Z+1)\alpha^2]^{-1}, \quad (13)\\ \langle\mu_t{}^2\rangle&=(Z+1)Ne^2[1+(Z+1)\alpha^2]^{-1},\end{aligned}$$

$$N\langle\cos\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_l)\rangle=-\alpha^2[1+(Z+1)\alpha^2]^{-1}. \quad (14)$$

Equation (14) merely represents a rederivation of the Fourier transform of the well-known Debye-Hückel two-particle correlation function for two electrons. If we had taken over this expression (and similar ones) from the Debye-Hückel theory, substitution into Eq. (11) would have given the desired results without requiring any other formulas of the present section.

For $\alpha\ll 1$ our results in Eq. (13) reduce to those for randomly distributed particles, as they should. For $\alpha\gg 1$ the total charge density fluctuations $\langle\mu_t{}^2\rangle$ are smaller than those for random distribution by a factor $\alpha^{-2}\propto k^2$. The electron charge density fluctuations $\langle\mu_e{}^2\rangle$, however, are reduced only by a factor $Z/(Z+1)$ even in the limit of $\alpha\rightarrow\infty$. Pines and Bohm[3] have carried out calculations for $\langle\mu_e{}^2\rangle$ for a model in which the positive charges are uniformly and continuously distributed. Their results can be obtained from our more general ones in Eq. (13) by making the formal substitution $Z=0$ in which case $\langle\mu_e{}^2\rangle=\langle\mu_t{}^2\rangle$ and $\langle\mu_i{}^2\rangle=0$. The expression for $\langle\mu_e{}^2\rangle$ for $Z=1$, the case of greatest interest for the ionospheric application, has also been derived by different methods by Fejer[4] and by Renau.[5] Kahn[6] has derived an expression for $\langle\mu_e{}^2\rangle$ for general values of Z and α which agrees with ours in the two limiting cases $\alpha\ll 1$ and $\alpha\gg 1$ but appears to be incorrect[7] for general values of α.

[3] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).
[4] J. A. Fejer, Can. J. Phys. (to be published).
[5] J. Renau (unpublished work).
[6] F. D. Kahn, Astrophys. J. **129**, 205 (1959).
[7] The fallacy in Kahn's derivation appears to lie in the use made of his Eq. (17): In this equation, two expressions for $\langle\mu_i{}^2\rangle$ occur,

We finally evaluate in a similar manner another statistical average which forms only part of $\langle\mu_e^2\rangle$. We first consider the positions of the ions, and hence $\rho_i=\mu_i e^{-i\delta_i}$, as fixed and ask for the average of the component of the electron quantity ρ_e which is in phase with the constant ρ_i. We first rewrite the expression in Eq. (10) for the average over the Nth electron (with $N-1$ electrons, as well as the ions, fixed), using the definitions in Eq. (6)

$$\langle\cos(\mathbf{k}\cdot\mathbf{r}-\delta)\rangle = (4\pi e/Vk^2\kappa T)[\mu_e\cos(\delta-\delta_e)+\mu_i\cos(\delta-\delta_i)].$$

We have $\mu_e\cos(\delta_e-\delta_i)=-e\sum_{j=1}^{N}\cos(\mathbf{k}\cdot\mathbf{r}_j-\delta_i)$ and for the jth term in this sum we first average over the jth electron, keeping the remaining electrons as well as ions fixed, and use the equation above. We average next over the remaining $N-1$ electrons, still keeping the ions fixed, and add the identical N terms $j=1$ to N. This gives

$$\begin{aligned}\langle\mu_e\cos(\delta_e-\delta_i)\rangle_{\rho_i} &= -\alpha^2[\langle\mu_e\cos(\delta_e-\delta_i)\rangle_{\rho_i}+\mu_i] \\ &= -\alpha^2(1+\alpha^2)^{-1}\mu_i,\end{aligned} \tag{15}$$

where the subscript ρ_i indicates that this quantity is kept fixed. We finally square the expression in Eq. (15) average over the positions of all the ions as well and use the explicit expression in Eq. (13) for $\langle\mu_i^2\rangle$ to obtain the desired result

$$\begin{aligned}\Theta_{ei}&\equiv\langle\langle\mu_e\cos(\delta_e-\delta_i)\rangle^2_{\rho_i}\rangle \\ &= ZNe^2\alpha^4(1+\alpha^2)^{-1}[1+(Z+1)\alpha^2]^{-1}, \qquad (16)\\ \Theta_{ee}&\equiv\langle\mu_c^2\rangle-\Theta_{ei}=Ne^2(1+\alpha^2)^{-1}.\end{aligned}$$

