considering D as negligible [Fig. 2(a)]. Correlation effects account for approximately 15 to 20% of the total shift between curves a and b in Fig. 2 for temperatures up to about $0.85T_c$ (the range of experimental interest), while for higher temperatures correlation effects account for as much as 40% of this shift. As in the case of sublattice magnetization, the pair correlation calculation probably underestimates the effect of correlation on the shift by a factor equal to the number of neighbors. If this expected further shift from curve b is added in, curve c is obtained.

4. DISCUSSION

We have shown that the combined effects of large uniaxial anisotropy and exchange correlation can account for a shift in the thermal dependence of both the sublattice magnetization and the antiferromagnetic resonance frequency from that predicted by a Brillouin function magnetization curve. Such a departure has been noted experimentally for FeF_2.[1,2] In FeF_2 the situation is more complicated than that described above because $S=2$, there is second-neighbor exchange, and the anisotropy is given by $-DS_{jz}^2-E(S_{jx}^2-S_{jy}^2)$. However,[3] E is only about 10% of D, so that the terms in E can be neglected to a good approximation. A calculation of the thermal dependence of the resonance frequency of the molecular field type discussed above has been carried out by Cooper and Ohlmann.[2] This accounts for perhaps 50% of the observed shift from behavior corresponding to Brillouin-function magnetization dependence. The previous discussion would indicate that the remaining shift can be accounted for by correlation effects.

ACKNOWLEDGMENTS

The author wishes to thank Professor F. Keffer for his valuable advice and encouragement. He wishes to thank Dr. V. Jaccarino and Dr. L. R. Walker for experimental information about the sublattice magnetization, Dr. R. Ohlmann for the resonance frequency data, and Professor C. Kittel for his interesting comments.

PHYSICAL REVIEW VOLUME 120, NUMBER 4 NOVEMBER 15, 1960

Electron Levels in a One-Dimensional Random Lattice

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(Received July 7, 1960)

Let the potential of a one-dimensional scalar particle be $V(x)=V_0\sum_{-\infty}^{\infty}\delta(x-x_j)$, $-\infty<x<\infty$, where $V_0<0$, and where the sequence (x_j) is random, with a Poisson distribution. The quantity of interest is a certain limiting level distribution, equal numerically to the node density of real solutions $\psi(x)$ of the Schrödinger equation. The random variables $z_j=\psi'(x_j-0)/\psi(x_j)$, $-\infty<j<\infty$, constitute an ergodic stationary Markov process. The stationary density $T(z)$ of the (z_j) satisfies a first-order linear differential-difference equation, and the node density is given (with probability 1) by $\lim_{z\to\infty} z^2T(z)$ (Rice's formula). Numerical results are obtained by integrating the second-order linear differential equation satisfied by the Fourier transform of $T(z)$.

1. INTRODUCTION

WE are concerned with the distribution of energy levels of a one-dimensional electron (scalar particle) moving in a one-dimensional random array of atoms. The atoms, all of one kind, have (randomly) fixed positions, and the electron-atom potentials are assumed to be δ functions. The Schrödinger equation for an electron of mass m and energy E is then

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V_0\sum_{j=-\infty}^{\infty}\delta(x-x_j)\psi(x)=E\psi(x), \quad (1)$$

where $V_0<0$ is the strength of each electron-atom interaction (attractive) and $\cdots\leq x_{-1}\leq x_0\leq x_1\leq\cdots$ are the positions of the atoms, randomly distributed on the infinite line. We consider in detail only the case where the x_j's have a Poisson distribution, although our methods are applicable to certain more general distributions (described in Sec. 6). Boundary conditions for (1) are discussed presently.

One model which gives rise to (1) (and the one which led to the present investigation) is the impurity band model of Lax and Phillips.[1] The "atoms" represent impurity atoms in a one-dimensional crystal; the periodic potential of the pure crystal is replaced by a constant (included in E). Another model to which (1) might be applied is a one-dimensional liquid metal. We do not discuss such applications, confining ourselves to a mathematical study of (1). A detailed discussion of the impurity band model, with references to the literature, is given in the Lax and Phillips article.

2. LEVEL DISTRIBUTION

The quantity we seek is the limiting distribution-in-energy of the eigenvalues of (1), defined as follows. Let

[1] M. Lax and J. C. Phillips, Phys. Rev. 110, 41 (1958).

ω denote a sequence $\omega=(\cdots,x_{-1},x_0,x_1,\cdots)$ of atom positions. We treat each sequence ω as a single point in an infinite-dimensional space Ω. This space Ω of all possible such sequences is a measure space, carrying a probability measure which we denote by $P\{\ \ \}$; measurable ω functions are "random variables."[2] Expectations [averages over ω with weighting $dP(\omega)$] will be denoted by $\langle\cdots\rangle$.

For each sequence ω let $E_1(L,\omega)\leq E_2(L,\omega)\leq\cdots$ denote the eigenvalues of (1) for a finite interval $0\leq x\leq L$. That is, for fixed ω, the $E_m(L,\omega)$'s are the values of E for which there exist solutions of (1) on $0\leq x\leq L$ satisfying, say, $\psi(0)=\psi(L)=0$.[3] Still with ω fixed, let $\mathcal{N}_L(E,\omega)$ be defined as a function of E by

$$\mathcal{N}_L(E,\omega)=(1/L)$$
$$\times[\text{number of } E_m(L,\omega)\text{'s which satisfy } E_m(L,\omega)\leq E],$$
$$-\infty<E<\infty;$$

[i.e., a nondecreasing step function which vanishes to the left of $E_1(L,\omega)$ and which jumps by $1/L$ at each $E_m(L,\omega)$]. By the "distribution of levels" we mean the limit

$$\mathcal{N}(E)=\lim_{L\to\infty}\mathcal{N}_L(E,\omega)\ \text{with probability 1.} \tag{2}$$

As we prove in Appendix 2, the right-hand side in (2) exists and is independent of ω with probability 1, as indicated.[4]

To obtain $\mathcal{N}(E)$ explicitly we will make use of the fact that $\mathcal{N}(E)$ is equal to the density of zeros of any real solution of (1).[5,6] For each E and ω let $\psi(x;E,\omega)$ denote the (real) solution of (1) which satisfies boundary conditions $\psi(0;E,\omega)=\xi_0$, $\psi'(0;E,\omega)=\eta_0$, where ξ_0 and η_0, real and not both zero, are arbitrarily given boundary values. If $\nu_L(E,\omega)$ denotes the number of zeros of $\psi(x;E,\omega)$ in the interval $0\leq x\leq L$, then

$$\lim_{L\to\infty}\mathcal{N}_L(E,\omega)=\lim_{L\to\infty}[\nu_L(E,\omega)/L]\ \text{with probability 1,} \tag{3}$$

independently of ξ_0 and η_0. That both limits exist and are equal whenever either one exists is a theorem in differential equations[5]; probability is not involved. Since the common limit in (3) is constant (as an ω function) with probability 1, it is equal to its expected value with probability 1. Our problem is thus reduced to finding the (ω,x)-average number of zeros per unit length of real solutions of (1).

In subsequent sections we work with the dimensionless quantity

$$N(\lambda)=(1/n)\mathcal{N}(\hbar^2\lambda/2m),$$

where n is the expected density of atoms. There being n atoms per unit length (on the average), $N(\lambda)$ may be regarded as the number of electron levels per atom below energy $E=(\hbar^2/2m)\lambda$. $N(\lambda)$ will depend only on a dimensionless energy ratio λ/κ_0^2 and a dimensionless density parameter n/κ_0, where $\kappa_0=-mV_0/\hbar^2(>0)$ is the inverse range of an electron of energy $-(\hbar^2/2m)\kappa_0^2$ bound to an isolated atom. (We follow the Lax and Phillips notation here.)

3. THE PHASE PROCESS

For expository purposes we discuss (1) with reference to still another model, namely, a classical harmonic oscillator disturbed by randomly occurring impulses. We substitute symbol t for symbol x in (1), and regard $\psi(t)$ (real) as the displacement at time t of a unit point mass from an equilibrium position $\psi=0$ on the ψ axis. With $\xi=\psi$, $\eta=\dot\psi$, (1) may be expressed as

$$\dot\xi(t)=\eta(t),$$
$$\dot\eta(t)=-[\lambda+2\kappa_0\sum_{j=-\infty}^{\infty}\delta(t-t_j)]\xi(t), \tag{4}$$

where, again, $\lambda=2mE/\hbar^2$ and $\kappa_0=-mV_0/\hbar^2$.

We refer to the times t_j as *hits*. At each hit, the particle coordinate is unchanged (continuity of ψ):

$$\xi(t_j+0)=\xi(t_j-0), \tag{5}$$

but the particle momentum receives an increment proportional to the displacement, directed toward the origin:

$$\eta(t_j+0)-\eta(t_j-0)=-2\kappa_0\xi(t_j) \tag{6}$$

[obtained by integrating (4) from t_j-0 to t_j+0]. Between hits the particle moves as a harmonic oscillator with force constant λ:

$$\dot\xi(t)=\eta(t),\quad \dot\eta(t)=-\lambda\xi(t),\quad t\neq t_j. \tag{7}$$

The probability is 0 that two or more hits are ever simultaneous, and we neglect such events.

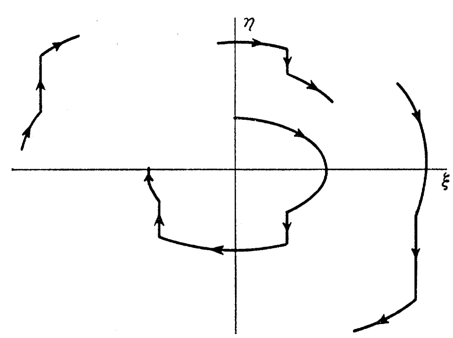

FIG. 1. Phase motion between hits on ellipses $\lambda\xi^2+\eta^2=$const for given $\lambda>0$. At hits, motion is instantaneous on vertical segments.

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[2] J. L. Doob, *Stochastic Processes* (John Wiley & Sons, Inc., New York, 1953), p. 599 ff.

[3] The numbers $E_m(L,\omega)$ depend only on those x_j's which happen to satisfy $0\leq x_j\leq L$, of course. The number of such x_j's is finite with probability 1, implying that the $E_m(L,\omega)$'s are defined with probability 1 (that is, except for a set of ω's of total probability 0).

