Cartesian Tensor Scalar Product and Spherical Harmonic Expansions in Boltzmann's Equation

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The electron or ion distribution function, expanded in a sum of spherical harmonics, is shown to be equivalent to a tensor scalar product expansion. As such, it can be readily substituted into the Boltzmann equation to give transport equations integrated over angle and also the necessary equations for determination of the expansion coefficients. This has been done for terms up to order three, the order of the pressure transport tensor.

I. TENSOR SCALAR PRODUCT EXPANSION OF THE DISTRIBUTION FUNCTION

Introduction

THE statement has been made that the expansion of the distribution function in Legendre polynomials is equivalent to an expansion in a vector form as

$$f=\sum_l f_l P_l(\cos\theta)=f_0+\frac{\mathbf{f}_1\cdot\mathbf{v}}{v}+\cdots. \tag{1}$$

This equation is limited to one dimension by the use of Legendre polynomials only on the left hand side. The more general and true statement is that the expansion of the distribution function in spherical harmonics Y_{lms} is equivalent to a Cartesian tensor scalar product expansion

$$f=\sum_{l,m,s} f_{lms}(v,r,t)Y_{lms}(\theta,\phi)=\sum_l \frac{\{\mathbf{f}_l\}\vdots_l\{\mathbf{v}^l\}}{v^l} \tag{2}$$

term by term, i.e., that

$$\sum_{m,s} f_{lms}Y_{lms}=\frac{\{\mathbf{f}_l\}\vdots_l\{\mathbf{v}^l\}}{v^l}. \tag{3}$$

Here

$$Y_{lms}(\theta,\phi)=\sin\theta P_l^m(\cos\theta)(\delta_{0s}\cos m\phi+\delta_{1s}\sin m\phi),$$

with the orthogonality relationship

$$\int Y_{lms}Y_{l'm's'}d^2\Omega=\int_0^{2\pi}\int_0^{\pi} Y_{lms}Y_{l'm's'}\sin\theta d\theta d\phi =\frac{2\pi}{2l+1}\frac{(l+m)!}{(l-m)!}(1+\delta_{0m}).$$

Also $\{\mathbf{f}_l\}$ is a symmetric lth order Cartesian tensor, and

$$\{\mathbf{v}^l\}=\mathbf{v}^i\mathbf{v}^j\cdots\mathbf{v}^k,\quad (l\text{ terms})$$

the lth order Cartesian tensor formed by the Cartesian components $\mathbf{v}^i$ of $\mathbf{v}$ along the 3 or z (polar), 1 or $x(\phi)$, and 2 or y axes, and

$$\{\mathbf{f}_l\}\vdots_l\{\mathbf{v}^l\}=\sum_{ijk} f_{(l)\cdots kji}\frac{v^i}{v}\frac{v^j}{v}\frac{v^k}{v}\cdots =\sum f_{(l)\cdots ji}\cos\theta^i\cos\theta^j\cdots,$$

where i, j, k, etc., may each refer to the x, y, or z axes. Later other letters will be used in the same way.

The f_l tensors are, of course, completely symmetric, but the components also have additional restrictions, because while v^x, v^y, v^z are independent, the $\cos\theta^i$'s (the velocity direction cosines) are not, since the sum of their squares is one. Because of the orthogonality of the spherical harmonics, they form the unique angle expansion for the distribution function, and in order to justify and discuss the tensor scalar (or dot) product expansion, it must be determined whether the tensor dot product expression can be converted to the spherical harmonic expansion, or vice versa.

The reverse proves somewhat easier. Consider the expansion

$$f=\sum_{l,m,s} f_{lms}Y_{lms}.$$

If we convert each Y_{lms} into a sum or products of l direction cosines, then we can take these product terms and write them as a result of a tensor dot product.

If we denote by f_{pqr} the value (all the same) of the elements in the f_l tensor with p x-subscripts, q y-subscripts, r z-subscripts, $(l=p+q+r)$ then, if

$$Y_{lms}=\sum_{pqr}\alpha_{pqr}{}^{lms}\cos^p\theta^x\cos^q\theta^y\cos^r\theta^z, \tag{4}$$

we can write

$$\sum_{m,s} f_{lms}Y_{lms}=\frac{\{f_l\}\vdots_l\{v^l\}}{v^l}, \tag{5}$$

with $\{f_l\}$ a fully symmetric tensor whose elements are

$$f_{(l)ijk\cdots}=f_{pqr}=\frac{p!q!r!}{(p+q+r)!}\sum_{m,s}\alpha_{p,q,r}{}^{l,m,s}f_{lms}. \tag{5a}$$

The expansion of Y_{lms} in lth order products of direction cosines [i.e. (4)] is a very easy matter.

