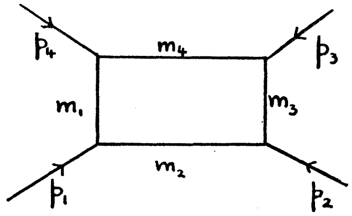

FIG. 1. Fourth-order diagram whose complex singularities are discussed in the text.

where $\alpha>0$, $\beta>0$, and γ is such that the line passes between the branches Γ_5 and Γ_1 of Γ (Fig. 2).

Suppose P is a complex singularity of F lying on C. Then, by (3) and (4), the coordinates of P depend on the m_i^2 only as differences: If we add an amount x to each m_i^2, the resultant $\Gamma(x)$ still passes through P, which lies on a line of the form (7) which is independent of x. As x increases, by (6), the θ_i remain real and tend to zero, and, as $\theta_1+\theta_2+\theta_3+\theta_4\rightarrow 2\pi+0$, the branches $\Gamma_1(x)$ and $\Gamma_5(x)$ join. Thus no line of the form (7)

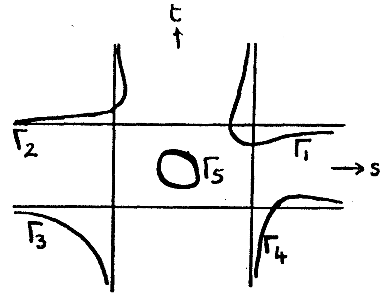

FIG. 2. Curves of real singularities of the fourth-order diagram of Fig. 1 in the s, t plane.

can meet all $\Gamma(x)$ in the same complex point, and so P is nonexistent, and $C_l(q^2)$ cannot have complex singularities in its physical sheet (defined by cuts on the real q^2 axis), except for the above-mentioned "kinematic" singularities.

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Oscillatory Character of Reissner-Nordström Metric for an Ideal Charged Wormhole*

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A transformation is presented to remove coordinate ("pseudo") singularities from metrics of a certain class, a special case of which is the transformation of Kruskal, extending the Schwarzschild metric beyond its pseudosingularity. The transformation is applied to the Reissner-Nordström metric, which describes a concentration of charge and mass in general relativity. On an initial surface this metric shows the same general behavior as the Schwarzschild metric, describing a "wormhole," or bridge, between two asymptotically flat spaces, but with electric flux flowing through the wormhole. It is found that the region of minimum radius, the so-called "throat" of the wormhole, begins to contract, but reaches a minimum and re-expands after a finite proper time, rather than pinching off as in the Schwarzschild-Kruskal case: *the raduis of the throat pulsates periodically in time*, "cushioned" by Maxwell pressure of the electric field through the throat. The motion of charged particles in this metric is investigated, and it is shown that no particle can hit the geometric singularity at $r=0$; (1) quite in general, provided only that the mass of the test particle exceeds the value associated in general relativity with its charge, and (2) in particular when the test particle has no charge at all, but (3) such collisions are *not* avoided when the throat itself is not endowed with any electric flux.

I. INTRODUCTION

MANY of the well-known solutions of Einstein's equations contain apparent singularities which have not been understood until recently. Such singularities seem to be a characteristic feature of solutions of the free-space Einstein and Einstein-Maxwell equations; they can often be prevented from occurring if one grants that near the singularity some field of nonzero rest-mass contributes to the curvature of space. However, it has been suggested[1] that in the domain of free-space gravitation and electromagnetism all "properly closed" spaces must *always* develop an intrinsic geometrical singularity as time evolves. It is therefore interesting to examine some of the known exact solutions of the Einstein-Maxwell equations for singularities, even in cases where they describe spaces which—instead of being closed—are asymptotically flat at great distances.

To do this, one must distinguish between true geometric singularities at which invariants of the Riemann curvature tensor become singular, and "pseudosingularities," which are due to an unfortunate choice of coordinate system. The well-known Schwarzschild solution provides a good example of both types of singularity. In the two most common coordinate systems, some of the metric coefficients vanish or

* Based in part on Chapters 2 and 6–10 of a thesis submitted by the first author in partial fulfillment of the requirements for the B.A. degree at Princeton University.

[1] J. A. Wheeler, Nuovo cimento.