If the ion mass M is large compared with the electron mass m, the expressions in Eq. (16) have the following physical significance. The ions move slowly so that ρ_i varies slowly with time and the frequency Fourier transform of μ_i^2 has a narrow spread and so does the Fourier transform of the square of that part of ρ_e which remains in phase with ρ_i. The full Fourier transform of $\mu_e^2(t)$ (which we shall analyze in detail in the next section) thus contains one part, with a narrow frequency spread, whose integrated intensity is given by Θ_{ei}. The remaining part represents the electron density fluctuations which are not correlated with the ions, has a wide frequency spread characteristic of electron thermal velocity Doppler broadening and integrated intensity given by Θ_{ee}. For $\alpha\ll1$ the dominant part is Θ_{ee} and the integrated intensity of the narrow part is small, $\Theta_{ee}\approx Ne^2$ and $\Theta_{ei}\approx ZNe^2\alpha^4$. For $\alpha\gg1$ the dominant part is Θ_{ei} and $\Theta_{ee}\approx\langle\mu_i^2\rangle\approx Ne^2\alpha^{-2}\ll\Theta_{ei}$.

In deriving Eq. (10) from Eq. (8) for a particular wave-mode $\mathbf{k}$ we had carried only the leading term in an expansion in powers of $1/\kappa T$ and thereby omitted all terms which involve ρ_{qt} for any other wave-mode $\mathbf{q}\neq\mathbf{k}$. The approximation made thereby is equivalent to the so-called "random phase approximation" which neglects correlations between different wave-modes. Some deviations from this approximation could also be calculated with methods similar to those of the present section. For instance, with ρ_{ke} and ρ_{ki} fixed, the average of $\mu_{qt}\mu_{q't}\cos(\delta-\delta_{qt}-\delta_{q't})$ for $\mathbf{q}+\mathbf{q}'=\mathbf{k}$ could be evaluated. Using such expressions and keeping terms of second power in $1/\kappa T$ in Eqs. (8) and (9), corrections to Eqs. (10) and (13) could then be obtained. For $\alpha\ll1$ the leading correction to Eq. (10) is probably of relative order $e^2k/\kappa T\sim(\Lambda\alpha)^{-1}$, the correction to $\langle\mu_e^2\rangle$ of relative order α/Λ where α and Λ are defined in Eqs. (4) and (5). For $\alpha\gg1$ the leading correction both to Eq. (10) and to $\langle\mu_e^2\rangle$ is probably of relative order $(\Lambda\alpha^3)^{-1}$.

3. THE FREQUENCY SPECTRUM

We have so far evaluated only the root mean square average of the quantity $\rho_e(t)$ defined in Eq. (1) and now wish to calculate its time-dependence, or rather its frequency Fourier transform. In this section we neglect collisions entirely (mean free path much larger than both k^{-1} and D) and assume that the only forces acting are those of the electric field $\mathbf{E}(\mathbf{r})$, given in Eq. (7), due to the charge density fluctuations themselves. We again assume that $\Lambda\gg1$ and also that $nk^{-3}\gg1$. In this case the use of a Boltzmann equation for a Boltzmann distribution function $f(\mathbf{r},\mathbf{v},t)$ is in general justified. Such an equation was used by Bhatnagar, Gross and Krook[8] and the spatial and frequency Fourier transforms of the distribution function f evaluated. We shall use a method similar to that of BGK but we shall have to take account of the discrete nature of the electrons more explicitly since we wish to retain also terms in an expression for $\rho_e^2(t)$ which are proportional to only N, the number of electrons in the volume V, rather than N^2.

We consider a fixed value of the wave vector $\mathbf{k}$, take its direction as the z axis, call the z component of velocity v and shall omit the subscript k in ρ_{ke}, etc. We define a quantity $\sigma_{ev}(t)$ by

$$dv\sigma_{ev}(t)=-e\sum_j{}^{(v)}\,e^{-ikz_j},\quad \rho_e(t)=\sum_v dv\sigma_{ev}(t), \tag{17}$$

where $\sum^{(v)}$ denotes summation over all electrons whose velocity $\dot{z}_j=v_j$ lies between v and $v+dv$. Our quantity $\sigma_{ev}(t)$ is essentially $-e$ times the spatial Fourier transform for wave vector $\mathbf{k}$ of the Boltzmann distribution function $j(\mathbf{r},\mathbf{v},t)$. The Boltzmann equation reads

$$\partial\sigma_{ev}/\partial t+ikv\sigma_{ev}=(e^2/m)\int d^3r\,E_z(\mathbf{r},t)[\partial f(\mathbf{r},\mathbf{v},t)/\partial v]e^{-ikz},$$

where the right-hand side represents the contribution from those electrons whose velocity was below v previously but passed into the $v\to v+dv$ bin due to the acceleration of the electric field E_z (minus those that have passed beyond $v+dv$); E_z is given by Eq. (7). We assume $\Lambda=nD^3\gg1$, $nk^{-3}\gg1$ and the absence of any

one for N electrons and one for $N+1$, which should be taken at constant density N/V (not at constant volume V). Since second differences occur in subsequent equations, the use of a constant volume V is not justified.