Expansion of Spherical Harmonics in Direction Cosine Products

The direction cosines themselves are given by:

$$\cos\theta^x=\sin\theta\cos\phi,\quad \cos\theta^y=\sin\theta\sin\phi,\quad \cos\theta^z=\cos\theta, \tag{6}$$

so that

$$Y_{lms}=\frac{(2l)!}{2^l l!(l-m)!}\left[\cos^{l-m}\theta-\frac{(l-m)(l-m-1)}{2(2l-1)}\cos^{l-m-2}\theta+\cdots\right]\sin^m\theta(\delta_{0s}\cos m\phi+\delta_{1s}\sin m\phi)$$

$$=\frac{(2l)!}{2^l l!(l-m)!}[\cos^{l-m}\theta-\cdots]\sin^m\theta\Big[\delta_{0s}\sum_{n=0}^{m/2;(m-1)/2}(-1)^n\,{}^mC_{2n}\cos^{m-2n}\phi\sin^{2n}\phi$$
$$+\delta_{1s}\sum_{n=0}^{(m/2)-1;(m-1)/2}(-1)^n\,{}^mC_{2n+1}\cos^{m-2n-1}\phi\sin^{2n+1}\phi\Big],$$

where Moivre's theorem has been used to expand the $m\phi$ terms. In direction cosines this is

$$Y_{lms}=\frac{(2l)!}{2^l l!(l-m)!}\left[\cos^{l-m}\theta^z-\frac{(l-m)(l-m-1)}{2(2l-1)}\cos^{l-m-2}\theta^z(\cos^2\theta^x+\cos^2\theta^y+\cos^2\theta^z)+\cdots\right]$$
$$\times\Big[\delta_{0s}\sum_{n=0}^{m/2;(m-1)/2}(-1)^n\,{}^mC_{2n}\cos^{m-2n}\theta^x\cos^{2n}\theta^y+\delta_{1s}\sum_{n=0}^{(m/2)-1;(m-1)/2}(-1)^n\,{}^mC_{2n+1}\cos^{m-2n-1}\theta^x\cos^{2n+1}\theta^y\Big],\quad(8)$$

where the terms in $\cos^{l-m-2g}\theta$, etc., have been, so to speak, filled out by multiplying by $(\cos^2\theta^x+\cos^2\theta^y+\cos^2\theta^z)^g=1$, to give l terms in each product.

Our expansion of (4) has been carried out, and so we can write the f_l tensors in terms of f_{lms} coefficients. (It can be readily seen that for all Y_{lms} of given l, we will use all lth-order cosine products.) As a result of the filling-out operation, the lth-order tensor contains f_{lms} terms of order l only, which makes for considerable simplicity. (It is evident that there is a certain degree of arbitrariness in how the tensors are constructed from the f_{lms} terms since one can go on multiplying by $(\cos^2\theta^x+\cos^2\theta^y+\cos^2\theta^z)=1$. The choice here seems to be the simplest and leads to quite elegant results.)

With this choice of the Y_{lms} expressions in $\cos\theta^i$'s the reverse process of converting the direction cosine products to spherical harmonic sums is very easy if approached in the right fashion.

Expansion of Direction Cosine Products in Spherical Harmonics

Initially the outlook seems unpromising, for it seems that there is a gap between the number of different $\cos\theta^i$ products of order l and the number of spherical harmonics of order l. The number of products is the same as the number of independent elements in the completely symmetric l tensor in three-dimensional space, namely $(l+1)(l+2)/2$, $((l+n-1)!/l!(n-1)!$ in n-dimensional space) while the number of lth-order spherical harmonics is only $2l+1$. The difficulty vanishes when it is realized that one needs to use all the lower order spherical harmonics of the same parity as l (i.e., even or odd as l is even or odd), as well as the lth-order harmonics. It is readily verified that the addition of these harmonics supplies the number of terms one needs, since

for $l=2\lambda$;

$$\sum_{\lambda=0}^{\lambda=\frac{1}{2}l}(2l+1)=\sum 4\lambda+1=(\lambda+1)1+4\frac{\lambda(\lambda+1)}{2}$$
$$=(\lambda+1)(2\lambda+1)=\frac{(l+1)(l+2)}{2},$$

for $l=2\lambda+1$;

$$\sum_{\lambda=0}^{\lambda=\frac{1}{2}l-1}(2l+1)=\sum 4\lambda+3=(\lambda+1)3+4\frac{\lambda(\lambda+1)}{2}$$
$$=\frac{(\lambda+1)(2\lambda+3)}{2}=\frac{(l+1)(l+2)}{2}.$$