[4] That is to say, there exists a number $\mathcal{N}(E)$ which has the following property: The set of all ω for which $\mathcal{N}_L(E,\omega)$ fails to converge to $\mathcal{N}(E)$ (as $L\to\infty$, with E and ω fixed) is a subset of Ω which has probability 0.

[5] H. M. James and A. S. Ginzbarg, J. Phys. Chem. **57**, 840 (1953).

[6] H. Schmidt, Phys. Rev. **105**, 425 (1957).

Orbits in phase space for $\lambda>0$ and $\lambda<0$ are shown in Figs. 1 and 2, respectively. It is seen that in the case $\lambda>0$ the η-axis crossings (zeros of $\dot{\psi}$) occur at a rate at least $\lambda^{\frac{1}{2}}/\pi$, the effect of the hits being to speed the angular motion. When $\lambda<0$, however, the motion (7) is retrograde (counterclockwise) in the sectors $|\eta/\xi|<\sqrt{-\lambda}$, and if there were no hits the phase point would stick at the asymptote $\eta/\xi=+\sqrt{-\lambda}$.

The phase variables $\{(\xi(t,\omega),\eta(t,\omega)),\ -\infty<t<\infty\}$ constitute a two-dimensional stochastic process. The η-axis crossings are determined by the angular part of the motion, however, and since (4) is homogeneous in ξ and η, we may treat the angular part separately.[7] Accordingly, we introduce the variable $z=\eta/\xi=\dot{\psi}/\psi$, and we will show now that the random variables $\{z(t,\omega),\ -\infty<t<\infty\}$ constitute a Markov process. Equations (5) and (6) give

$$z(t_j+0)=z(t_j-0)-2\kappa_0 \tag{8}$$

at hits, while (7) gives

$$\dot{z}=-(z^2+\lambda),\quad t\neq t_j, \tag{9}$$

for the motion between hits. Note that at an η-axis crossing the variable z flies off to $z=-\infty$ and instantly reappears at $z=+\infty$ (compare $z=-c\tan ct$ at its singularities). For any given time τ, Eqs. (8)–(9) determine $z(t,\omega)$ for all times $t>\tau$ as a function of $z(\tau,\omega)$ and the times of the hits occurring after τ. It is a property of the Poisson process, however, that the times of the hits occurring after τ are statistically independent of the times of the hits occurring before τ.[8] From this it can be shown that the times of the hits occurring after τ are independent of the random variables $z(t',\omega)$, $-\infty<t'\leq\tau$. It follows that for any $t>\tau$ the conditional probability distribution of $z(t,\omega)$ given all values $\{z(t',\omega),\ -\infty<t'\leq\tau\}$ is the same as the conditional distribution of $z(t,\omega)$ given only $z(\tau,\omega)$, and this is the Markov property.[9] The z process has

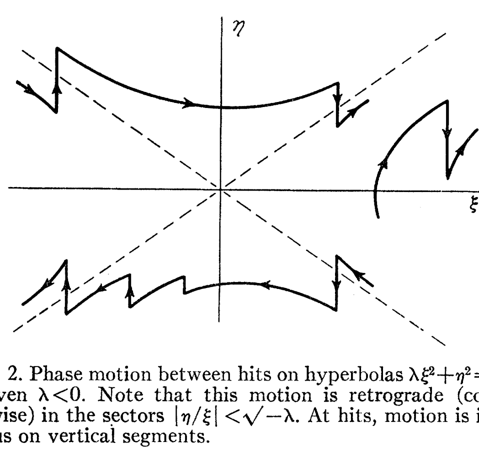

FIG. 2. Phase motion between hits on hyperbolas $\lambda\xi^2+\eta^2=$const for given $\lambda<0$. Note that this motion is retrograde (counterclockwise) in the sectors $|\eta/\xi|<\sqrt{-\lambda}$. At hits, motion is instantaneous on vertical segments.

[7] The full (ξ,η) process is discussed in Appendix 1.
[8] See reference 2, pp. 398 ff.
[9] See reference 2, p. 80.

stationary (independent of time origin) transition probabilities; the hit process is stationary, whence (4) admits time shifts.

We wish to find the probability distribution of $z(t,\omega)$, and for this we use the following characterization of the Poisson process.[10] The number of hits in any small time interval $(t,\ t+dt)$ is statistically independent of the numbers in any other disjoint measurable t sets disjoint from $(t,\ t+dt)$, and has the probability distribution

$$P\{(t,\ t+dt)\text{ contains no hit}\}=1-ndt+o(dt),$$

$$P\{(t,\ t+dt)\text{ contains exactly one hit}\}=ndt+o(dt),$$

$$P\{(t,\ t+dt)\text{ contains more than one hit}\}=o(dt),$$

where n is the expected number of hits per unit time and $o(dt)$ are negligible higher order quantities.

To simplify the derivation, we assume that the probability distribution of $z(t,\omega)$ has a density:

$$P\{z(t,\omega)\leq\zeta\}=\int_{-\infty}^{\zeta}T(z,t)dz,\quad -\infty<\zeta<\infty,$$

where, of course,

$$T(z,t)\geq 0,\quad \int_{-\infty}^{\infty}T(z,t)dz=1,\quad -\infty<t<\infty. \tag{10}$$

Abusing the terminology somewhat, we refer to the z axis also as "phase space" and a point $z(t,\omega)$ as a "phase point." Let us regard $T(z,t)\delta z$ as the fraction of phase points in an ensemble which lie in an interval $(z,\ z+\delta z)$ at time t. The phase points which lie in $(z,\ z+\delta z)$ at time $t+dt$ consist of

(a) those which were in $(z',\ z'+\delta z')$ at time t and (with probability about $1-ndt$) received no hit during $(t,\ t+dt)$, where, from (9),

$$z'=z+(z^2+\lambda)dt,$$
$$\delta z'=(dz'/dz)\delta z=(1+2zdt)\delta z;$$

(b) those which were in $(z'',\ z''+\delta z'')$ at time t and (with probability about ndt) received exactly one hit during $(t,\ t+dt)$, where, from (8),

$$z''=z+2\kappa_0,$$
$$\delta z''=(dz''/dz)\delta z=\delta z;$$

(c) a negligible proportion which reach $(z,\ z+\delta z)$ after being hit more than once during $(t,\ t+dt)$. Conservation of phase points requires

$$T(z,\ t+dt)\delta z=(1-ndt)T(z',t)\delta z'+ndtT(z'',t)\delta z'',$$

to within higher order differentials. From this we obtain

$$\frac{\partial T(z,t)}{\partial t}=\frac{\partial}{\partial z}[(z^2+\lambda)T(z,t)]+n[T(z+2\kappa_0,\ t)-T(z,t)]. \tag{11}$$

[10] W. Feller, *Probability Theory and its Applications* (John Wiley & Sons, Inc., New York, 1950), p. 366.

If an initial distribution[11] of phase points is given, say $T(z,0)$, then (11) determines $T(z,t)$ uniquely for all $t>0$. Every solution tends to a limiting density

$$T(z)=\lim_{t\to\infty} T(z,t),$$

the limit being independent of the initial distribution. This stationary density $T(z)$ is the (unique) solution of

$$\frac{d}{dz}[(z^2+\lambda)T(z)]+n[T(z+2\kappa_0)-T(z)]=0 \qquad (12)$$

which satisfies the normalizing condition

$$\int_{\infty}^{\infty} T(z)dz=1. \qquad (13)$$

Proofs of these statements involve ergodic and other properties of the z process, and are given in Appendix 2. We assume in all that follows that the distribution of phase points is the stationary distribution, i.e., $T(z,t)=T(z)$, $-\infty<t<\infty$.

Differentiation of

$$(z^2+\lambda)T(z)+n\int_z^{z+2\kappa_0} T(\zeta)d\zeta=\text{const} \qquad (14)$$

shows that (14) is an integral-equation version of (12). Equation (14) is simply the assertion that the flux of phase points is constant. For the phase points which pass point z leftward during a small time interval of duration dt due to the motion (9) is $-(1-ndt)T(z)(\dot{z}dt)\approx(z^2+\lambda)T(z)dt$, and the leftward flux at z due to hits is $(ndt)T(\zeta)d\zeta$ from all elements $d\zeta$ in the interval $z\leq\zeta\leq z+2\kappa_0$. The leftward flux at $z=-\infty$ is the expected η-axis crossing rate $nN(\lambda)$, so that (14) may be put into the form

$$T(z)=\frac{n}{z^2+\lambda}\left[N(\lambda)-\int_z^{z+2\kappa_0} T(\zeta)d\zeta\right]. \qquad (15)$$

Since the integral here vanishes as $z\to\pm\infty$, it follows that

$$N(\lambda)=(1/n)\lim_{z\to\pm\infty} z^2T(z), \qquad (16)$$

which we use later.

Another integrated version of (12) is obtained by integrating with a factor which integrates all terms except the one $T(z+2\kappa_0)$. We define

$$u(z)=e^{(n/k)\arctan(z/k)} \quad \text{if} \quad \lambda=k^2>0,$$

$$u(z)=\left|\frac{z-\kappa}{z+\kappa}\right|^{n/(2\kappa)} \quad \text{if} \quad \lambda=-\kappa^2<0,$$

[11] We treat ξ_0 and η_0 (and hence $z_0=\eta_0/\xi_0$) as random quantities, statistically independent of the t_j's. The basic random element ω now consists of a specification of ξ_0 and η_0 as well as all of the t_j's.

with $|\arctan x|<\frac{1}{2}\pi$ in the first case. We have $u'(z)/u(z)=n/(z^2+\lambda)$ in either case, and it is easily verified that (12) is equivalent to

$$\frac{d}{dz}\left[\frac{T(z)}{u'(z)}\right]=-\frac{T(z+2\kappa_0)}{u(z)}. \qquad (17)$$

For the case $\lambda=k^2$ we find

$$T(z)=u'(z)\left[e^{-\frac{1}{2}n\pi/k}N(k^2)+\int_z^{\infty}\frac{T(\zeta+2\kappa_0)}{u(\zeta)}d\zeta\right]$$
$$=u'(z)\left[e^{\frac{1}{2}n\pi/k}N(k^2)-\int_{-\infty}^{z}\frac{T(\zeta+2\kappa_0)}{u(\zeta)}d\zeta\right],$$
$$-\infty<z<\infty, \qquad (18)$$

integrating (17) over the indicated intervals and using (16). If we eliminate $N(k^2)$ in (18) we have

$$T(z)=\frac{u'(z)}{1-e^{-n\pi/k}}\left[\int_z^{\infty}\frac{T(\zeta+2\kappa_0)}{u(\zeta)}d\zeta\right.$$
$$\left.+e^{-n\pi/k}\int_{-\infty}^{z}\frac{T(\zeta+2\kappa_0)}{u(\zeta)}d\zeta\right], \quad -\infty<z<\infty, \qquad (19)$$

while elimination of $T(z)$ gives

$$N(k^2)=\int_{-\infty}^{\infty}T(\zeta+2\kappa_0)\frac{e^{-(n/k)\arctan(\zeta/k)}}{e^{\frac{1}{2}n\pi/k}-e^{-\frac{1}{2}n\pi/k}}d\zeta. \qquad (20)$$

In the case $\lambda=-\kappa^2$ we avoid integrating (17) over $z=\kappa$, since $u(\kappa)=0$. Integrating (17) over $(-\kappa, z)$ for $z<\kappa$ and over (z,∞) for $z>\kappa$, we have

$$T(z)=-u'(z)\int_{-\kappa}^{z}\frac{T(\zeta+2\kappa_0)}{u(\zeta)}d\zeta, \quad -\infty<z<\kappa,$$
$$T(z)=u'(z)\left[N(-\kappa^2)+\int_z^{\infty}\frac{T(\zeta+2\kappa_0)}{u(\zeta)}d\zeta\right], \qquad (21)$$
$$\kappa<z<\infty.$$

The above relations will be used in Sec. 5 to obtain approximations and bounds for $N(\lambda)$. A probabilistic interpretation of (19) and (21) is given in Appendix 2.