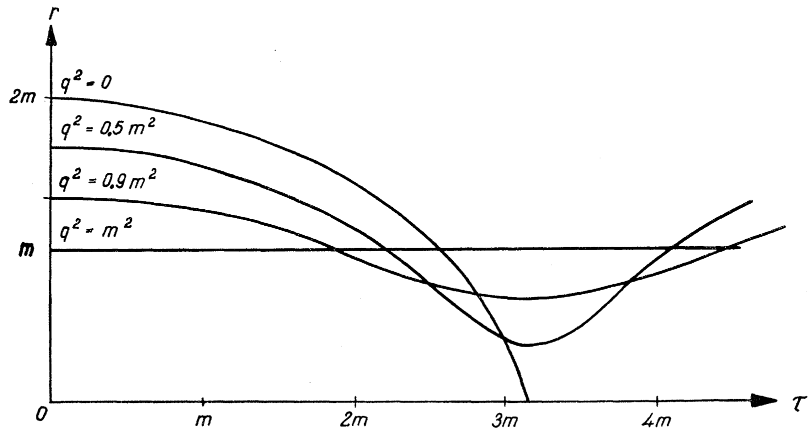

FIG. 1. Radius r of the throat of the Reissner-Nordström wormhole as a function of proper time τ. The radius of the throat, as measured by its circumference $2\pi r$, or by the area $4\pi r^2$ of the minimum sphere, pulsates periodically between maximum radius r_1 and minimum radius r_2, as seen by an observer stationed at the throat in his proper time τ. The curves r vs τ are various cycloids; their period $2\pi m$ is independent of the value of q. For $q=0$ the cycloid reaches the singularity $r=0$ at its cusp. As q increases, the amplitude of the oscillation decreases, so that the observer at the throat does not reach the geometric singularity. In the limit $q^2 \to m^2$ the Maxwell pressure of the electromagnetic field through the wormhole just balances the gravitational forces tending to contract the throat. [Also see B. Bertotti, Phys. Rev. **116**, 1331 (1959)].

become infinite at a finite value of the radial coordinate as well as $r=0$. The former is a pseudosingularity, whereas the latter is a true geometric singularity. Although in physical bodies the presence of matter changes the field equations from their free-space form much before the singularity is reached, these singularities do present a problem from a more fundamental point of view.

It has been shown by a number of authors[2] that one can continue the Schwarzschild solution across the pseudosingularity. Kruskal[3] has found a single coordinate system in which to write the maximum analytic extension of the Schwarzschild solution. Of course, no change of coordinate system can remove the singularity at $r=0$, which can actually be reached by particle and light geodesics.

The geometric singularity at $r=0$ has physical effects whose interpretation is not yet clear. Consider, for example, the space-like surface about which the solution is time-symmetric.[4] This surface has the topology of an Einstein-Rosen bridge[5] between two asymptotically flat spaces. The bridge, like a throat, has a narrowest region. The circumference of this narrowest region at the Schwarzschild time $t=0$ is $2\pi r_{\mathrm{Sch}}=2\pi(2GM/c^2)=4\pi m$ (here m is the mass measured in units of cm). Fix attention on an observer at this throat of the bridge, where r takes on its minimum value. For this observer, r is a time-like coordinate, which decreases as he moves along his geodesic. Thus he finds that the throat, whose radius is also given by r, shrinks continually, until it "pinches off" at $r=0$. One might hope that if the throat contained some electric flux, as in the case of the Reissner-Nordström solution,

$$ds^2=(1-2m/r+q^2/r^2)^{-1}dr^2+r^2d\Omega^2 - (1-2m/r+q^2/r^2)dt^2,$$

the stress of the field in the shrinking throat would keep it from pinching off. In this case the observer stationed at the throat would never reach the singularity at $r=0$ (Fig. 1).

In this paper we shall find a coordinate system for the Reissner-Nordström solution which is analogous to Kruskal's coordinates for the Schwarzschild solution. In Sec. II we shall develop a procedure to eliminate the pseudosingularities, applicable to a general class of static spherically symmetric metrics. In Sec. III we apply this procedure to some well-known solutions, in particular to the Reissner-Nordström solution. Section IV is devoted to a more detailed discussion of the geometry of the Reissner-Nordström metric; finally, Sec. V deals with trajectories of uncharged and charged particles in this metric.

II. TRANSFORMATION TO NONSINGULAR COORDINATES

We consider in this paper line elements which can, in a suitable coordinate system, be written in the form

$$\begin{aligned} ds^2&=\phi^{-1}dr^2+r^2d\Omega^2-\phi dt^2,\\ d\Omega^2&=d\theta^2+\sin^2\theta d\varphi^2, \qquad (1)\\ \phi&=\phi(r). \end{aligned}$$

This metric is spherically symmetric and "static" in the sense that it does not depend on the coordinate t.