The introduction of these lower order harmonics is carried out by filling out the lower order spherical harmonics (expressed in direction cosines) by repeated multiplications with the sum of the squares of the cosines, to reach order l.

When this has been done there are $(l+1)(l+2)/2$ linear equations expressing all Y_{lms}'s of the same parity as l and of order l or lower, in terms of the $(l+1)(l+2)/2$ independent direction cosines products, all products of order l. To find the inverse set of equations we invert the matrix of the set of equations and thus express all $\cos\theta^i$ products of order l in terms of spherical harmonics of the same parity in l and of order l or less.

An example to make this process clear is the following for $l=2$. We write:

$Y_{000}=1=\cos^2\theta^x+\cos^2\theta^y+\cos^2\theta^z$,
(Note the "filling out" of $Y_{000}=1$.)

$Y_{200}=\frac{1}{2}(3\cos^2\theta-1)=\cos^2\theta^z-\frac{1}{2}\cos^2\theta^x-\frac{1}{2}\cos^2\theta^y$,

$Y_{220}=3\sin^2\theta\cos 2\phi=3(\cos^2\theta^x-\cos^2\theta^y)$,

$Y_{210}=3\sin\theta\cos\theta\cos\phi=3\cos\theta^z\cos\theta^x$,

$Y_{211}=3\sin\theta\cos\theta\sin\phi=3\cos\theta^z\cos\theta^y,$

$Y_{221}=3\sin^2\theta\sin 2\phi=6\cos\theta^x\cos\theta^y.$

A set of equations with six even Y's of order 2 and 0 and six distinct $\cos\theta^i$ products (three squares, three cross products). If we solve this set of equations for the $\cos\theta^i$ products or invert the matrix, we obtain:

$$\cos^2\theta^3=\cos^2\theta^z=\tfrac{1}{3}Y_{000}+\tfrac{2}{3}Y_{200},$$
$$\cos^2\theta^1=\cos^2\theta^x=\tfrac{1}{3}Y_{000}-\tfrac{1}{3}Y_{200}+\tfrac{1}{6}Y_{220},$$
$$\cos^2\theta^2=\cos^2\theta^y=\tfrac{1}{3}Y_{000}-\tfrac{1}{3}Y_{200}-\tfrac{1}{6}Y_{220},$$
$$\cos\theta^1\cos\theta^3=\cos\theta^x\cos\theta^z=\tfrac{1}{3}Y_{210},$$
$$\cos\theta^2\cos\theta^3=\cos\theta^y\cos\theta^z=\tfrac{1}{3}Y_{211},$$
$$\cos\theta^1\cos\theta^2=\cos\theta^x\cos\theta^y=\tfrac{1}{6}Y_{221},$$

i.e., $\cos^p\theta^x\cos^q\theta^y\cos^r\theta^z=\sum_{l',m,s}\beta_{pqr}{}^{l'ms}Y_{l'ms}$ where $l'=l-2n$ (n integral) and $p+q+r=l$. This operation has been carried out for $l=0$, 1, 2, 3, 4, and the matrices have been determined.[1] The $l=0$ matrices are 1 and the $l=1$ matrices are three by three unit diagonal matrices. The $l=2$ matrices can be written down by inspection from the examples, and the $l=3$ and $l=4$ matrices are available in reference 1 or from the American Documentation Institute.[2] The considerable simplicity of the zero and first order equations as developed by various authors is really a result of the simplicity of the matrices for $l=0$, 1.