[8] P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954) (hereafter referred to as BGK).

large scale macroscopic deviations from thermal equilibrium and replace f in the right-hand side of the Boltzmann equation by the equilibrium distribution function

$$f(\mathbf{r},\mathbf{v},t)\rightarrow nF_e(v);$$
$$F_e(v)=(2\kappa T/m\pi)^{\frac{1}{2}}\exp(-mv^2/2\kappa T), \quad (18)$$

where $n=N/V$ is the electron particle density. This gives the so-called Vlasov equation,

$$\partial\sigma_{ev}/\partial t+ikv\sigma_{ev}=i(4\pi e^2n/\kappa T)(v/k)\rho_t F_e(v), \quad (19)$$

where ρ_t is ρ_{kt} for our fixed value of $\mathbf{k}$. This substitution is equivalent to the "random phase approximation" since we have neglected fluctuations in $f(\mathbf{r},\mathbf{v},t)$ with wave vector $\mathbf{q}$ together with components for $\mathbf{E}$ in Eq. (7) with wave vector $\mathbf{k}-\mathbf{q}$.

We define next the frequency Fourier transforms, or rather Laplace transforms, of the time-dependent quantities $\sigma_{ev}(t)$ and $\rho_e(t)$,

$$q_{ev}(\omega)=\int_0^\infty dt\,\sigma_{ev}(t)e^{-(i\omega+\gamma)t},$$
$$Q_e(\omega)=\int_0^\infty dt\,\rho_e(t)e^{-(i\omega+\gamma)t}, \quad (20)$$

where γ is a real, positive, infinitesimally small constant. In a radar experiment, where the frequency spectrum of $|\rho_e(t)|^2$ is obtained during a large but finite time interval $(2\gamma)^{-1}$, a quantity essentially like $|Q_e(\omega)|^2$ is measured and we also have for the time average of $|\rho_e|^2$,

$$\langle|\rho_e(t)|^2\rangle\approx 2\gamma\int_0^\infty|\rho_e(t)|^2e^{-2\gamma t}dt$$
$$=(\gamma/\pi)\int_{-\infty}^\infty|Q_e(\omega)|^2d\omega. \quad (21)$$

Using the identity

$$\int_0^\infty\dot\sigma_{ev}(t)e^{-(i\omega+\gamma)t}dt=-\sigma_{ev}(0)+(i\omega+\gamma)q_{ev}(\omega),$$

we derive from Eq. (19) the relation

$$q_{ev}(\omega)=(\omega+kv-i\gamma)^{-1}[-i\sigma_{ev}(0)$$
$$+(4\pi e^2n/\kappa T)(v/k)F_e(v)Q_t(\omega)], \quad (22)$$

where σ_{tv}, ρ_t, q_{tv}, and Q_t are defined by equations analogous to Eqs. (17) and (20) but with all the charges, electrons and positive ions of charge Ze, included. For the ions alone one obtains an equation similar to Eq. (22) with ne^2 replaced by Zne^2 and with a distribution function $F_i(v)$ occurring which is defined as in Eq. (18) but with the ion mass M replacing the electron mass m. In this expression we also allow the ion temperature T_i to differ from the electron temperature T. For the electron and ions combined we then find

$$q_{tv}(\omega)=(\omega+kv-i\gamma)^{-1}[-i\sigma_{tv}(0)$$
$$+(4\pi ne^2/k\kappa T)(F_e+ZTT_i^{-1}F_i)Q_t(\omega)]. \quad (23)$$

Summing Eq. (23) over all velocity groups v and using the fact that $Q_t=\sum_v dvq_{tv}$ we obtain an explicit expression for $Q_t(\omega)$ in terms of the quantities $\sigma_{tv}(0)$ at the initial time $t=0$. The terms involving $F_e(v)$ and $F_i(v)$ are smoothly varying functions of v and we can replace the summation over v by an integration. In the term involving $\sigma_{tv}(0)$, however, we must be careful to preserve the discreteness of the summation and of expressions like Eq. (17) and write this term as a summation over individual electrons and ions. This gives