[2] G. E. Lemaitre, Ann. soc. Sci. Bruxelles Ser. I **53**, 51 (1933); J. L. Synge, Proc. Roy. Irish Soc. **53**, 83 (1950); D. Finkelstein, Phys. Rev. **110**, 965 (1958); J. Ehlers, dissertation, University of Hamburg, 1958 (unpublished); C. Fronsdal, Phys. Rev. **116**, 778 (1959).

[3] M. D. Kruskal.

[4] For details, see Kruskal's paper, reference 3.

[5] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935). See also C. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957).

Equation (1) does not represent the most general metric of this type, but it includes some well-known metrics (Schwarzschild, Reissner-Nordström, de Sitter) as special cases. We assume that the function $\phi(r)$ has zeros or poles, representing pseudosingularities, which are to be eliminated by a change of coordinates. In the following we confine attention to the neighborhood of *one* of the pseudosingularities and look for a coordinate patch which will continue the metric analytically across that pseudosingularity; the entire analytic extension may consist of several such patches, overlapping in finite regions.

Following Kruskal we shall try to determine a simultaneous transformation of r and t to new coordinates $u(r,t)$ and $v(r,t)$, in terms of which the light cones are lines with slope ± 1. In such coordinates the metric (1) takes the form

$$ds^2 = f^2(u,v)(du^2 - dv^2) + r^2(u,v)d\Omega^2, \tag{2}$$

where $f^2(u,v)$ is to be regular at the pseudosingularity. By comparing (2) and (1) we find the conditions on u and v that this transformation be possible:

$$\begin{aligned} &f^2(u_r^2 - v_r^2) = 1/\phi(r),\\ &f^2(u_t^2 - v_t^2) = -\phi(r),\\ &u_r u_t - v_r v_t = 0. \end{aligned} \tag{3}$$

The subscripts denote ordinary differentiation with respect to the corresponding variable. Equations (3) are to be considered as three differential equations to be solved simultaneously for u, v, and f. First eliminate f to obtain equations for u and v alone:

$$\frac{u_t^2 - v_t^2}{u_r^2 - v_r^2} = \frac{u_t^2[1-(v_t/u_t)^3]}{-v_r^2[1-(u_r/v_r)^2]} = -\frac{u_t^2}{v_r^2} = -\phi^2(r), \tag{4}$$

hence

$$u_t = \phi(r) v_r; \quad v_t = \phi(r) u_r. \tag{5}$$

In terms of a new radial coordinate r^*, defined by $dr^* = \phi^{-1}(r)dr$, these equations take on the simple form

$$u_t = v_{r^*} \quad v_t = u_{r^*}. \tag{6}$$

The general solution of these equations is

$$\begin{aligned} u &= h(r^*+t) + g(r^*-t),\\ v &= h(r^*+t) - g(r^*-t), \end{aligned} \tag{7}$$

where h and g are arbitrary functions of one variable. Below, prime will denote differentiation with respect to this variable. Eq. (3) can now be solved for f^2,

$$f^2 = -\frac{\phi(r)}{u_t^2 - v_t^2} = \frac{\phi(r)}{4h'(r^*+t)g'(r^*-t)}. \tag{8}$$

The variables $\alpha = r^*+t$ and $\beta = r^*-t$ that occur in the argument of h and g are "lightlike" coordinates in r, t space, because from Eq. (1),

$$ds^2 = \phi(dr^{*2} - dt^2) = \phi d\alpha d\beta. \tag{9}$$

In order that f^2 in Eq. (8) be nonsingular, any singularity in the numerator $\phi(r)$ must be cancelled by the denominator, for all t. We must therefore have

$$\begin{aligned} h(r^*+t) &= Ae^{\gamma\cdot(r^*+t)},\\ g(r^*-t) &= Be^{\gamma\cdot(r^*-t)}. \end{aligned} \tag{10}$$

Here A and B are arbitrary scale-factors, and we shall take $A=B$ below; γ is a constant whose value must be chosen such that f^2 of Eq. (8),

$$f^2 = \phi(r)e^{-2\gamma r^*}/4A^2\gamma^2 \tag{11}$$

is regular and positive throughout the coordinate patch. In the general case, expand ϕ as a power series near the pseudosingularity. One finds that a choice of γ which makes f^2 of Eq. (11) nonsingular is possible only whenever the singularity in ϕ is a zero of the first order. To correlate our result with Kruskal's form of the Schwarzschild solution, put

$$\begin{aligned} &\gamma = \tfrac{1}{4}m; \quad A = \tfrac{1}{2};\\ &\phi(r) = 1-(2m/r); \quad r^* = r + 2m\ln(r-2m). \end{aligned}$$