Tensor Elements Expressed in Spherical Harmonic Coefficients

Having shown the equivalence of the expressions, the next step is to unite them. Since it is not proposed to carry any fourth order terms in the actual expansion, only the tensors up to rank three are given from the α matrices and (5a).

$$f_0=f_0=f_{000}, \tag{9a}$$

$$\mathbf{f}_1=f_1=f_{110}\mathbf{i}_x+f_{111}\mathbf{i}_y+f_{100}\mathbf{i}_z, \tag{9b}$$

$$\{\mathbf{f}_2\}=\begin{matrix}\mathbf{i}_x\\ \mathbf{i}_y\\ \mathbf{i}_z\end{matrix}\begin{bmatrix}-\tfrac{1}{2}f_{200}+3f_{220} & 3f_{221} & \tfrac{3}{2}f_{210}\\ 3f_{221} & -\tfrac{1}{2}f_{200}-3f_{220} & \tfrac{3}{2}f_{211}\\ \tfrac{3}{2}f_{210} & \tfrac{3}{2}f_{211} & f_{200}\end{bmatrix}, \tag{9c}$$

$\{\mathbf{f}_3\}$ Here the independent elements will be listed only, because of the difficulty of representing the array on the flat page.

$$f_{xxx}=-\tfrac{3}{2}f_{310}+15f_{330},\ f_{yyy}=-\tfrac{3}{2}f_{311}-15f_{331},\ f_{zzz}=f_{300},$$
$$f_{x^2y}=-\tfrac{1}{2}f_{311}+15f_{331},\ f_{xy^2}=-\tfrac{1}{2}f_{310}-15f_{330},\ f_{xz^2}=2f_{310},$$
$$f_{x^2z}=-\tfrac{1}{2}f_{300}+5f_{320},\quad f_{y^2z}=-\tfrac{1}{2}f_{300}-5f_{320},\quad f_{yz^2}=2f_{311},$$
$$f_{xyz}=5f_{321}. \tag{9d}$$

[1] T. W. Johnston, Research Report (7–801,6) RCA Victor Research Laboratory, Montreal, Canada (unpublished).

[2] The matrices have also been deposited as Document No. 6415 with the ADI Auxiliary Publications project, Photoduplication Service, Library of Congress, Washington 25, D. C. A copy may be secured by citing the Document number and by remitting $1.25 for photoprints or $1.25 for 35-mm microfilm. Advance payment is required. Make checks payable to: Chief, Photoduplication Service, Library of Congress.

Velocity Averages Expressed in Tensors

The next step is to ask how average quantities may be represented in this system. The results are very simple and can be readily seen to be:

For a scalar,

$$\langle\phi\rangle=\frac{4\pi}{n}\int\phi f_0v^2dv, \tag{10a}$$

where

$$n=4\pi\int f_0v^2dv.$$

For a vector,

$$\langle\mathbf{Q}\rangle=Q(v)\mathbf{v}/v,$$
$$\mathbf{Q}=\frac{4\pi}{3n}\int\mathbf{f}_1Qv^2dv. \tag{10b}$$

For a dyad,

$$\{\mathbf{QQ}\}=Q^2(v)\{\mathbf{vv}\}/v^2,$$
$$\langle\{\mathbf{QQ}\}\rangle=\frac{4\pi}{3n}\int f_0\{\mathbf{I}_2\}Q^2v^2dv+4\pi\frac{2}{3\times5}\int\{\mathbf{f}_2\}Q^2v^2dv. \tag{10c}$$

For a triad,

$$\{\mathbf{QQQ}\}=Q^3(v)\{\mathbf{vvv}\}/v^3,$$
$$\langle\{\mathbf{QQQ}\}\rangle=\frac{4\pi}{5n}\int[(\mathbf{f}_1\cdot\mathbf{f}_1)\cdot\mathbf{f}_1\{\mathbf{I}_2\}]_3Q^3v^2dv$$
$$+\frac{4\pi}{n}\frac{2\times3}{3\times5\times7}\int\{\mathbf{f}_3\}Q^3v^2dv,$$

where $[\ \ ]_l$ means permute **ijk** for each element in all ways ($l!$ ways), add the result and divide by $l!$ (to produce a completely symmetric lth-order tensor), and $\{\mathbf{I}_l\}$ is the unit diagonal lth-order tensor, $\langle\ \ \rangle$ denotes velocity average.