$$Q_t(\omega)=ie[1-G_e(\omega)-G_i(\omega)]^{-1}$$
$$\times\{\sum_{j=1}^N(\omega+kv-i\gamma)^{-1}e^{-ikz_j}$$
$$-Z\sum_{j=1}^{N/Z}(\omega+kv-i\gamma)^{-1}e^{-ikZ_j}\}, \quad (24)$$

where

$$G_e(\omega)=\int_{-\infty}^\infty(4\pi e^2nv/k\kappa T)(\omega+kv-i\gamma)^{-1}F_e(v)dv,$$
$$G_i(\omega)=\int_{-\infty}^\infty(4\pi Ze^2nv/k\kappa T_i)(\omega+kv-i\gamma)^{-1}F_i(v)dv, \quad (25)$$

and z_j, Z_j denote the position of the jth electron or ion, respectively, at time $t=0$ (and v_j is the corresponding velocity). After summing Eq. (22) over velocity v and making use of Eq. (25), we also find

$$Q_e(\omega)=ie\left[\frac{1-G_i}{1-G_e-G_i}\sum_{j=1}^N\frac{e^{-ikz_j}}{\omega+kv_j-i\gamma}\right.$$
$$\left.-Z\frac{G_e}{1-G_e-G_i}\sum_{j=1}^{N/Z}\frac{e^{-ikZ_j}}{\omega+kv_j-i\gamma}\right]. \quad (26)$$

Equation (26) expresses $Q_e(\omega)$ explicitly in terms of the positions and velocities of all the electrons and ions at the initial time $t=0$. If we were to put $Z=0$ and replace the summation in Eq. (26) by an integral over $\sigma_{ev}(0)dv$ we essentially would obtain Eq. (44) of BGK and we shall return to such an equation in Sec. 5. At the moment, however, we want an expression for $|Q_e(\omega)|^2$ under conditions of thermal equilibrium (except that T_i may differ from T) and therefore take the modulus squared of the right-hand side of Eq. (26) and average over initial conditions. In the double summation over particles j and l we separate out the terms with $j=l$ and obtain

$$\left(\left|\sum_{j=1}^N\frac{e^{-ikz_j}}{\omega+kv_j-i\gamma}\right|^2\right)$$
$$=N\int_{-\infty}^\infty\frac{F_e(v)dv}{(\omega+kv)^2+\gamma^2}+N(N-1)$$
$$\times\int\int\frac{F_e(v_j)F_e(v_l)\langle e^{ik(z_l-z_j)}\rangle_{v_j,\,v_l}dv_jdv_l}{(\omega+kv_j-i\gamma)(\omega+kv_l-i\gamma)}. \quad (27)$$

The quantity $\langle\ \rangle_{vj,vl}$ indicates an average over the positions of the two electrons with their velocities kept constant and is of the same order of magnitude as the expression in Eq. (14). With the density fixed, the second term on the right-hand side of Eq. (27) is then proportional to N, just as the first term, but as $\gamma\to 0$ the second term tends to a constant limit whereas the first term is proportional to γ^{-1}. Keeping only terms of order γ^{-1} we can neglect the second term in Eq. (27) and evaluate the integral in the first term and obtain from Eq. (26)

$$(\gamma/\pi Ne^2)\langle|Q_e(\omega)|^2\rangle = |1-G_e-G_i|^{-2}k^{-1}[|1-G_i|^2F_e(-\omega/k) +Z|G_e|^2F_i(-\omega/k)], \quad (28)$$

where G is defined in Eq. (25) and F in Eq. (18).

4. RESULTS

Equation (28) is the essential result in its most general form. For $Z=T/T_i=1$ this result has also been obtained by Dougherty and Farley[9] and, for $\alpha\gg 1$, by Fejer.[4] Using Eqs. (54) and (56) of BGK the expressions for G in Eq. (25) can be expressed in terms of tabulated functions,

$$G_e(\omega)=-\alpha^2[1-f(x)+i(\pi)^{\frac{1}{2}}x\exp(-x^2)], \quad x=\omega/\omega_e,\quad \omega_e=(2k^2\kappa T/m)^{\frac{1}{2}},$$
$$G_i(\omega)=-(ZT\alpha^2/T_i)[1-f(y)+i(\pi)^{\frac{1}{2}}y\exp(-y^2)], \quad y=\omega/\omega_i,\quad \omega_i=(2k^2\kappa T_i/M)^{\frac{1}{2}}, \quad (29)$$

$$f(x)=2x\exp(-x^2)\int_0^x \exp(t^2)dt, \quad (30)$$

and the integral in Eq. (30) is tabulated.[10] For $x<1$ the Taylor series (convergent for all x) converges rapidly,

$$f(x)=2x^2\{1-(2/3)x^2+(4/15)x^4 +\cdots(-2x^2)^n[3\cdot 5\cdots(2n+1)]^{-1}+\cdots\}. \quad (31)$$