The explicit form of the transformation to the coordinates u, v in which the metric is nonsingular is then obtained by substitution into (7),

$$\begin{aligned} u &= Ae^{\gamma r^*}[e^{\gamma t} + e^{-\gamma t}] = 2Ae^{\gamma r^*}\cosh\gamma t,\\ v &= 2Ae^{\gamma r^*}\sinh\gamma t. \end{aligned} \tag{12}$$

The inverse transformation can, in general, only be given implicitly:

$$\begin{aligned} F(r) &\equiv 4A^2 e^{2\gamma r^*} = u^2 - v^2 \equiv w,\\ t &= (1/2\gamma)\tanh^{-1}[2uv/(u^2-v^2)]. \end{aligned} \tag{13}$$

These transformations differ from Kruskal's transformation for the Schwarzschild solution only in the form of the function $F(r)$. Therefore, the relation between new and old coordinates near the pseudosingularity is qualitatively the same as for the Schwarzschild case near $r=2m$. At this point r and t interchange rôles as spacelike and timelike coordinates, but u remains spacelike and v timelike.

III. EXAMPLES

As we have seen, for the Schwarzschild metric the above transformation is identical with that given by Kruskal. To illustrate the procedure by another well-known case, consider the metric of the de Sitter universe, written in the "static" frame.[6] This metric is of form (1), with

$$\phi = 1 - r^2/R^2, \quad 0 < r < R. \tag{14}$$

[6] See, for example, E. Schrödinger, *Expanding Universes* (Cambridge University Press, Cambridge, 1956).

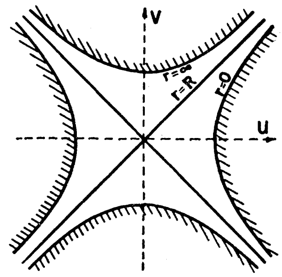

FIG. 2. Kruskal diagram for the de Sitter metric. The scale factor has been chosen $A=B=1$. The static metric was given in the region $u>|v|$, $u^2-v^2<1$; the new coordinates give an analytic extension in the u, v plane except for the shaded region. A typical space-surface $u=0$ is bounded by two points $r=0$, indicating that if the coordinates θ, φ are taken into account this surface has the topology of a three-sphere. An infinite proper time is needed to reach the hyperbola $r=\infty$. Geodesics can be extended to arbitrary length, and the nonsingular metric is complete in this sense. This coordinate system does not show the invariance of the geometry under space translations. Near the point $u=v=0$ the relationship between old and new coordinates is similar to that shown by Schrödinger, reference 7, Fig. 2.

The new measure of radial distance becomes

$$r^*=\int dr/\phi(r)=\frac{R}{2}\ln\frac{R+r}{R-r}. \tag{15}$$

For regularity in the u, v coordinates the numerator of (11),

$$\phi e^{-2\gamma r^*}=\frac{(R+r)^{1-\gamma R}(R-r)^{1+\gamma R}}{R^2} \tag{16}$$

must be everywhere finite in the range of r considered; therefore,

$$1+\gamma R=0,\quad \gamma=-1/R, \tag{17}$$

and the complete transformation, Eqs. (11) and (12), becomes explicitly

$$\begin{gathered} f^2=\left[\frac{R+r}{2A}\right]^2=\frac{R^2A^2}{(A^2+w)^2}, \\ \begin{Bmatrix} u \\ v \end{Bmatrix}=A\left(\frac{R-r}{R+r}\right)^{\frac{1}{2}}\begin{Bmatrix} \cosh \\ \sinh \end{Bmatrix}(-t/R), \end{gathered} \tag{18}$$

with the inverse, from Eq. (13),

$$\begin{gathered} r=R\left(\frac{A^2-w}{A^2+w}\right), \\ t=\frac{1}{2\gamma}\tanh^{-1}\left(\frac{2uv}{u^2-v^2}\right). \end{gathered} \tag{19}$$