These averages can be derived either by expressing the direction cosines of the polyadic tensors in spherical harmonics and using the orthogonality relation, or by using the tensor expression and the integration formula

$$\int\cos^p\theta^x\cos^q\theta^y\cos^r\theta^z d^2\Omega$$
$$=\frac{[1+(-1)^p][1+(-1)^q][1+(-1)^r]}{4}$$
$$\times\frac{\Gamma(\tfrac{1}{2}p+\tfrac{1}{2})\Gamma(\tfrac{1}{2}q+\tfrac{1}{2})\Gamma(\tfrac{1}{2}r+\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}p+\tfrac{1}{2}q+\tfrac{1}{2}r+\tfrac{3}{2})}, \tag{11a}$$

i.e., a nonzero result only for p, q, r all even, which is

$$\int \cos^p\theta^x \cos^q\theta^y \cos^r\theta^z d^2\Omega = \frac{4\pi 1\times 1\times 3\times 5\cdots(p-1)1\times 1\times 3\times 5\cdots(q-1)1\times 1\times 3\times 5\cdots(r-1)}{1\times 3\times 5\cdots(p+q+r+1)}. \tag{11b}$$

Both, of course, give the same answer. In this case the spherical harmonic expansion is slightly easier to apply.

Because of the identity of α and β for $l=0$, 1, the first two averages (10a) and (10b) are the same as those given by Allis,[3] while (10c) has been given by Delcroix.[4] The application of these averages to pressure tensors, pressure transport tensors, etc., is obvious.

Applying the Tensor Form to Boltzmann's Equation

This polyadic dot product formulation allows a sidestep around the problem of solving Boltzmann's equation by expressing the distribution function in spherical harmonics. One substituted the expansion into Boltzmann's equation, attempted to juggle the result until the only angle terms (including the collision term) were all linear combinations of spherical harmonics, and then, using the orthogonality relationship one was to obtain a set of $(l+1)^2$ independent equations to solve for the $(l+1)^2$ unknown f_{lms}'s in terms of collision coefficients and the like. The difficulty that hindered this program beyond the first order (i.e., $\mathbf{f}_1$) has always been the presence of terms such as $\cos\theta[\partial P_{lm}(\cos\theta)/\partial\theta]$, for which no simple recursion relations exist. We can sidestep this difficulty by using the Cartesian polyadic tensor distribution function.

Previous consideration of more or less limited cases of spherical harmonic expansion for first order have been treated by Allis[3] and various investigators at Yale[5–8] under Margenau. The author has previously treated[9] the general case of Boltzmann's equation up to the first order. It is believed that the present paper is the first explicit treatment of second and third order terms in the Boltzmann equation using spherical harmonics. Delcroix[4] implicitly introduced $\{\mathbf{f}_2\}$ in the distribution function only, without putting it into the equation.

The treatment given here is not to be confused with the particular expansion which uses Sonine or Laguerre polynomials[10–12] and spherical harmonics, and which is employed for like-particle interactions.[13] That expansion can be put directly into the Boltzmann equation, and with recursion relations given by Burnett,[10] the spherical harmonic terms can be collected quite neatly as Kelly[14] has done. The drawback is that if the distribution function is far from Maxwellian the expansion converges rather slowly. The spherical harmonics of order l are associated with an exponential with v^2 and Sonine polynomials (in v^2) of order $l+\frac{1}{2}$ rather than arbitrary function of velocity magnitude. For the markedly non-Maxwellian case one would like to be free to choose the velocity magnitude terms in other ways than those dictated by the Sonine expansion to obtain more rapid convergence.

The point of the treatment given here is that one can choose the velocity-magnitude expansion one wishes, and that the nature and interpretation of the separation of f into velocity magnitude and angle terms is shown in the construction and use of the f_l tensors. The Sonine expansion is a particular case, other expansions may well be more suited to other circumstances; the significance of the f_{lms} terms is the same no matter how they are expanded.

An interesting resemblance is shown between the averages in (10), the f_l tensors and the terms in Grad's n-dimensional Hermite expansion,[15] which is, by the nature of Hermite polynomials, linked specifically with a Maxwellian weighting function. This presumably indicates some degree of kinship, but the resemblance will not be discussed further here.

II. USE OF TENSOR FORM IN THE BOLTZMANN EQUATION

Substitution of Tensor Form into Boltzmann's Equation

Assume that the right hand side of the Boltzmann equation can be expanded in spherical harmonics. In the same manner as for the distribution function we can express this as a Cartesian tensor scalar product expression from (5a) using C_{lms} for f_{lms} and hence deriving $\{\mathbf{C}_l\}\vdots_l\{\mathbf{v}^l\}/v^l$ tensor forms.