4. NUMERICAL METHODS

The methods used to obtain $N(\lambda)$ numerically are based on the system

$$\frac{d^2\varphi(s)}{ds^2}=a(s)\varphi(s), \quad s\neq 0,$$
$$a(s)=\lambda+n\frac{1-e^{-2i\kappa_0 s}}{is}, \qquad (22)$$
$$\varphi(0)=1,$$
$$\lim_{s\to\pm\infty}\varphi(s)=0,$$

satisfied by the characteristic function (Fourier transform) $\varphi(s)$ of the probability density $T(z)$:

$$\varphi(s)=\int_{-\infty}^{\infty} e^{isz}T(z)dz.$$

The level distribution $N(\lambda)$ is obtained from

$$\mathrm{Re}\varphi'(0+)=-\mathrm{Re}\varphi'(0-)=-\pi n N(\lambda). \quad (23)$$

(Proof appears in Appendix 3.)

Three integration schemes used each involve use of an asymptotic formula for $\varphi'(s)/\varphi(s)$, and the accuracy of the numerical results is limited by the accuracy of the asymptotic formula. The values given for $N(\lambda)$ are least accurate when λ is near $-\kappa_0^2$ and μ is small. (Unfortunately, this is a region of considerable interest.)

One method involves numerical integration of (22) from some large s in to $s=0$, the asymptotic formula being used to obtain a starting value for $\varphi'(s)/\varphi(s)$; Eq. (23) then gives $N(\lambda)$. In a closely related method the Riccati equation associated with (22),

$$y'(s)+y^2(s)=\lambda+n\frac{1-e^{-2i\kappa_0 s}}{is}, \quad s\neq 0$$

$$\varphi(s)=\exp\left(\int_0^s y(s')ds'\right),$$

is integrated numerically from some large s to $s=0$, the asymptotic formula now giving the starting value of $y(s)$.

A third method is based on the fact that if $\theta(s)$, $0\leq s<\infty$, is an unbounded, zero-free solution of differential equation (22) then

$$\varphi(s)=\theta(s)\int_s^\infty \frac{ds'}{\theta^2(s')}\Big/\theta(0)\int_0^\infty\frac{ds'}{\theta^2(s')}$$

satisfies the system (22), so that from (23),

$$N(\lambda)=\frac{1}{n\pi}\mathrm{Re}\left[1\Big/\theta^2(0)\int_0^\infty\frac{ds'}{\theta^2(s')}-\frac{\theta'(0)}{\theta(0)}\right]. \quad (24)$$

The integral in (24) can be brought into the form

$$\int_0^\infty \frac{ds'}{\theta^2(s')}=\int_0^s\frac{ds'}{\theta^2(s')}+\frac{1}{\theta^2(s)[\theta'(s)/\theta(s)-\varphi'(s)/\varphi(s)]}, \quad (25)$$

and is evaluated numerically by integrating Eq. (22) from $s=0$ $[\theta(0)=1, \theta'(0)=i]$ to some large s and using (57)–(58) to approximate $\varphi'(s)/\varphi(s)$ in the denominator in (25). The numerical solutions $\theta(s)$ were found to be zero-free.

The numerical results are given in Tables I–III, and appear as triangles in Figs. 3–5. For comparison, the open circles and solid circles represent the Monte Carlo and local density model results, respectively, of Lax and Phillips.[1]

TABLE I. Level distribution for $n/\kappa_0=10$.

κ/κ_0	$N(-\kappa^2)$
20.00	0.000346
5.00	0.0315
4.672	0.0444
4.472	0.054
4.272	0.063
0.0	0.142
k/κ_0	$N(k^2)$
0.2	0.142
3.162	0.1741
4.472	0.201

TABLE II. Level distribution for $n/\kappa_0=1$.

κ/κ_0	$N(-\kappa^2)$
5.00	0.027
2.50	0.043
1.37	0.258
1.18	0.298
1.02	0.335
1.002	0.3349
1.00	0.345
0.98	0.352
0.8	0.392
0.02	0.451
k/κ_0	$N(k^2)$
0.2	0.454
1.0	0.581
1.414	0.629

TABLE III. Level distribution for $n/\kappa_0=0.1$. Since $N(\lambda)$ is nondecreasing in λ, the value given above for $N(-\kappa^2)$ at $\kappa/\kappa_0=1.0001$ is surely incorrect.

κ/κ_0	$N(-\kappa^2)$
0	0.86
0.67	0.857
0.98	0.729
0.99	0.701
0.999	0.658
0.9999	0.588
1.0001	0.2443
1.001	0.2887
1.01	0.2599
1.02	0.233
1.33	0.0754
2.00	0.00186

5. APPROXIMATIONS

(a) The Optical and Diffusion Approximations

When the hits are fast and weak ($n\to\infty$, $\kappa_0\to 0$, $n\kappa_0\to$ finite) the integral in (15) may be approximated as

$$n\int_z^{z+2\kappa_0} T(\zeta)d\zeta\approx 2n\kappa_0 T(z), \quad (26)$$

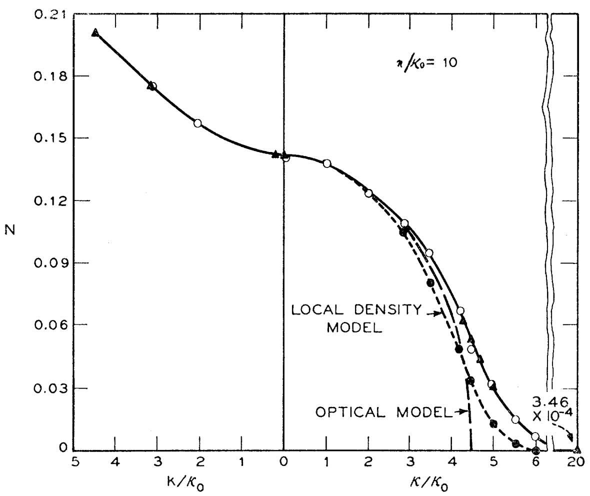

FIG. 3. Level distribution versus $|\lambda|^{\frac{1}{2}}/\kappa_0$ for $n/\kappa_0=10$. ▲=exact value from (22), ○=Monte Carlo value (Lax and Phillips) ●="local density model" (Lax and Phillips).

and the resulting equation is easily solved for

$$T(z)\approx\frac{nN(\lambda)}{z^2+\lambda+2n\kappa_0}. \tag{27}$$

Imposing the normalizing condition (13), we have the optical approximation:

$$N(\lambda)\approx\frac{(\lambda+2n\kappa_0)^{\frac{1}{2}}}{n\pi},\quad \lambda\geq-2n\kappa_0, \qquad (\approx 0,\quad \lambda\leq -2n\kappa_0). \tag{28}$$

This is also the result given by the "optical model" of Lax and Phillips,[1] whence the terminology.

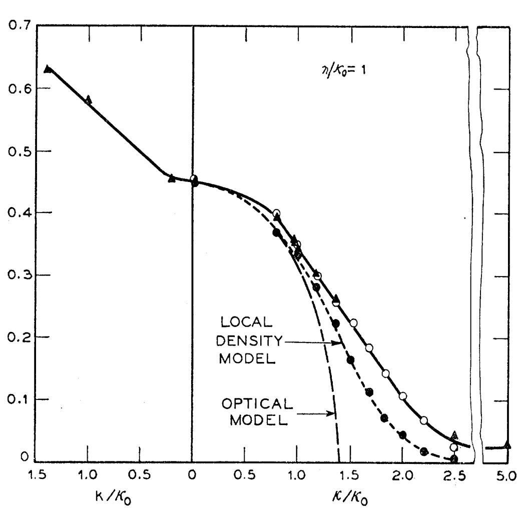

FIG. 4. Level distribution versus $|\lambda|^{\frac{1}{2}}/\kappa_0$ for $n/\kappa_0=1$. ▲=exact value from (22), ○=Monte Carlo value (Lax and Phillips) ●="local density model" (Lax and Phillips).