The Kruskal $u-v$ diagram for this form of the de Sitter metric is shown in Fig. 2. It extends the metric of the static frame, which is well known to describe only part of the de Sitter space, to a description of the complete de Sitter space. Near the origin $u, v=0$ our transformation coincides with the usual transformation[7] between the static and the de Sitter frame. The main difference between the above and the de Sitter frame is the occurrence of the point $r=\infty$ at a finite value of u, v. However, the proper time needed to reach $r=\infty$ from the origin tends toward infinity:

$$\tau=\int_0^1 f dv=\int^1 R dv/(1-v^2)\to\infty. \tag{20}$$

The transformation for the Reissner-Nordström metric differs from the above only in complexity. The metric is of form (1) with

$$\phi(r)=1-2m/r+q^2/r^2=(r-r_1)(r-r_2)/r^2, \tag{21}$$

which vanishes at two pseudosingularities,

$$r_1=m+(m^2-q^2)^{\frac{1}{2}},\quad r_2=m-(m^2-q^2)^{\frac{1}{2}}. \tag{22}$$

In addition, a true geometric singularity occurs at $r=0$, just as in the Schwarzschild case. The new radial coordinate is

$$r^*=r+\frac{r_1^2}{r_1-r_2}\ln(r-r_1)-\frac{r_2^2}{r_1-r_2}\ln(r-r_2). \tag{23}$$

Again we must make the denominator of (11) nonsingular,

$$\phi e^{-2\gamma r^*}=\frac{1}{r^2}e^{-2\gamma r}(r-r_1)^{1-[2\gamma r_1 2/(r_1-r_2)]}(r-r_2)^{1+[2\gamma r_2 2/(r_1-r_2)]}. \tag{24}$$

From this expression it is clear that we cannot avoid singularities at *both* r_1 and r_2. Therefore, the Reissner-Nordström metric will be described in terms of two coordinate patches, neighborhoods of r_1 and r_2, respectively. Quantities referring to only one of these patches will be denoted by corresponding subscripts 1 or 2, or generally by i or j where $i, j=1$, 2 or 2, 1. The values of γ needed to avoid zeros in Eq. (24) are

$$\gamma_i=(r_i-r_j)/2r_i^2, \tag{25}$$

so that the metric coefficient in the new coordinates becomes

$$f_i^2=\frac{\exp(-2\gamma_i r)}{(2A\gamma_i r)^2}(r-r_j)^{2(1-2\gamma_j m)}. \tag{26}$$

In this case, Eq. (13) cannot be solved explicitly for r as a function of W. As in the Schwarzschild case, we cannot therefore give an *explicit* expression for f^2 in terms of u and v. The transformation of the coordinates is again obtained by substitution into Eq. (12):

$$\begin{Bmatrix} u_i \\ v_i \end{Bmatrix}=2A(r-r_i)^{\frac{1}{2}}(r-r_j)^{-\frac{1}{2}(r_j/r_i)2}e^{\gamma_i r}\begin{Bmatrix} \cosh\gamma_i t \\ \sinh\gamma_i t \end{Bmatrix}. \tag{27}$$

[7] See, for example, R. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, New York, 1934).

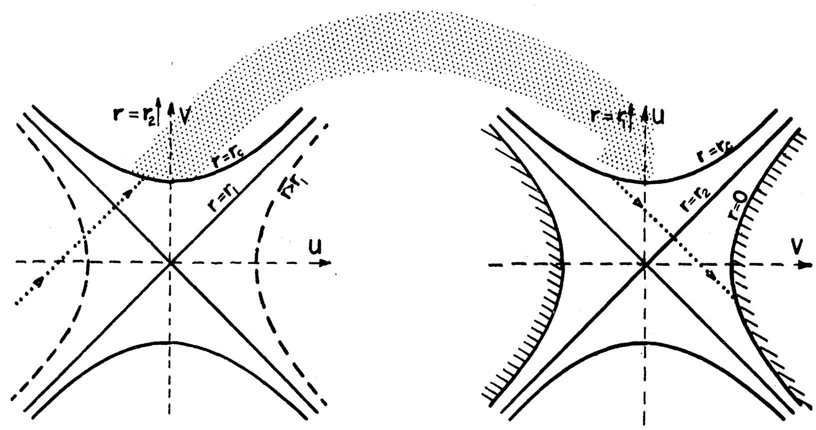

FIG. 3. The two coordinate patches describing the Reissner-Nordström geometry.