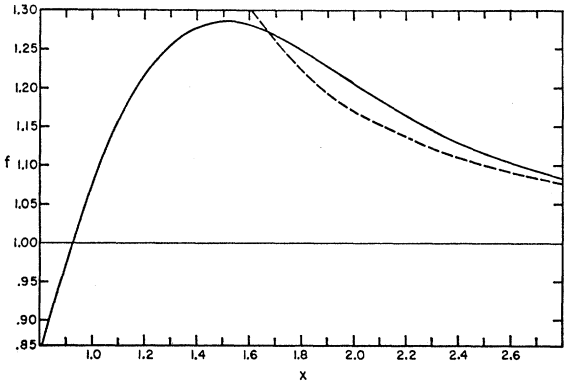

FIG. 1. The function $f(x)$, defined in Eq. (30), plotted against x. The dashed curve denotes the asymptotic expansion carrying only the first two terms on the right-hand side of Eq. (32).

[9] J. P. Dougherty and D. T. Farley (unpublished work).
[10] K. A. Karpov, *Tablitsi funktsii v kompleksnoi oblasti* (Akad. Nauk. U.S.S.R., Moscow, 1958).

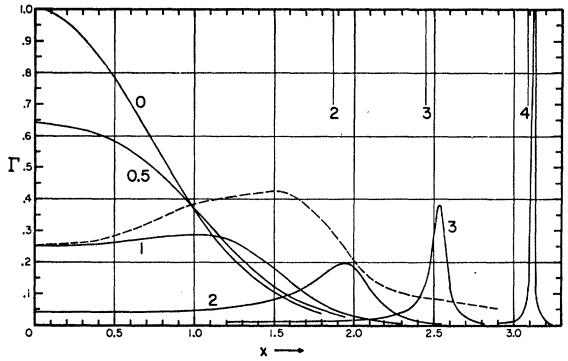

FIG. 2. The function $\Gamma_\alpha(x)$ plotted against x for $\alpha=0$, 0.5, 1, 2, 3 and 4. The vertical lines near the top of the figure denote x_0, Eq. (38), for $\alpha=2$, 3 and 4. The dashed curve denotes $\Gamma_\alpha^{(1)}(x)$, Eq. (41), for $\alpha=1$.

For $x\gg 1$ we have the asymptotic expansion

$$f(x)-1\approx(2x^2)^{-1}[1+(3/2x^2)+(15/4x^4)+\cdots]. \quad (32)$$

For intermediate values of x, the function $f(x)$ is plotted in Fig. 1, as is (dashed curve) the function obtained from the first two terms on the right-hand side of Eq. (32). The velocity distribution functions F in Eq. (28) can be written explicitly as $k^{-1}F_e=\exp(-x^2)/(\pi)^{\frac{1}{2}}\omega_e$ and $k^{-1}F_i=\exp(-y^2)/(\pi)^{\frac{1}{2}}\omega_i$.

The constants ω_e and ω_i represent Doppler spread frequencies characteristic of thermal velocities of the electrons and ions, respectively. For most cases of practical interest $m\ll M$ and $T_i\lesssim T$ so that

$$\eta\equiv\omega_i/\omega_e=(mT_i/MT)^{\frac{1}{2}}\ll 1. \quad (33)$$

In this case a good approximation (except for some special cases discussed below) can be given for Eq. (28), in terms of a single-parameter family of functions Γ of one variable, as follows. The first term in Eq. (28) involves $F_e\propto\exp(-x^2)$ and is of most interest for $|x|\sim 1$. Disregarding the narrow region $|x|=|y|\eta\lesssim\eta$ we have $|y|\gg 1$ for this term and $G_i\approx ZT\alpha^2/2T_iy^2$ can be neglected compared with unity and with G_e. The second term in Eq. (28) involves $F_i\propto\exp(-y^2)$ and is unimportant if $|y|=|x|\eta^{-1}\gtrsim\eta^{-1}$. In the important regions we then have $|x|\ll 1$ and $G_e\approx-\alpha^2[1+i(\pi)^{\frac{1}{2}}x]$. Neglecting also the term $i(\pi)^{\frac{1}{2}}x$, we obtain finally[11]

$$\frac{\gamma}{(\pi)^{\frac{1}{2}}Ne^2}\langle|Q_e(\omega)|^2\rangle d\omega=\Gamma_\alpha(x)\frac{d\omega}{\omega_e}+Z\left(\frac{\alpha^2}{1+\alpha^2}\right)^2\Gamma_\beta(y)\frac{d\omega}{\omega_i}, \quad \beta^2=\frac{ZT\alpha^2}{T_i(1+\alpha^2)}, \quad (34)$$

$$\Gamma_\alpha(x)=\exp(-x^2)\{[1+\alpha^2-\alpha^2f(x)]^2 +\pi\alpha^4x^2\exp(-2x^2)\}^{-1}, \quad (35)$$

where α is defined in Eq. (5) and x, y in Eq. (29).