The approximate solution (27) is not everywhere non-negative when λ is below the optical band edge $-2n\kappa_0$. That this singularity is an artifact of the approximation method is shown by including another term in (26), viz.,

$$n\int_z^{z+2\kappa_0}T(\zeta)d\zeta\approx 2n\kappa_0T(z)+2n\kappa_0^2T'(z).$$

We substitute this in (15), solve the resulting differential equation for $T(z)$, and then impose the normalizing condition (13). Omitting the details, the result is the diffusion approximation:

$$N(\lambda)\approx\frac{1}{2\pi^{1/2}3^{1/6}\epsilon^{2/3}\int_0^\infty\exp(-t^6-\mu t^2)dt}, \tag{29}$$

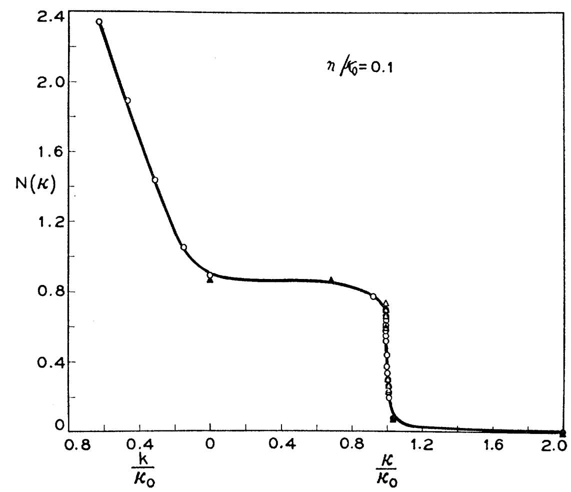

FIG. 5. Level distribution versus $|\lambda|^{\frac{1}{2}}/\kappa_0$ for $n/\kappa_0=0.1$. ▲=exact value from (23), △=exact value from (24), ○=Monte Carlo value (Lax and Phillips).

where

$$\epsilon=\frac{n}{\kappa_0},\quad \mu=\left(\frac{3}{\epsilon^2}\right)^{\frac{1}{3}}\left(\frac{\lambda}{\kappa_0^2}+2\epsilon\right).$$

(We explain the appellation presently.) In Fig. 6 we show the quantity

$$A(\mu)=\left[\int_0^\infty\exp(-t^6-\mu t^2)dt\right]^{-1},$$

together with the approximation

$$B(\mu)=\left[\int_0^\infty\exp(-\mu t^2)dt\right]^{-1}=(4\mu/\pi)^{\frac{1}{2}}$$

[which yields (28)]. We note in passing that according to (29), the number of levels per atom below the optical band edge is

$$N(-2n\kappa_0)\approx\frac{1}{2\pi^{1/2}3^{1/6}\epsilon^{2/3}\Gamma(7/6)}$$
$$\approx 0.2532/\epsilon^{2/3},$$

in agreement with an observation made by Lax and Phillips[1] on the basis of their Monte Carlo results. A comparison of the various computations of $N(\lambda)$ is shown in Fig. 7.

The optical approximation (28) corresponds to replacing the random potential in (1) by its (ω,x)-average value nV_0. Our diffusion approximation (29) corresponds to adding a white Gaussian noise correction term to nV_0, making $(\psi(x),\psi'(x))$ a two-dimensional diffusion process, as follows. The integral of the potential in (1) is

$$Q(x)=\int_0^x V_0\sum_{j=-\infty}^{\infty}\delta(x'-x_j)dx'=V_0P(x),$$

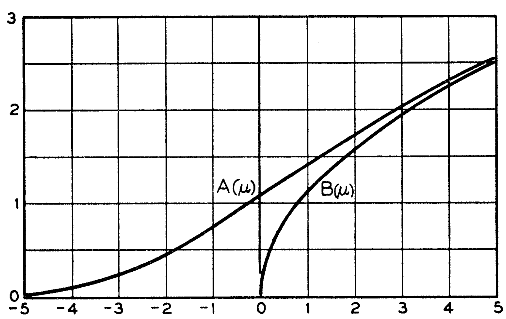

FIG. 6. Ordinates $A(\mu)$ and $B(\mu)$, proportional to the diffusion and optical approximations (29) and (28), respectively. Definition of μ appears with (29).

where $P(x)$ is a Poisson process[12] with density parameter n. If we define a process $\gamma(x)=[P(x)-nx]/\sqrt{n}$ then $\gamma(x)$ is nearly a Brownian motion (Wiener process) when n is large,[13] and the corresponding potential is $nV_0+n^{\frac{1}{2}}V_0\gamma_0'(x)$, where $\gamma_0(x)$ is a Brownian motion ["$\gamma_0'(x)$"="white Gaussian noise"]. This approximation can be used to obtain (29); we omit the details.[14]

(b) The Case $\lambda=k^2>0$

The integral on the right-hand side in (15) lies between 0 and 1, whence

$$\frac{n[N(k^2)-1]}{z^2+k^2}\leq T(z)\leq\frac{nN(k^2)}{z^2+k^2}. \qquad (30)$$

We integrate this over $(-\infty,\infty)$ and impose (13),

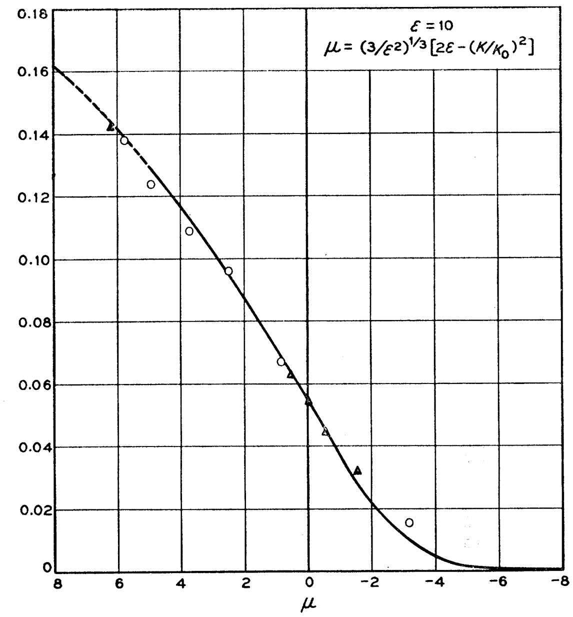

FIG. 7. Comparison of diffusion approximation (29) with exact distribution for $n/\kappa_0=10$. ▲=exact value from (23), ○=Monte Carlo value (Lax and Phillips).

obtaining the bounds $k/(n\pi)\leq N(k^2)\leq 1+k/(n\pi)$. Similarly, from (18),

$$e^{-\frac{1}{2}n/k}N(k^2)u'(z)\leq T(z)\leq e^{\frac{1}{2}n\pi/k}N(k^2)u'(z),$$

giving bounds $(e^{n\pi/k}-1)^{-1}\leq N(k^2)\leq 1+(e^{n\pi/k}-1)^{-1}$. Using $e^x\geq 1+x$, we have the bounds ($\lambda>0$)

$$\frac{k}{n\pi}\leq N(k^2)\leq\frac{1}{1-e^{-n\pi/k}}.$$

For a refined version of the above, we rewrite (20) as

$$N(k^2)=\frac{e^{\frac{1}{2}n\pi/k}}{e^{\frac{1}{2}n\pi/k}-e^{-\frac{1}{2}n\pi/k}}-\int_{-\infty}^{\infty}T(z+2\kappa_0)\frac{e^{\frac{1}{2}n\pi/k}-e^{-(n/k)\arctan(z/k)}}{e^{\frac{1}{2}n\pi/k}-e^{-\frac{1}{2}n\pi/k}}dz.$$

Substituting for $T(z+2\kappa_0)$ the bounds given by (30), we obtain the bounds ($\lambda>0$)

$$\frac{1}{1-e^{-n\pi/k}+\Delta}\leq N(k^2)\leq\frac{1+\Delta}{1-e^{-n\pi/k}+\Delta},$$
$$\Delta=\frac{n}{k}\int_{-\infty}^{\infty}\frac{1-e^{-(n/k)(\frac{1}{2}\pi+\arctan x)}}{[x+(2\kappa_0/k)]^2+1}dx. \qquad (31)$$

When $k\gg n$ the numerator in the integral for Δ may be expanded in powers of n/k to give the result

$$N(k^2)\approx\frac{k}{n\pi}+\frac{1}{\pi}\arctan\frac{\kappa_0}{k}+O\left(\frac{n}{k}\right).$$

[12] That is, $P(x)$ increases by unity at each x_j. See reference 2, pp. 98 and 398.
[13] See reference 2, pp. 434–435.
[14] See M. C. Wang and G. E. Uhlenbeck, Revs. Modern Phys. **17**, 332 (1945).

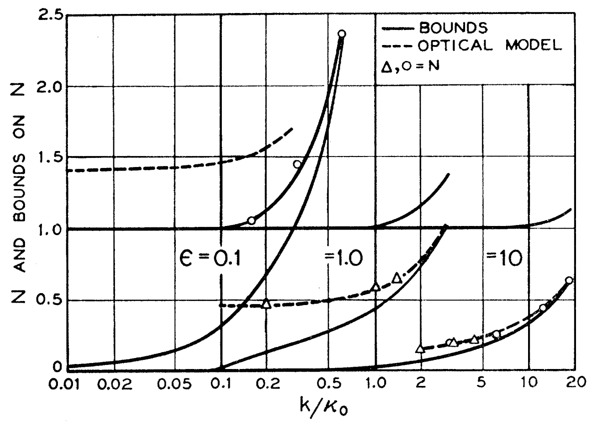

FIG. 8. The bounds (31) for the level distribution. △ = exact value from (22), ○ = Monte Carlo value (Lax and Phillips).

[This can also be obtained by iterating (15).] The bounds (31) are shown in Fig. 8; the integral Δ was evaluated numerically.

(c) The Case $\lambda=-\kappa^2<0$

The denominator on the right-hand side in (15) is negative when $|z|<\kappa$, and since $T(z)$ is never negative we must have

$$N(-\kappa^2)\leq\int_z^{z+2\kappa_0} T(\zeta)d\zeta, \quad -\kappa\leq z\leq\kappa. \tag{32}$$

We fit as many disjoint subintervals $z<\zeta<z+2\kappa_0$ as possible into the interval $(-\kappa, \kappa+2\kappa_0)$, add integrals (32) for the subintervals, and impose (13), obtaining the bound $(\lambda<0)$

$$N(-\kappa^2)\leq 1/(q+1),$$
$$q=\text{integer part of } (\kappa/\kappa_0).$$

A bound which is much stronger when q is large is obtained as follows. We rewrite (21) as

$$T(z)=\frac{nu(z)}{\kappa^2-z^2}\int_{-\kappa}^{z}\frac{T(\zeta+2\kappa_0)}{u(\zeta)}d\zeta, \quad -\infty<z<\kappa.$$

Now, $u(\zeta)$ is decreasing on $-\kappa<\zeta<\kappa$, whence

$$T(z)\leq\frac{n}{\kappa^2-z^2}\int_{-\kappa}^{z}T(\zeta+2\kappa_0)d\zeta, \quad -\kappa<z<\kappa.$$

When $q>1$ we have

$$\kappa^2-z^2\geq 4\kappa_0(\kappa-\kappa_0) \quad \text{if} \quad -\kappa+2\kappa_0\leq z\leq\kappa-2\kappa_0,$$

so that

$$T(z)\leq\alpha\int_{-\kappa}^{z}T(\zeta+2\kappa_0)d\zeta, \quad -\kappa+2\kappa_0\leq z\leq\kappa-2\kappa_0,$$
$$\alpha=n/4\kappa_0(\kappa-\kappa_0). \tag{33}$$

It is not hard to verify that (33) implies

$$T(z)\leq\frac{\alpha^{r+1}(z+\kappa)(z+\kappa+2r\kappa_0)^{r-1}}{r!},$$
$$-\kappa+2\kappa_0\leq z\leq\kappa-2(r+1)\kappa_0, \tag{34}$$
$$r=0, 1, \cdots, q-2.$$

[The case $r=0$ follows from (33) and (13), and the others by induction.] We now apply (32) to the estimate (34), choosing for r the largest value for which (34) holds on an interval of length at least $2\kappa_0$, viz., $r=q-3$. There results the bound $(\lambda<0)$

$$N(-\kappa^2)\leq\frac{[2\kappa_0\alpha(q-1)]^{q-2}}{(q-1)!}$$
$$\leq\frac{(\frac{1}{2}\epsilon)^{q-2}}{(q-1)!}, \tag{35}$$

valid for $q\geq 2$, where, again, $\epsilon=n/\kappa_0$ and $q=$integer part of (κ/κ_0).