IV. THE REISSNER-NORDSTRÖM GEOMETRY

In the discussion of the Reissner-Nordström solution we shall always assume that the mass parameter m exceeds the minimum value associated in general relativity with the charge q, (m and q in units of cm)

$$m^2>q^2. \tag{28}$$

In this case the two roots r_i of Eq. (21) are real, and our analysis yields two coordinate patches; in both the relationship between old coordinates, r, t and new coordinates u, v is qualitatively the same as for the Schwarzschild case. As the parameter r decreases from infinity in the first patch we approach the first pseudosingularity r_1 (see Fig. 2); as r decreases below r_1 it becomes the timelike coordinate. Since the second pseudosingularity is not removed in the first patch, we cannot let r decrease below r_2 in this patch. Actually, the values of the new coordinate v tends to infinity as r approaches r_2, as is seen from Eq. (28).[8] Thus the metric is actually regular for all finite values of u and v in the first patch, but it is not complete since, as we shall see, the proper distance between r_1 and r_2 is finite.

It therefore becomes advisable to change over to the second patch at some larger value of r, r_c, as shown in Fig. 3. Since r continues to be the timelike coordinate, we have drawn u as ordinate and v as abscissa in the second patch. The second patch, similar to the first, has a regular metric for arbitrarily large values of $u>v$; but since we are crossing over at some value r_c, $r_1>r_c>r_2$, we enter this patch somewhere along the hyperbola shown in the figure. It does not matter whether we choose to enter along the lower branch, in the direction of increasing u, or along the upper branch, in the direction of decreasing u. For purposes of easy visualization we have chosen the latter possibility and picked a value of r_c such that the two hyperbolas can be made to coincide by laying one patch on top of the other.[9] Points with identical t coordinates then lie one above the other if the two patches are thus superimposed and "glued together" along the boundary r_c. As r decreases further to r_2 it reverts to a spacelike coordinate, until we reach the reverse geometric singularity at $r=0$.

To visualize the manifold it is natural, in general relativity, to consider it as a succession of spacelike surfaces. The manifold described by the two patches contains two surfaces about which the metric is time-symmetric,[10] $u_1=0$ and $v_2=0$. The geometry of the first surface is already well understood.[11] Like the analogous surface in the Schwarzschild-Kruskal manifold, it represents a "wormhole," or bridge, between two asymptotically flat spaces. The asymptotically flat parts are described by the region of large positive and negative values of u. At $u=0$, r reaches its minimum value r_1 on the surface. The sphere of minimum area $4\pi r^2$, or the "throat" of the wormhole, therefore, is described by $r=r_1$, or $u=0$. This picture agrees exactly with the geometry found by solving the time-symmetric initial value problem of gravitation and electromagnetism. The initial metric obtained from this point of view is just the space-part of the Reissner-Nordström metric, written in isotopic coordinates,

$$ds^2=[(1+m/2\rho)^2-(q/2\rho)^2]^2(d\rho^2+\rho^2d\Omega^2),$$

with

$$r=\rho[(1+m/2\rho)^2-(q/2\rho)^2]. \tag{29}$$

This metric has been discussed in some detail by Misner and Wheeler.[11] For purposes of visualization, it can be imbedded in 4-dimensional space, giving a surface of revolution much like the well-known surface for the Schwarzschild case, obtained by rotating a parabola about a line perpendicular to its axis.[12]

[8] We are indebted for this remark to R. Lindquist.

[9] This requirement, that $4A_1{}^2e^{+2\gamma_1 r^*}=4A_2{}^2e^{2\gamma_2 r^*}$ can be satisfied by choice of A_1, A_2, or r_c.

[10] For definition see, for example, D. R. Brill, Ann. Phys. **7**, 456 (1959).

[11] See, for example, C. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957).

[12] See, for example, Hermann Weyl, *Space—Time—Matter* (Dover Publications, New York, 1952), pp. 259–260; also see L. Flamm, Physik. Z. **17**, 448 (1916). In the Reissner-Nordström case the curve to be rotated in the θ and φ directions is given, in the $R-r$ plane, by

$$R(r)=\int[(1-\phi)/\phi]^{\frac{1}{2}}dr=\int[(2mr-q^2)/(r-r_1)(r-r_2)]^{\frac{1}{2}}dr.$$