[11] This result had been stated previously by E. E. Salpeter, J. Geophys. Research **65**, 1851 (1960). In Eq. (1) of this paper $\frac{1}{2}\pi\beta^2\theta^2$ should read $\frac{1}{2}\pi\beta^4\theta^2$.

Each function $\Gamma_\alpha(x)$ is even in x. It is plotted for positive x in Fig. 2 for $\alpha=0$, 0.5, 1, 2, 3, and 4. For $\alpha\ll1$ the function is close to the Gaussian $\Gamma_0(x)=\exp(-x^2)$. This is, of course, the characteristic Doppler spread spectrum for noninteracting electrons. For $\alpha\gg1$, on the other hand, $\Gamma_\alpha(x)$ has a very sharp maximum near $x=\pm x_0$, where x_0 is the solution of the dispersion relation,

$$f(x_0)-1=\alpha^{-2}. \tag{36}$$

For x very near x_0, $\Gamma_\alpha(x)$ can then be approximated by the Lorentzian shape,

$$\Gamma_\alpha(x)\approx\tfrac{1}{2}\alpha^2\exp(-x_0^2)\{4(x-x_0)^2+[\tfrac{1}{2}(\pi)^{\frac{1}{2}}\alpha^4e(-x_0^2)]^2\}^{-1}, \tag{37}$$

for $\alpha\gg1$, where we have used the approximate relation $f-1\approx1/2x^2$ in evaluating coefficients. If Eq. (36) is solved[12] approximately by using the first two terms in the asymptotic expansion, Eq. (32), we obtain

$$x_0^2\approx\tfrac{1}{2}(\alpha^2+3),\quad \omega_0^2\equiv(x_0\omega_e)^2\approx\omega_p^2+3\kappa Tk^2/m. \tag{38}$$

This expression for ω_0^2 is the well-known[13] dispersion relation for longitudinal (electrostatic) plasma oscillations.[14] The Lorentzian shape of Eq. (37) is characteristic of the resonance spectrum for a long-lived oscillation. The width of the spectrum, the expression in the second round bracket in Eq. (37) comes from the so-called Landau[15] (or "drift") damping which is contributed by those few electrons in the tail of the Maxwell distribution whose velocity equals the (very large) phase velocity of the plasma oscillation. As α increases the width decreases and the maximum of $\Gamma_\alpha(x)$ increases sharply, even though the integrated intensity

$$\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}\Gamma_\alpha(x)dx\approx\alpha^{-2} \tag{39}$$

decreases. It should be remembered that, for a practical problem, collisions also contribute a very small width to the spectrum which dominates the Landau damping for very large values of α and that small slow variations of the over-all electron density will vary ω_p and broaden the spectrum. As the discussion in Sec. 2 shows, the integrated intensity in Eq. (39) should not depend on collisions or on the width.

We have discussed so far only the first term, involving $\Gamma_\alpha(x)$, in Eq. (34) which represents the part of the frequency spectrum which is important at large frequencies, of the order of ω_e or of ω_p, and whose integrated intensity is given by Θ_{ee} in Eq. (16). We turn now to the second term, involving $\Gamma_\beta(y)$, in Eq. (34) which is important only for small frequencies and whose integrated intensity is given by Θ_{ei} in Eq. (16). For $\alpha\ll1$ we have $\Theta_{ei}\approx Z\alpha^4\Theta_{ee}\ll\Theta_{ee}$ but the width of the second term is smaller by a factor of $\eta=\omega_i/\omega_e\ll1$ than that of the first and the peak of the second term will dominate for small ω as long as $Z\alpha^4\gg\eta$. For $\alpha\gg1$ the integrated intensity Θ_{ei} of the second term dominates that of the first term. For the case of greatest interest, $Z=T_i/T=1$, we then have $\beta=1$ as $\alpha\to\infty$ and $\Gamma_\beta(y)$ has the almost flat-topped shape plotted in Fig. 2. In this case we have almost complete charge neutrality, the electron density mainly follows that of the ions which can change only slowly and leads to a narrow frequency width of order ω_i. The shape of $\Gamma_1(y)$ differs from the Gaussian for noninteracting ions because electrostatic potentials of order κT are set up by the requirement that the electrons follow the charge density of the (slow) ions.