The following intuitive argument gives an approximation even smaller than (35) for large q. Without hits, the velocity of a phase point is $\dot{z}=\kappa^2-z^2$, and the time required to move from $z=-\kappa+2\kappa_0$ to $z=\kappa-2\kappa_0$ is $T=(1/\kappa)\log[(\kappa/\kappa_0)-1]$, assuming $q>1$. This time is small when κ is large, and a phase point will spend most of the time in the region $\kappa-2\kappa_0\leq z\leq\kappa$, waiting for a favorable succession of hits (see Fig. 2). To escape the interval $-\kappa\leq z\leq\kappa$ the particle must receive about q (or more) hits within a time interval T. (Fewer than q will not carry the point past $z=-\kappa$, and it will slip back to the vicinity of $z=+\kappa$.) Now, the probability that an interval of duration T after any hit contains $q-1$ or more further hits is[9]

$$p=\frac{e^{-nT}(nT)^{q-1}}{(q-1)!}\left[1+\frac{nT}{q}+\frac{(nT)^2}{q(q+1)}+\cdots\right]$$
$$\leq\frac{e^{-nT}(nT)^{q-1}}{(q-1)!}\cdot\frac{1}{1-(nT/q)}.$$

If such bursts of hits are infrequent enough their rate of occurrence will be about np. Since each η-axis crossing requires about one burst, we have, finally, the approximation $(\lambda<0)$

$$N(-\kappa^2)\approx p\approx e^{-nT}(nT)^{q-1}/(q-1)!,$$
$$nT=(n/\kappa)\log(\kappa/\kappa_0-1)\approx(\epsilon/q)\log(q-1),$$

valid for $q>1$, $nT/q\ll 1$.

6. GENERALIZATIONS

There seems to be no way of extending our methods to the corresponding three-dimensional problem; there is no Markov property when the parameter is three-dimensional, and (3) does not generalize (to the best of the authors' knowledge). The one-dimensional case can be generalized somewhat, however.

First, the strength of each interaction can be random. That is, instead of the potential $-(\hbar^2/m)\kappa_0 \sum \delta(x-x_j)$ of (1) we may consider the potential

$$-(\hbar^2/m) \sum \kappa_{0j}\delta(x-x_j),$$

where the κ_{0j}'s are independent random variables with common probability distribution, say $P\{\kappa_{0j} \leq u\} = F(u)$, $-\infty < u < \infty$.[16] The x_j's are to have the Poisson distribution with density n, as before. The analysis follows closely that of the previous sections. In place of (12) we find

$$\frac{d}{dz}[(z^2+\lambda)T(z)]+n\int_{-\infty}^{\infty}[T(z+2\kappa_0)-T(z)]dF(\kappa_0)=0,$$

and the only change in (22)–(23) is

$$a(s)=\lambda+n\int_{-\infty}^{\infty}\frac{1-e^{-2i\kappa_0 s}}{is}dF(\kappa_0).$$

The optical approximation is just (28) again, except that

$$\langle\kappa_0\rangle=\int_{-\infty}^{\infty}\kappa_0 dF(\kappa_0)$$

replaces κ_0. Similarly, the diffusion approximation is given by (29), except that the parameters are

$$\epsilon=\frac{n}{\langle\kappa_0^2\rangle^{\frac{1}{2}}},\quad \mu=\left(\frac{3}{\epsilon^2}\right)^{\frac{1}{3}}\frac{\lambda+2n\langle\kappa_0\rangle}{\langle\kappa_0^2\rangle},$$

with

$$\langle\kappa_0^2\rangle=\int_{-\infty}^{\infty}\kappa_0^2 dF(\kappa_0).$$

With the strengths randomized this way, our Eq. (1) is the Schrödinger equation of the one-dimensional version of a scalar meson pair theory Hamiltonian discussed by Montroll and Potts[17] in their study of interactions of lattice defects.

The distribution of atom positions can also be generalized. We use the temporal description of Sec. 3. Let $1, 2, \cdots, \nu$ denote the states of an ergodic Markov chain[18] with transition probabilities p_{ij} and stationary probabilities p_i; that is,

$$p_{ij}\geq 0,\quad \sum_{j=1}^{\nu} p_{ij}=1,\quad i=1,\cdots,\nu$$

$$\sum_{i=1}^{\nu} p_i p_{ij}=p_j,\quad j=1,\cdots,\nu.$$

[16] The case treated in previous sections results when $F(u)=0$ on $-\infty<u<\kappa_0$ and $F(u)=1$ on $\kappa_0\leq u<\infty$.

[17] E. W. Montroll and R. B. Potts, Phys. Rev. **102**, 72 (1956), Eq. (7.1).

[18] See reference 2, Chap. V.

Let n_i denote a hit rate for each state $i=1, 2, \cdots, \nu$. The hit process is determined by the following properties:

(a) Random changes of "state" occur at hits. If the state is i just before a hit, then the probability is p_{ij} that the state is j just after the hit, and the state remains j until the next hit.

(b) If the state is i at time t, the probability that during $(t, t+dt)$ there is

no hit is $1-n_i dt$,
exactly one hit is $n_i dt$,
more than one hit is 0,

neglecting higher order infinitesimals.

(c) The strengths (κ_0) of hits are random variables. At hits where the state changes from i to j, the strength of the hit has (given) probability distribution $P\{\kappa_0\leq u\} = F_{ij}(u)$.

Let $T_i(z)$ denote the stationary joint mixed density of z and i; that is

$$P\{(z\leq\zeta)\ \&\ (\text{state is } i)\}=\int_{-\infty}^{\zeta}T_i(z)dz$$

(at any time). Then, corresponding to (12), we have

$$\frac{d}{dz}[(z^2+\lambda)T_j(z)]-n_jT_j(z)+\sum_{i=1}^{\nu}n_ip_{ij}\times\int_{-\infty}^{\infty}T_i(z+2\kappa_0)dF_{ij}(\kappa_0)=0,\quad j=1,2,\cdots,\nu. \tag{36}$$

The normalization is[19]

$$\int_{-\infty}^{\infty}T_i(z)dz=(p_i/n_i)/\sum_{i=1}^{\nu}(p_j/n_j),\quad i=1,\cdots,\nu, \tag{37}$$

and the η-axis crossing rate is

$$\mathfrak{N}(E)=\lim_{z\to\pm\infty}z^2\sum_{i=1}^{\nu}T_i(z). \tag{38}$$

Even though the methods of Sec. 4 are applicable, solution of (36)–(38) appears to be difficult in any cases of interest.

7. ACKNOWLEDGMENTS

The authors wish to acknowledge interesting and helpful discussions and correspondence with Melvin Lax and James C. Phillips, and with Professor Mark Kac and his student Peter Mengert. The numerical analysis described in Sec. 4 was programmed for the IBM 704 by H. T. O'Neil.

[19] The right-hand side is the fraction of time spent in state i (sojourns in state i having mean duration $1/n_i$); p_i is the numerical frequency of i in long random sequences of states.

APPENDIX 1

It should be clear that the two-dimensional phase process $\{(\xi(t,\omega),\eta(t,\omega)), -\infty<t<\infty\}$ is a Markov process. Let us assume for simplicity that the random variables $\xi(t,\omega)$ and $\eta(t,\omega)$ have at time t a joint probability density $R(\xi,\eta,t)$ in the ξ, η-plane, one smooth enough to insure the validity of the following derivation.[20] We regard $R(\xi,\eta,t)\delta A$ as the fraction of phase points in an ensemble which at time t fall in an element of area δA around the point (ξ,η). As in Sec. 3, we obtain easily

$$R(\xi,\eta,t+dt)\delta A=(1-ndt)R(\xi',\eta',t)\delta A'+(ndt)R(\xi'',\eta'',t)\delta A'',$$

where, for the motion (7),

$$\xi'=\xi-\eta dt,$$
$$\eta'=\eta+\lambda\xi dt,$$
$$\delta A'=\frac{\partial(\xi',\eta')}{\partial(\xi,\eta)}\delta A=\delta A,$$

and for the motion (5)–(6),

$$\xi''=\xi,$$
$$\eta''=\eta+2\kappa_0\xi$$
$$\delta A''=\frac{\partial(\xi'',\eta'')}{\partial(\xi,\eta)}\delta A=\delta A,$$

neglecting higher order differentials throughout. There follows

$$\frac{\partial R(\xi,\eta,t)}{\partial t}=\left(\lambda\xi\frac{\partial}{\partial\eta}-\eta\frac{\partial}{\partial\xi}\right)R(\xi,\eta,t)+n[R(\xi,\eta+2\kappa_0\xi,t)-R(\xi,\eta,t)],\quad(39)$$

subject to the normalization conditions

$$R(\xi,\eta,t)\geq 0,\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R(\xi,\eta,t)d\xi d\eta=1.\quad(40)$$

In contrast to (10)–(11), the system (39)–(40) has no stationary solution.[21] The explanation is that each random orbit spirals out to infinity in the ξ, η-plane, exponentially fast in t, with probability 1. (We prove this assertion in Appendix 2.) In terms of the impurity band model, the amplitude of each random wave function (viz., the quantity $|E[\psi(x;E,\omega)]^2+(\hbar^2/2m)[\psi'(x;E,\omega)]^2|^{\frac{1}{2}}$) increases exponentially fast in x with probability 1, for any E and any initial value and slope.[22]

[20] Both here and in Sec. 3, the use of probability measures (set functions) would give only an essentially trivial gain in generality.