The second surface of time-symmetry, $v_2=0$, can be analyzed similarly. It is bounded at two ends by the singularity $r=0$, and has a maximum radius at $r=r_2$. The geometry of this surface is also described by the metric (29), where ρ now takes on those negative values which make r positive,

$$\rho_0=\tfrac{1}{2}(q-m)>\rho>-\tfrac{1}{2}(q+m)=\rho_0'.$$

A point whose ρ coordinate differs by some small number ϵ from one of these limits ρ_0 has the r-coordinate, given by Eq. (29), $r=(q/\rho_0)\epsilon$, and geodesic distance from the singularity $s=\int ds=(q/\rho_0^2)\epsilon^2$. Hence, the area of the sphere drawn at that distance is $A=4\pi r^2=(4\pi q^2/\rho_0^2)\epsilon^2=4\pi qs$ (for a regular point we should have $A=4\pi s^2$). This property characterizes the singularity at $r=0$ on the surface $v_2=0$.

By symmetry it is clear that the line $u=0$ in the first patch, and its continuation $v=0$ in the second patch is a geodesic, the worldline of an observer who initially, and permanently, is situated at the "throat" of the "wormhole." The radius of the wormhole as defined from the surface area of the extremal sphere, is simply given by the r value at the observer. As in the pure Schwarzschild case, the observer sees this radius begin to decrease from its maximum value r_1 as time proceeds from the initial point, $v_1=0$: the throat begins to shrink. However, for the Reissner-Nordström case it does *not* pinch off, but reaches a minimum r_2 at the second pseudosingularity. From there on it increases again to r_1, and so forth (see Fig. 1). Thus our observer sees the wormhole pulsate in time, kept from collapsing by the pressure of the electric flux through it. The proper time interval from r_1 to r_2, or half the pulsation period T, is readily calculated:

$$\frac{T}{2}=\int_{r_1}^{r_2}ds=\int_0^\infty fdv=\int_{r_1}^{r_2}\frac{dr}{[-\phi(r)]^{\frac{1}{2}}}=\pi m. \qquad (30)$$

Thus the period $T=2\pi m$ is independent of the charge, and is the same as the time required for pinch-off in the pure Schwarzschild case.

V. TRAJECTORIES OF PARTICLES

We have seen that the "electromagnetic cushion" provided by the electric flux through the Reissner-Nordström "wormhole" effectively prevents an observer stationed at the throat and following a geodesic from reaching any pinch-off at the singularity $r=0$. Will observers moving along less symmetrically situated geodesics likewise avoid this singularity?

The Reissner-Nordström metric is invariant under t translations, which have the form of Lorentz transformations on u and v. Since any geodesic starting at the throat on the initial surface $v_1=0$, at arbitrary velocity, is related to the geodesic of the "stationary" observer discussed above by such a transformation, observers on all such geodesics will see the same time-development of the space, and none will hit the singularity at $r=0$. It is also clear that light rays, which are described simply by lines of slope ± 1 in the u, v coordinates, *will* hit this singularity (see Fig. 2). To discuss trajectories other than these simplest examples, we examine the law of motion of test particles[13] of mass μ and charge ϵ,

$$\frac{d^2x^\nu}{d\tau^2}+\Gamma_{\alpha\beta}{}^\nu\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=-\frac{\epsilon}{\mu}F_\sigma{}^\nu\frac{dx^\sigma}{d\tau}. \qquad (31)$$

In the Reissner-Nordström metric written in the r, t coordinates the electromagnetic field tensor has the nonvanishing components

$$F_{rt}=-F_{tr}=q/r^2. \qquad (32)$$

When this tensor is transformed to the new u, v coordinates, one immediately obtains its analytic extension into the regions beyond the pseudosingularities,

$$\begin{aligned}F^{uv}=-F^{vu}&=(u_rv_t-v_ru_t)F^{rt}=\phi(u_r^2-v_r^2)F^{rt}\\&=F^{rt}/f^2=-q/r^2f^2,\\F_{uv}=f^2F_{rt}&=f^2q/r^2;\quad F_{uu}=F_{vv}=0.\end{aligned} \qquad (33)$$

The electric field in the surface $v_1=0$, defined as the force of electromagnetic origin on a stationary unit charge, has contravariant components

$$F_\alpha{}^\nu\frac{dx^\alpha}{d\tau}=F_\alpha{}^\nu\delta^\alpha{}_u/f=(q/r^2f)\delta_u{}^\nu. \qquad (34)$$

If $q>0$, this force always points in the *positive u* direction on the initial surface. On the sheet $u<0$, it points toward the wormhole throat, and on the sheet $u>0$ it points away from the throat. The "charged wormhole" therefore looks like a negative charge viewed from one sheet, and like a positive charge when viewed from the other sheet; the flux flows continuously from one sheet to the other through the wormhole.