If the ion temperature T_i is lower than the electron temperature T, as well as $\alpha\gg1$, we have $\beta=(ZT/T_i)^{\frac{1}{2}}\gg1$ and the "ion component" $\Gamma_\beta(y)$ also has a Lorentzian shape like Eq. (37). This sharp "resonance curve" represents the so-called positive-ion oscillations[13,16] whose frequency is the same as that of a plasma oscillation for ficticious particles with the ion charge and mass but with the electron temperature. If the quantity η, defined in Eq. (33), is negligibly small and there are no collisions the width of the frequency spectrum for the positive-ion oscillation is given by $\Gamma_\beta(y)$ no matter how large β. However, if $(\eta T_i/ZT)\gg\exp(-\beta^2/2)$, then the replacement in the derivation of Eq. (34) of $G_e\approx-\alpha^2[1+i(\pi)^{\frac{1}{2}}x]$ by $-\alpha^2$ is not justified and the actual width, although still small, is larger than that given by $\Gamma_\beta(y)$.

To summarize the results so far for the most important case of $T_i=T$, $Z=1$ and $m\ll M$: We have defined the dimensionless parameter α in Eq. (5) and have $\beta=\alpha(1+\alpha^2)^{-\frac{1}{2}}$, $\eta\equiv\omega_i/\omega_e=(m/M)^{\frac{1}{2}}$. The frequency distribution is given by Eq. (34) with x and y defined in Eq. (29). The integral over $d\omega$ of the first term in Eq. (34) is $(\pi)^{\frac{1}{2}}(1+\alpha^2)^{-1}$, that of the second term is $(\pi)^{\frac{1}{2}}\alpha^4(1+\alpha^2)^{-1}(1+2\alpha^2)^{-1}$. The sum of the two integrals is $(\pi)^{\frac{1}{2}}(1+\alpha^2)(1+2\alpha^2)^{-1}$ which decreases only by a factor of 2 as α goes from zero to infinity. The function $\Gamma_\alpha(x)$ is even in x and is plotted for positive x in Fig. 2, has Gaussian shape for $\alpha=0$, is almost flat-topped for $\alpha=1$ and has a maximum at a nonzero value of x for larger values of α. For $\alpha\gtrsim4$, the function $\Gamma_\alpha(x)$ has the Lorentzian shape of Eq. (37) and a very sharp peak of height greater than unity and Eq. (38) is a very good approximation. For $\alpha<(m/M)^{1/8}$, the maximum of the first term in Eq. (34) is larger than that of the second term. This is again the case for large values of α ($\alpha\sim5$ for $M/m\sim10^4$) but it should be remembered that in a practical problem such as the ionosphere application there are other causes, besides Landau damping,

[12] Equation (36) also has a second solution with $x_0\sim1$. This solution is of no interest since $\Gamma_\alpha\sim\alpha^{-4}\ll1$.

[13] See reference 2, Chap. 4.

[14] The relevance of such plasma oscillations to the radar backscatter problem was first pointed out by A. I. Akhiezer, I. G. Prokgoda, and A. G. Sitenko, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 750 (1957) [translation: Soviet Phys.-JETP **6**, 576 (1958)], who used a model in which the ions are replaced by a uniform charge distribution.

[15] L. Landau, J. Phys. U.S.S.R. **10**, 25 (1946).

[16] E. P. Gross and M. Krook, Phys. Rev. **102**, 593 (1956).

broadening the "resonance peak" and the actual maximum will be lower than that given by $\Gamma_\alpha(x)$.

5. SOME DEVIATIONS FROM THERMAL EQUILIBRIUM

We have discussed so far only cases in which complete thermodynamic equilibrium holds (except that the ion and electron temperatures T_i and T may differ). We now consider one very special kind of small deviation from equilibrium. We assume that equilibrium has been established but that at some time $t=0$ the electron and ion charge densities are suddenly both altered in a nonhomogeneous manner (but keeping charge neutrality) by some external agent. This might be accomplished, for instance, by the sudden passage of fast ionizing particles with a patchy spatial distribution. The newly created patchy electron charge distribution is assumed to be small compared with the uniform density n but its spatial Fourier transform $\rho_{ke}(0)=-\rho_{ki}(0)$, Eq. (1), is assumed to be larger than ρ_{ke} for purely thermal density fluctuations. We further assume the absence of collisions and $\Lambda\gg1$ in Eq. (4), will use the random phase approximation and consider only one particular value of the wave vector $\mathbf{k}$ (and drop the subscript k).[17] We assume next that $\sigma_{ev}(0)$ and $\sigma_{iv}(0)$, Eq. (17), are smoothly varying functions of the velocity v. We also assume that after the initial time $t=0$ there are no external forces or disturbances (except for the possibility of another sudden burst of ionization *after* the effects of the original disturbance have died down).