[21] If $T(z)$ satisfies (12), then $R(\xi,\eta,t)=(1/\xi^2)T(\eta/\xi)$ satisfies (39), but is not normalizable.

[22] This is different from the case of periodic potentials, where wave functions are bounded in the allowed bands $[N'(\lambda)>0]$ and unbounded in the forbidden bands $[N'(\lambda)=0]$.

This is reflected in the behavior of the second moments of ξ and η, as follows. Let $M_{\alpha\beta}(t)$ denote the (α,β) moment of (ξ,η):

$$M_{\alpha\beta}(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi^\alpha\eta^\beta R(\xi,\eta,t)d\xi d\eta.$$

If the $r+1$ integrals $M_{\alpha\beta}(t)$ of total weight $\alpha+\beta=r$ are absolutely convergent, then they satisfy

$$\dot{M}_{\alpha\beta}=-\beta\lambda M_{\alpha+1,\beta-1}+\alpha M_{\alpha-1,\beta+1}+n\sum_{\gamma=0}^{\beta-1}\frac{(-2\kappa_0)^{\beta-\gamma}\beta!}{\gamma!(\beta-\gamma)!}M_{\alpha+\beta-\gamma,\gamma},$$
$$\beta=r-\alpha,$$
$$\alpha=0,1,\cdots,r,$$
$$r=0,1,\cdots,$$

obtained by integrating (39) with $\xi^\alpha\eta^\beta d\xi d\eta$.

For $r=1$ we have

$$\dot{M}_{10}=M_{01},$$
$$\dot{M}_{01}=-(\lambda+2n\kappa_0)M_{10},$$

with solution

$$M_{10}(t)=M_{10}(0)\cos\Lambda t+M_{01}(0)\Lambda^{-1}\sin\Lambda t$$
$$M_{01}(t)=-M_{10}(0)\Lambda\sin\Lambda t+M_{01}(0)\cos\Lambda t,$$
$$\Lambda=(\lambda+2n\kappa_0)^{\frac{1}{2}}.$$

We note that the approximate $nN(\lambda)$ given by the optical model, (28), is the η-axis crossing rate of the average phase point $(\langle\xi(t,\omega)\rangle,\langle\eta(t,\omega)\rangle)=(M_{10}(t),M_{01}(t))$.

For $r=2$ there obtains

$$\begin{aligned}\dot{M}_{20}&=2M_{11},\\ \dot{M}_{11}&=-(\lambda+2n\kappa_0)M_{20}+M_{02}\\ \dot{M}_{02}&=4n\kappa_0^2M_{20}-2(\lambda+2n\kappa_0)M_{11}.\end{aligned}\quad(41)$$

Solutions proportional to $e^{\Lambda t}$ exist if Λ satisfies

$$\Lambda^3+4(\lambda+2n\kappa_0)\Lambda-8n\kappa_0^2=0;$$

there is always one positive root, say Λ_1, and roots Λ_2, Λ_3 either both negative or complex conjugates with negative real part. A solution of (41) is

$$M_{20}(t)=\sum k(\Lambda_1,\Lambda_2,\Lambda_3)e^{\Lambda_1 t},$$
$$M_{11}(t)=\sum \tfrac{1}{2}\Lambda_1 k(\Lambda_1,\Lambda_2,\Lambda_3)e^{\Lambda_1 t},$$
$$M_{02}(t)=\sum\left[\frac{\Lambda_1^2}{4}+\frac{2n\kappa_0^2}{\Lambda_1}\right]k(\Lambda_1,\Lambda_2,\Lambda_3)e^{\Lambda_1 t},$$
$$k(\Lambda_1,\Lambda_2,\Lambda_3)=\frac{(8n\kappa_0^2/\Lambda_1)\langle\xi^2\rangle_0+\langle(\Lambda_1\xi+2\eta)^2\rangle_0}{2(\Lambda_1-\Lambda_2)(\Lambda_1-\Lambda_3)},$$

where $\langle\xi^2\rangle_0$ denotes $M_{20}(0)$, etc., and where the summation in each case is over the cyclic permutations of (1,2,3). We observe that the coefficient of $e^{\Lambda_1 t}$ is always nonvanishing in each moment, from which it follows that no stationary solution of (39) could have finite second moments. However, as mentioned previously, we prove a stronger result in Appendix 2.

Rice's formula[23] for the expected η-axis crossing rate at time t is

$$\mathfrak{N}_t=\int_{-\infty}^{\infty} R(0,\eta,t)|\eta|d\eta, \tag{42}$$

and we show now that (16) is but a disguised version of the Rice formula. We introduce coordinates

$$\rho=(\xi^2+\eta^2)^{\frac{1}{2}},\quad z=\eta/\xi,$$

and the marginal density of z, defined as

$$T(z,t)=\int_0^{\infty}\left[R\left(\frac{\rho}{(1+z^2)^{\frac{1}{2}}},\frac{\rho z}{(1+z^2)^{\frac{1}{2}}},t\right)+R\left(\frac{-\rho}{(1+z^2)^{\frac{1}{2}}},\frac{-\rho z}{(1+z^2)^{\frac{1}{2}}},t\right)\right]\frac{\rho d\rho}{1+z^2}; \tag{43}$$

the factor $\rho/(1+z^2)$ is the Jacobian $\partial(\xi,\eta)/\partial(\rho,z)$. From (43) we have, at least formally,

$$\lim_{z\to\pm\infty} z^2T(z,t)=\int_0^{\infty}[R(0,\rho,t)+R(0,-\rho,t)]\rho d\rho,$$

and the right-hand side is Rice's integral (42).

APPENDIX 2

The continuous parameter process $\{z(t,\omega), -\infty<t<\infty\}$ is of a type which seems not to have been treated in much detail in the literature. However, there is associated with it a certain discrete parameter process of familiar type.[1] Let the indexing of the hits be such that $\cdots\leq t_0(\omega)\leq 0<t_1(\omega)\leq\cdots$, and consider the random variables

$$z_0(\omega)=z(0+0,\omega)+2\kappa_0,$$
$$z_j(\omega)=z(t_j(\omega)-0,\omega),\quad j=1,2,\cdots.$$

That is, $z_j(\omega)$ for $j>0$ is the value of $z(t,\omega)$ at the instant before the jth hit. The process $\{z_j(\omega), j=0,1,\cdots\}$ is a Markov process, whose transition probabilities we now ascertain.

We first find the probability $p(\zeta,z)dz$ that a phase point starting at ζ and moving according to $\dot{z}=-(z^2+\lambda)$ receives its next hit when it is in the interval $(z,z+dz)$. The duration θ of the time interval to the next hit has probability density $ne^{-n\theta}$ on $0\leq\theta<\infty$,[8] so that if $\theta(\zeta,z)$ denotes the time required to get from ζ to z, then

$$p(\zeta,z)dz=ne^{-n\theta(\zeta,z)}\left|\frac{\partial\theta(\zeta,z)}{\partial z}\right|dz=\left|\frac{\partial}{\partial z}e^{-n\theta(\zeta,z)}\right|dz. \tag{44}$$

[23] S. O. Rice, Bell System Tech. J. **23**, 282 (1944).

For the motion $\dot{z}=-(z^2+\lambda)$ we have formally

$$\theta(\zeta,z)=-\int_{\zeta}^{z}\frac{dx}{x^2+\lambda}, \tag{45}$$

but some care is required at this point.

In the case $\lambda=k^2>0$ we obtain from (45)

$$\theta(\zeta,z)=\frac{1}{k}\left[\arctan\frac{\zeta}{k}-\arctan\frac{z}{k}\right],$$

where by arctan we must understand a multiple-valued version of the inverse tangent function. In the ξ, η-plane, a phase point may wind around the origin several times without being hit. This corresponds to traversing the whole z-axis several times without hits, the time required to go from $z=+\infty$ to $z=-\infty$ being π/k. If we add the contributions from the various appropriate branches, there results

$$p(\zeta,z)=\frac{u'(z)}{u(\zeta)}\frac{1}{1-e^{-n\pi/k}}\times\begin{Bmatrix}1 & \text{if} & -\infty<z\leq\zeta<\infty\\ e^{-n\pi/k} & \text{if} & -\infty<\zeta<z<\infty\end{Bmatrix}, \tag{46}$$

$$u(z)=e^{(n/k)\arctan(z/k)},$$

where now $|\arctan|<\frac{1}{2}\pi$.

When $\lambda=-\kappa^2<0$ we obtain from (45), formally,

$$\theta(\zeta,z)=\frac{1}{2\kappa}\left\{\log\left|\frac{\zeta-\kappa}{\zeta+\kappa}\right|-\log\left|\frac{z-\kappa}{z+\kappa}\right|\right\},$$

but there are complications because the velocity κ^2-z^2 changes sign at $z=\pm\kappa$, and because a point starting at $\zeta<-\kappa$ may reach $z>\kappa$ by passing $-\infty$. Taking these into account, we have

$$p(\zeta,z)=\left|\frac{u'(z)}{u(\zeta)}\right|\quad\text{if}\quad -\infty<z<\zeta<-\kappa,$$
$$\text{or if}\quad -\kappa<\zeta<z<\kappa,$$
$$\text{or if}\quad \kappa<z<\zeta<\infty,$$
$$\text{or if}\quad -\infty<\zeta<-\kappa\quad\text{and}\quad \kappa<z<\infty,$$
$$=0\quad\text{otherwise},$$

where

$$u(z)=\left|\frac{z-\kappa}{z+\kappa}\right|^{\frac{1}{2}n/\kappa}.$$

A point starting at $\zeta=\pm\kappa$ will stay there until the next hit, but this will occur with probability 0 if ζ has a probability density.

Let $T_j(z)$ denote the probability density of the random variable $z_j(\omega)$; we assume that $T_0(z)$ is given. It should be clear that the $T_j(z)$ satisfy

$$T_{j+1}(z)=\int_{-\infty}^{\infty} T_j(\zeta)\rho(\zeta-2\kappa_0, z)d\zeta, \quad j=0, 1, \cdots.$$

If there is a stationary density $T(z)$ of the z_j's then it satisfies

$$T(z)=\int_{-\infty}^{\infty} T(\zeta+2\kappa_0)\rho(\zeta,z)d\zeta. \tag{47}$$

Using the explicit form of $\rho(\zeta,z)$ given above, it is readily verified that (47) is just the Eq. (19) or (21) satisfied by the $T(z)$ of Sec. 3. [To bring (21) to the form (47) one uses

$$N(-\kappa^2)=\int_{-\infty}^{-\kappa} \frac{T(\zeta+2\kappa_0)}{u(\zeta)}d\zeta, \tag{48}$$

obtained by integrating (17) over $(-\infty, -\kappa)$.]