Although the electromagnetic field distribution in the Reissner-Nordström space becomes clearer in the u, v coordinates, it is more difficult to write Eq. (31) in these coordinates, because Eq. (26) cannot be solved explicitly for r in terms of u and v. We shall therefore use the r, t system to discuss trajectories of test particles.

The four equations (31) specify the development in proper time of the four coordinates r, θ, φ, and t. Two of these equations express conservation of "angular momentum": if we satisfy the θ equation by restricting attention to orbits in the $\theta=\pi/2$ plane, the φ equation,

$$(1/r^2)d^2(r^2\varphi_\tau)/d\tau^2=0 \qquad (35)$$

implies that the angular momentum parameter

$$h=r^2\varphi_\tau \qquad (36)$$

is a constant of the motion. One of the remaining two

[13] See, for example, D. M. Chase, Phys. Rev. **95**, 243 (1954).

equations, viz., that for r, may be replaced by the condition that the four-vector velocity be a unit vector. This follows from differentiation of the metric (1) with respect to proper time τ,

$$1=\phi t_\tau^2-\phi^{-1}r_\tau^2-(h^2/r^2). \tag{37}$$

The remaining equation,

$$\phi^{-1}d(\phi t_\tau)/d\tau=(\epsilon/\mu)(q/\phi r^2)r_\tau \tag{38}$$

can be integrated to give

$$\phi t_\tau=-(\epsilon q/\mu r)+k, \tag{39}$$

where k is a constant of integration. By eliminating t_τ from (37) and (39) one obtains the radial equation

$$\phi[1+(h^2/r^2)]=[-(\epsilon q/\mu r)+k]^2-r_\tau^2. \tag{40}$$

This equation is analogous to the energy integral of the corresponding classical radial equation of motion in a central force field. The analogy can be made more explicit by rewriting (40) as follows,

$$\tfrac{1}{2}\mu(k^2-1)=\tfrac{1}{2}\mu r_\tau^2+(-\mu m+\epsilon qk)/r \\ +(1-\epsilon^2/\mu^2)(\mu q^2/2r^2)+\mu h^2\phi/2r^2. \tag{41}$$

To find the range of variation of r, we use the standard method of setting $r_\tau=0$ to obtain the turning points. Equation (40) shows that at these points ϕ is positive, hence r is space-like, as is necessary for the turning points of a timelike trajectory. From Eq. (41) we can verify the well-known result for the Schwarzschild case ($q=0$): If the angular momentum parameter h does not vanish, the positive term on the right-hand side of (41) may dominate at small r, and prevent the particle from reaching $r=0$; but it must reach the singularity at $r=0$ in a finite proper time if $h=0$. In the Reissner-Nordström case, the next to last term of Eq. (41) will always dominate the $1/r$-term at small r, and it is positive if $\mu^2>\epsilon^2$. Thus in this case no test particle will ever reach the singularity at $r=0$, provided only that the mass of the test particle exceeds the minimum associated in general relativity with its charge.

V. CONCLUSIONS

The analytic continuation of the usual Reissner-Nordström metric for a "point" charge presented here shows many features similar to the Schwarzschild-Kruskal metric. In some respects it prevents fewer difficulties than the Schwarzschild metric; in particular, if we restrict attention to particles whose mass exceeds the minimum associated with their charge ($m>|q|$, $\mu>|\epsilon|$), the throat of the wormhole representing the particle pulsates in time and does not pinch off; and no test particle ever hits the remaining geometric singularity at $r=0$.

Our analysis has been confined to the case $|q|<m$. If we formally let q *exceed* m in the metric (22), the character of the solution at small r changes radically, since now the intrinsic geometrical singularity occurs on the initial surface of time-symmetry. In this case a singular initial geometry can only be avoided by assuming some "real" charge and mass distribution of finite extent near the origin. The physical situation obtained by shrinking such a charge distribution to a point has been discussed by Arnowitt *et al.*[14] It is quite different from the wormhole topology we have discussed here, although the two agree perfectly at large distances from the concentration of mass and charge.

In conclusion, the discovery that the Reissner-Nordström metric—known since 1916—is endowed with a pulsating throat, is in full accord with the concept of a dynamic balance between gravitational pull and Maxwell pressure, and adds to the physical interest of this standard solution of Einstein's equations.

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[14] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. Letters **4**, 375 (1960).