If we assume the disturbance at $t=0$ occurs instantaneously, then the frequency Fourier transform of $\rho_e(t)$ is again given by Eq. (26) but we can replace the summations by integrations over the smoothly varying functions $\sigma_{ev}(0)$ and $\sigma_{iv}(0)$. We specialize further by assuming that $\sigma_{iv}(0)$ is proportional to the Maxwell distribution function for ions at temperature $T=T_i$ and $\sigma_{ev}(0)$ that for electrons at temperature $T\delta^{-2}$ where $\delta\lesssim1$. Carrying out the integrations over dv we find

$$Q_e(\omega)=\frac{a}{1-G_e-G_i}\left\{\frac{1-G_i}{\omega_e}\left[i\exp(-\delta^2x^2)+\frac{f(x\delta)}{x\delta(\pi)^{\frac{1}{2}}}\right]-\frac{G_e}{\omega_i}\left[i\exp(-y^2)+\frac{f(y)}{(\pi)^{\frac{1}{2}}y}\right]\right\}, \quad (40)$$

where a is a constant. Using the fact that $m\ll M$, we can simplify this expression as we did in Sec. 4 to obtain

$$|Q_e(\omega)|^2\propto\Gamma_\alpha^{(1)}(x\delta)\omega_e^{-2}+Z\alpha^4(1+\alpha^2)^{-2}\Gamma_\beta^{(1)}(y)\omega_i^{-2},$$
$$\Gamma_\alpha^{(1)}(x)=[\exp(-x^2)+\exp(x^2)f^2(x)/\pi x^2]\Gamma_\alpha(x). \quad (41)$$

As Eq. (41) shows, the function $\Gamma_\alpha^{(1)}(x)$ decreases much less rapidly for large x than the function $\Gamma_\alpha(x)$, as x^{-2} rather than as $\exp(-x^2)$. This slow falloff is due to our special assumption of a sudden onset of the disturbance which contributes Fourier components of large frequency. If the onset occupies a finite time duration T, as it would in practice, our Eq. (41) breaks down for $\omega>T^{-1}$ and the actual spectral intensity would be lower than in our approximation. Note also that $|Q_e(\omega)|^2$ in Eq. (41) is independent of γ^{-1}, the length of time over which the frequency spectrum is accumulated (as $\gamma\to0$), rather than being proportional to γ^{-1} as is the expression in Eq. (34) for the case of thermodynamic equilibrium. This is due to the fact that we assumed only one single external creation of a disturbance which dies down in a finite time period and very much later times do not contribute to Eq. (41).

For $\alpha\lesssim1$ (and $\delta\sim1$) the peak values of $\Gamma_\alpha^{(1)}$ do not differ very greatly from those of Γ_α. For $\alpha\gg1$ the second term in Eq. (41) contains $\Gamma_1^{(1)}(y)$. The dashed curve in Fig. 2 depicts this function $\Gamma_1^{(1)}(x)$ which is seen to be similar to $\Gamma_1(x)$ except for its longer tail. The first term $\Gamma_\alpha^{(1)}$ in Eq. (41), however behaves rather differently for $\alpha\gg1$ (even with $\delta\sim1$). It has the Lorentzian shape of the expression in square brackets in Eq. (37) but the very small multiplying factor $x_0^2\exp(-x_0^2)$ is missing. The integral of $\Gamma_\alpha^{(1)}$ over x thus *increases* with α roughly as $\exp(\alpha^2/2)$ rather than decreasing as α^{-2}. Physically this means that our assumed sudden onset of the disturbance can excite a plasma oscillation no matter how much larger ω_p is than ω_e, but this oscillation persists for a length of time (the inverse of the Landau damping frequency width) which increases with α as $\exp(\alpha^2/2)$. It should be noted again that, in practice, collisions will put an upper limit to the persistence time of the plasma oscillations and that the excitation of the oscillations would be strongly depressed if the onset time T of the disturbance is large compared with the oscillation period ω_p^{-1}. The assumptions in this section were chosen not so much because they are physically reasonable but because their consequences follow readily from previous work in this paper.

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[17] Because of the absence of collisions our case is quite different from those involving turbulence (with a short mean free path) where large eddies feed small ones and $\rho_{ke}(t)$ also depends on $\rho_{qe}(0)$ with $\mathbf{q}\neq\mathbf{k}$; see, for instance, F. Villars and V. Weisskopf, Proc. Inst. Radio Engrs. **43**, 1232 (1955); R. A. Silverman, J. Appl. Phys. **28**, 506 (1957).