The z_j process satisfies "Condition (Σ)" of Doob,[24] and the z axis constitutes one ergodic set.[25] Furthermore, there exists a finite measure φ of z sets relative to which the higher-order transition probabilities have a uniformly bounded density. [The transition probability densities relative to Lebesgue measure are given by

$$p^{(1)}(\zeta,z)=\rho(\zeta-2\kappa_0, z),$$

$$p^{(j+1)}(\zeta,z)=\int_{-\infty}^{\infty} p^{(j)}(\zeta,\zeta')p^{(1)}(\zeta',z)d\zeta', \quad j=1, 2, \cdots.$$

For the measure φ of Doob[26] we may use

$$\varphi\{A\}=\int_A \varphi'(z)dz,$$

with

$$\varphi'(z)=1/(1+z^2) \quad \text{if} \quad -(n/2)^2\leq\lambda<\infty,$$

$$\varphi'(z)=\frac{1}{|z-\kappa|^{1-\frac{1}{2}(n/\kappa)}[(z^2+1)^{\frac{1}{2}}]^{1+\frac{1}{2}(n/\kappa)}} \quad \text{if} \quad \kappa>\tfrac{1}{2}n,$$

$$(\lambda=-\kappa^2).$$

For $\lambda>0$ it is trivial that

$$\sup_{\substack{-\infty<z<\infty\\-\infty<\zeta<\infty}} \frac{p^{(1)}(\zeta,z)}{\varphi'(z)}<\infty,$$

and for $\lambda=-\kappa^2<0$, $\kappa\neq\kappa_0$, an extremely tedious calculation shows that

$$\sup_{\substack{-\infty<z<\infty\\-\infty<\zeta<\infty}} \frac{p^{(2)}(\zeta,z)}{\varphi'(z)}\frac{const}{1\kappa-\kappa_0}<\infty;$$

for $\kappa=\kappa_0$ it seems likely that $p^{(3)}(\zeta,z)/\varphi'(z)$ is bounded, but the authors have not attempted to verify this.]

The implication of all this is that there exists a unique normalized solution of (47) [and (12)] given by

$$T(z)=\lim_{j\to\infty} p^{(j)}(\zeta,z).$$

The non-negative limit, independent of ζ, is nonvanishing for every z except possibly $z=\kappa$, and the convergence of $p^{(j)}(\zeta,z)/\varphi'(z)$ to $T(z)/\varphi'(z)$ is exponentially fast in j, uniformly in ζ and z.[26] It is possible to express the solution $T(z,t)$ of (10)–(11) as a functional of $T_0(z)$ and the $p^{(j)}(\zeta,z)$, and to deduce therefrom the limiting behavior of $T(z,t)$. We are not concerned with nonstationary distributions, however, and we omit the details.

Rigorously, the symbol $N(\lambda)$ appearing in various expressions in Secs. 3–5 is to be regarded as the quantity $(1/n)\lim z^2T(z)$. [It is obvious from (14) that the limit exists.] We now give a proof that the limiting t-average η-axis crossing rate exists and is constant with probability 1, the constant being $\lim z^2T(z)$. Let $\gamma_j(\omega)$ denote the number of times that $z(t,\omega)$ reaches $+\infty$ during $t_j(\omega)\leq t<t_{j+1}(\omega)$. [We redefine $t_0(\omega)$ as $t_0(\omega)\equiv0$.] When $\lambda=-\kappa^2$ we have $\gamma_j(\omega)=\gamma(z_j(\omega), z_{j+1}(\omega))$, with

$$\begin{aligned}\gamma(\zeta,z)&=1 \quad \text{if} \quad \zeta<-\kappa+2\kappa_0 \quad \text{and} \quad z>\kappa\\ &=0 \quad \text{otherwise.}\end{aligned}$$

When $\lambda=k^2$, the random variables $\gamma_0(\omega)$, $\gamma_1(\omega)$, $\cdots$ are conditionally independent given all of the $z_k(\omega)$, $k=0, 1, \cdots$, and the conditional distribution of each $\gamma_j(\omega)$ is of geometric type with ratio $e^{-n\pi/k}$ [see (44)–(46)]. The conditional expectation of $\gamma_j(\omega)$ given $z_0(\omega)$, $z_1(\omega)$, $\cdots$ is

$$\langle\gamma_j(\omega)\,|\,z_0(\omega),z_1(\omega),\cdots\rangle=\gamma(z_j(\omega),z_{j+1}(\omega)),$$

where now

$$\begin{aligned}\gamma(\zeta,z)&=\frac{e^{-n\pi/k}}{1-e^{-n\pi/k}} \quad \text{if} \quad z\leq\zeta-2\kappa_0\\ &=\frac{1}{1-e^{-n\pi/k}} \quad \text{if} \quad z>\zeta-2\kappa_0.\end{aligned}$$

The random variable

$$\mathfrak{M}(j,\omega)=\frac{1}{j}\sum_{k=0}^{j-1}\gamma_j(\omega)$$

is the average number of η-axis crossings per hit after j hits, and a slightly modified version of a well-known

[24] See reference 2, p. 195.

[25] That is, there exists an integer ν such that for every ζ and every set Z of positive Lebesgue measure, $P\{z_\nu(\omega)\epsilon Z|z_0(\omega)=\zeta\}>0$. For $\lambda>0$ one may take $\nu=1$; for $\lambda=-\kappa^2$ any $\nu>(\kappa/\kappa_0)+1$ will do [see Sec. 5(c)].

[26] See reference 2, pp. 215–217.

ergodic theorem[27] gives

$$\lim_{j\to\infty} \mathfrak{M}(j,\omega)=\langle\gamma(z_k(\omega),z_{k+1}(\omega))\rangle \quad \text{with probability 1,}$$

where the expectation on the right-hand side is to be taken with respect to the stationary distribution of the z_j's. Since the stationary joint density of z_k and z_{k+1} is $T(z_k)\rho(z_k-2\kappa_0,\, z_{k+1})$, we have

$$\lim_{j\to\infty} \mathfrak{M}(j,\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(\zeta)\rho(\zeta-2\kappa_0,\, z)\gamma(\zeta,z)d\zeta dz \quad \text{with probability 1.} \tag{49}$$

Let now $N(L,\omega)$ denote the number of hits in a time interval $0\leq t<L$, and let $\nu_L(\omega)$ denote the number of η-axis crossings during $0\leq t<L$. Since

$$N(L,\omega)\mathfrak{M}(N(L,\omega),\omega)\leq\nu_L(\omega) \leq[N(L,\omega)+1]\mathfrak{M}(N(L,\omega)+1,\,\omega),$$

we have for the number of crossings per unit time,

$$\lim_{L\to\infty}\frac{\nu_L(\omega)}{L}=\lim_{L\to\infty}\left[\frac{N(L,\omega)}{L}\cdot\mathfrak{M}(N(L,\omega),\omega)\right] =n\lim_{j\to\infty}\mathfrak{M}(j,\omega) \quad \text{with probability 1,}$$

using $\lim N(L,\omega)/L=n$ and $\lim N(L,\omega)=\infty$ with probability 1 as $L\to\infty$. Using the explicit forms given for $\rho(\zeta,z)$ and $\gamma(\zeta,z)$, it is easily verified that the integral on the right-hand side in (49) is just the integral on the right-hand side in (20) or (48). We have thus proved

$$\lim_{L\to\infty}\frac{\nu_L(\omega)}{L}=\lim_{z\to\pm\infty} z^2T(z) \quad \text{with probability 1.}$$

Similar methods show that the (ξ,η) process diverges radially with probability 1. Consider the random variable $\sigma(t,\omega)=|\lambda\xi^2(t,\omega)+\eta^2(t,\omega)|$. For given ω, $\sigma(t,\omega)$ is constant between hits, from (7), while at hits we have (suppressing ω for a moment)

$$\frac{\sigma(t_j+0)}{\sigma(t_j-0)}=\frac{|\lambda\xi^2(t_j)+[\eta(t_j-0)-2\kappa_0\xi(t_j)]^2|}{|\lambda\xi^2(t_j)+\eta^2(t_j-0)|} =\frac{|\lambda+(z_j-2\kappa_0)^2|}{|\lambda+z_j^2|}$$

in terms of the random variables z_j previously defined. Assuming that $\sigma(0+0)\neq0$ holds with probability 1, we have

$$\frac{1}{j}\log\frac{\sigma(t_j+0)}{\sigma(0+0)}=\frac{1}{j}\sum_{k=1}^{j}\log\frac{|\lambda+(z_k-2\kappa_0)^2|}{|\lambda+z_k^2|},$$

using $\sigma(t_k-0)=\sigma(t_{k-1}+0)$. The ergodic theorem gives

$$\begin{aligned}\beta&\equiv\lim_{j\to\infty}\frac{1}{j}\log\frac{\sigma(t_j+0)}{\sigma(0+0)}\\ &=\int_{-\infty}^{\infty}T(z)\log\frac{|\lambda+(z-2\kappa_0)^2|}{|\lambda+z^2|}dz \quad \text{with probability 1}\\ &=\int_{-\infty}^{\infty}[T(z+2\kappa_0)-T(z)]\log|\lambda+z^2|\,dz\\ &=\frac{2}{n}\int_0^{\infty} z[T(z)-T(-z)]dz.\end{aligned}$$

[We have used (12) and (14) to obtain the last expression.] It follows that for each $\epsilon>0$ and each ω in a set of probability 1 there exists an integer $J(\epsilon,\omega)$ such that

$$\sigma(t_j(\omega)+0,\,\omega)>\sigma(0+0,\,\omega)e^{j(\beta-\epsilon)} \quad \text{for } every \quad j>J(\epsilon,\omega).$$

As we prove in Appendix 3, the number β is always positive; we choose some $\epsilon<\beta$, and our result follows.

APPENDIX 3

We define the function $\varphi_Z(s)$ as

$$\varphi_Z(s)=\int_{-Z}^{Z}e^{isz}T(z)dz,$$

noting that $\varphi_Z(s)$ converges uniformly in s to $\varphi(s)$ as $Z\to\infty$, and that $\varphi_Z(s)$ is an entire function of s for any fixed finite Z. (Here and throughout, s is real.) We multiply (12) by $e^{isz}dz$ and integrate over $(-Z,Z)$; integration by parts and rearrangement gives

$$\begin{aligned}&e^{isZ}(Z^2+\lambda)T(Z)-e^{-isZ}(Z^2+\lambda)T(-Z)\\ &\qquad+n\left(\int_{-Z+2\kappa_0}^{Z+2\kappa_0}e^{-2is\kappa_0}-\int_{-Z}^{Z}\right)e^{isz}T(z)dz\\ &=is\int_{-Z}^{Z}e^{isz}(z^2+\lambda)T(z)dz\\ &=is\left[\lambda-\frac{d^2}{ds^2}\right]\varphi_Z(s).\end{aligned}$$

For the boundary terms we substitute values given by (14), viz.,

$$(Z^2+\lambda)T(\pm Z)=n\left[N(\lambda)-\int_{\pm Z}^{\pm Z+2\kappa_0}T(z)dz\right],$$

and rearrange to obtain

$$\frac{d^2\varphi_Z(s)}{ds^2}=a(s)\varphi_Z(s)-nI(s)-2nN(\lambda)\frac{\sin sZ}{s}, \tag{50}$$

[27] Reference 2, pp. 218–232, 464–469.

where

$$I(s)=\int_Z^{Z+2\kappa_0}\frac{e^{is(z-2\kappa_0)}-e^{isZ}}{is}T(z)dz$$

$$-(\text{expression with } Z\rightarrow -Z),$$

and where $a(s)$ is given in (22). We have

$$|I(s)|\leq\int_Z^{Z+2\kappa_0}\left|\frac{2\sin\frac{1}{2}s(z-2\kappa_0-Z)}{s}\right|T(z)dz$$

$$+(\text{expression with } Z\rightarrow -Z)$$

$$\leq 2\kappa_0\left(\int_Z^{Z+2\kappa_0}+\int_{-Z}^{-Z+2\kappa_0}\right)T(z)dz,$$

whence $I(s)$ vanishes uniformly in s as $Z\rightarrow\infty$. We now integrate (50) twice from some fixed s_0:

$$\varphi_Z(s)=\varphi_Z(s_0)+(s-s_0)\varphi_Z'(s_0)$$

$$+\int_{s_0}^{s}\int_{s_0}^{s'}[a(s'')\varphi_Z(s'')-nI(s'')]ds''ds'$$

$$-2nN(\lambda)\int_{s_0}^{s}\int_{s_0}^{s'}\frac{\sin s''Z}{s''}ds''ds'. \quad (51)$$

As $Z\rightarrow\infty$, the first and third terms on the right-hand side of (51) converge to obvious limits, from the uniform convergence properties noted previously. It is easy to show that the fourth term has a limit as $Z\rightarrow\infty$, expressible in terms of the Dirichlet integral. Since $\varphi_Z(s)$ itself converges to $\varphi(s)$, it follows that the second term on the right-hand side of (51) also converges to a limit as $Z\rightarrow\infty$, of the form $(s-s_0)\times$(function of s_0). {We do not need the fact that the limit is actually $(s-s_0)[\varphi'(s_0+0)+\varphi'(s_0-0)]/2$.} Thus, passing to the limit in (51),

$$\varphi(s)=\varphi(s_0)+(s-s_0)\times(\text{function of } s_0)$$

$$+\int_{s_0}^{s}\int_{s_0}^{s'}a(s'')\varphi(s'')ds''ds'$$

$$-\pi nN(\lambda)s[\operatorname{sgn}s-\operatorname{sgn}s_0], \quad (52)$$

where $\operatorname{sgn}x=x/|x|$ if $x\neq 0$, $=0$ if $x=0$.

One differentiation shows that $\varphi'(s)$ is continuous except at $s=0$, where we have

$$\operatorname{Re}\varphi'(0+0)=-\operatorname{Re}\varphi'(0-0)=-\pi nN(\lambda)$$

and $\operatorname{Im}\varphi'(0+0)=\operatorname{Im}\varphi'(0-0)$. A second differentiation gives (22).

The quantity $\operatorname{Im}\varphi'(0)$ is of some interest, since

$$\operatorname{Im}\varphi'(0)=\int_0^\infty z[T(z)-T(-z)]dz, \quad (53)$$

and this is $\frac{1}{2}n\beta$ of Appendix 2. The following proof of (53) is a slightly modified version of one due to Mengert.[28] We have

$$\operatorname{Im}\varphi(s)=\int_0^\infty [T(z)-T(-z)]\sin sz dz, \quad (54)$$

whence, formally,

$$\operatorname{Im}\varphi'(s)=\int_0^\infty z[T(z)-T(-z)]\cos sz dz. \quad (55)$$

However, it is not hard to show from (15) that $z^4[T(z)-T(-z)]$ is bounded as $z\rightarrow\infty$. It follows that the integral on the right-hand side in (55) is uniformly absolutely convergent, and hence that (55) is valid for every s; for $s=0$ we have (53).

To prove that $\operatorname{Im}\varphi'(0)$ is positive, we multiply (22) by $\bar{\varphi}(s)$ and subtract the complex conjugate equation, obtaining

$$\frac{d}{ds}\operatorname{Im}[\varphi'(s)\bar{\varphi}(s)]=-\frac{n(1-\cos 2\kappa_0 s)}{s}|\varphi(s)|^2.$$

Integrating over $(0,\infty)$, we have

$$\operatorname{Im}\varphi'(0)=n\int_0^\infty\frac{1-\cos 2\kappa_0 s}{s}|\varphi(s)|^2ds,$$

using $\bar{\varphi}(0)=1$ and the fact that $\bar{\varphi}(s)\varphi'(s)\rightarrow 0$ as $s\rightarrow\infty$ (proved in the next paragraph). Since $\varphi(s)$ is continuous and $\varphi(0)=1$, it is clear that $\operatorname{Im}\varphi'(0)>0$.

As the Fourier transform of an integrable function, $\varphi(s)$ vanishes as $s\rightarrow\infty$, by the Riemann-Lebesgue theorem. However, the formal derivative of the Fourier integral for $\varphi(s)$ is not convergent, so we proceed as follows. We have, for any fixed $Z>0$, $Z^2+\lambda>0$,

$$\varphi(s)=\int_{-Z}^{Z}e^{isz}T(z)dz+\left(\int_{-\infty}^{-Z}+\int_Z^\infty\right)e^{isz}T(z)dz$$

$$=\int_{-Z}^{Z}e^{isz}T(z)dz+nN(\lambda)\left(\int_{-\infty}^{-Z}+\int_Z^\infty\right)\frac{e^{isz}}{z^2+\lambda}dz$$

$$-n\left(\int_{-\infty}^{-Z}+\int_Z^\infty\right)\frac{e^{isz}}{z^2+\lambda}\int^{z+2\kappa_0}T(x)dxdz,$$

using (15). The first and third terms can now be differentiated under the integral sign, the integrals remaining absolutely convergent after differentiation. According to the Riemann-Lebesgue theorem, these contributions to $\varphi'(s)$ vanish as $s\rightarrow\infty$. The second term can be evaluated explicitly in terms of the exponential integral $\int(e^{ix}/x)dx$, and it is not hard to

[28] Peter Mengert (private communication).

verify that the derivative of the second term also vanishes as $s\to\infty$, whence, finally, $\varphi'(s)\to 0$ as $s\to\infty$.

To investigate the asymptotic behavior of $\varphi(s)$, we bring (22) to the Riccati form

$$y'(s)+y^2(s)=\lambda+n\frac{1-e^{-2i\kappa_0 s}}{is},\quad s\neq 0,$$

with the substitution

$$\varphi(s)=\exp\left(\int_0^s y(s')ds'\right);$$

the boundary condition $\varphi(\infty)=0$ becomes

$$\lim_{s\to\infty}\int_0^s \mathrm{Re}\,y(s')ds'=-\infty. \tag{56}$$

Assuming that $y(s)$ can be expanded according to powers of n,

$$y(s)=y_0(s)+ny_1(s)+n^2y_2(s)+\cdots, \tag{57}$$

we find

$$y_0'+y_0^2=\lambda,$$

$$y_1'+2y_0y_1=\frac{1-e^{-2i\kappa_0 s}}{is},$$

$$\vdots\ .$$

These have as solutions

$$y_0(s)=\frac{h'(s)}{h(s)}$$

$$y_1(s)=\frac{1}{h^2(s)}\int^s h^2(s')\frac{1-e^{-2i\kappa_0 s'}}{is'}ds', \tag{58}$$

$$\vdots\ ,$$

where $h(s)=ae^{cs}+be^{-cs}$, $c^2=\lambda$, and a and b are arbitrary constants. The constants and limits of integration will be chosen so that not only is (56) satisfied but also $y_j(\infty)=0$, $j>0$.

For $\lambda=k^2>0$ ($c=k$, $a=0$) there results

$$y_0(s)=-k$$

$$y_1(s)=-\int_s^\infty e^{-2k(s'-s)}\frac{1-e^{-2i\kappa_0 s'}}{is'}ds',$$

$$\vdots\ ,$$

while for $\lambda=-\kappa^2<0$, $\kappa\neq\kappa_0$ ($c=i\kappa$, $b=0$) we have

$$y_0(s)=i\kappa,$$

$$y_1(s)=-\int_s^\infty e^{2i\kappa(s'-s)}\frac{1-e^{-2i\kappa_0 s'}}{is'}ds'$$

$$=-\frac{1}{2\kappa s}+(\text{function integrable at infinity}),$$

$$\vdots\ .$$

In the exceptional case $\kappa=\kappa_0$ the boundary conditions require $2b=a$, giving

$$y_0(s)=i\kappa_0\frac{2e^{i\kappa_0 s}-e^{-i\kappa_0 s}}{2e^{i\kappa_0 s}+e^{-i\kappa_0 s}},$$

$$y_1(s)=\frac{1}{(2e^{i\kappa_0 s}+e^{-i\kappa_0 s})^2}\int_s^\infty\frac{e^{-4i\kappa_0 s'}+3e^{-2i\kappa_0 s'}-4e^{2i\kappa_0 s'}}{is'}ds'$$

$$=-\frac{1}{2\kappa_0 s}+(\text{function integrable at infinity}),$$

$$\vdots\ .$$

In the numerical calculation described in Sec. 4, $y_0(s)+ny_1(s)$ was used as an approximation to $y(s)$. The proof that the expansion $y(s)=y_0(s)+ny_1(s)+\cdots$ is asymptotic in s in the usual sense appears to be difficult.