Let us substitute the tensor form into the Boltzmann equation, which is written here as

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla_r f+(\mathbf{a}+\mathbf{v}\times\boldsymbol{\omega}_b)\cdot\boldsymbol{\nabla}_v f=C, \tag{12}$$

[3] W. P. Allis, *Handbuch der Physik,* edited by S. Flügge (Springer-Verlag, Berlin, 1956), p. 404, Vol. 21.
[4] J.-L. Delcroix, *Introduction a la Theorie des Gaz Ionises* (Monographie Dunod, Paris, 1959), p. 69.
[5] H. Margenau, Phys. Rev. **69**, 508 (1946).
[6] P. Rosen, Phys. Rev. **103**, 390 (1956).
[7] T. E. Van Zandt, Ph.D. thesis, Yale University, 1954 (unpublished).
[8] D. C. Kelly, H. Margenau and S. C. Brown, Phys. Rev. **108**, 1367 (1957).
[9] M. P. Bachynski, I. P. Shkarofsky, and T. W. Johnston, *Plasmas and the Electromagnetic Field* [McGraw-Hill Book Company, Inc., New York (to be published)].
[10] D. Burnett, Proc. London Math. Soc. **39**, 385 (1934).
[11] D. Burnett, Proc. London Math. Soc. **40**, 382 (1934).
[12] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, 1958), 2nd ed., 3rd reprint, p. 123.
[13] R. Landshoff, Phys. Rev. **76**, 907 (1949); and **82**, 442 (1951).
[14] D. C. Kelly, Phys. Rev. **119**, 27 (1960).
[15] H. Grad, Communs. Pure and Applied Math. **2**, 325, 331 (1949).

with $\mathbf{a}=(q/m)\mathbf{E}$, $\boldsymbol{\omega}_b=(q/m)\mathbf{B}$, or in Cartesian form with the summation of repeated indices,

$$\frac{\partial f}{\partial t}+v_i\frac{\partial f}{\partial v^i}+(a^i+e_{ijk}v^j\omega_b{}^k)\frac{\partial f}{\partial v^i}-C=0, \quad (12a)$$

where e_{ijk} is the unit alternating tensor in the Cartesian system ($=0$ unless $i\neq j\neq k$, $=\pm1$ for even/odd permutation of ijk from the normal 1, 2, 3 order). Substitute

$$f=\sum\nolimits_l f_{(l)w\cdots ut}v^tv^u\cdots v^w/v^l,$$
$$C=\sum\nolimits_l C_{(l)w\cdots ut}v^tv^u\cdots v^w/v^l,$$

where $f_{(l)w\cdots t}(x^i,v^jv^j,t)$ is a function of time, position and velocity magnitude only. There results

$$\sum_l \frac{\partial}{\partial t}\left(\frac{f_{(l)w\cdots t}v^t\cdots v^w}{v^l}\right)+v^i\frac{\partial}{\partial x_i}\left(\frac{f_{(l)w\cdots t}v^t\cdots v^w}{v^l}\right)$$
$$+(a^i+e_{ijk}v^j\omega_b{}^k)\frac{\partial}{\partial v^i}\left(\frac{f_{(l)w\cdots t}v^t\cdots v^w}{v^l}\right)$$
$$-\frac{C_{(l)w\cdots t}v^t\cdots v^w}{v^l}=0,$$

where the velocity derivatives can be split up to yield

$$\sum_l\left[\frac{\partial}{\partial t}(f_{(l)w\cdots t})\frac{v^t\cdots v^w}{v^l}+\frac{\partial}{\partial x_i}(f_{(l)w\cdots t})\frac{v^t\cdots v^wv^i}{v^l}\right.$$
$$+\frac{a^iv^i}{v}\frac{\partial}{\partial v}\left(\frac{f_{(l)w\cdots t}}{v^l}\right)v^t\cdots v^w+l\frac{a^if_{(l)iv\cdots t}}{v}\frac{v^t\cdots v^v}{v^{l-1}}$$
$$\left.+le_{ijk}\omega_b{}^kf_{(l)iv\cdots t}\frac{v^t\cdots v^vv^j}{v}-C_{(l)w\cdots t}\frac{v^t\cdots v^w}{v^l}\right]$$

where for the velocity derivative of $a_i(\partial v^t\cdots v^w/\partial v^i)=\sum a^iv^u\cdots v^w+v^ta^iv^v\cdots v^w+\cdots$, the symmetry of $f_{(l)w\cdots t}$, has been used to add the terms up. The velocity gradient of the magnitude-of-velocity functions is in the direction of v and so gives zero when scalar-multiplied with $\mathbf{v}\times\boldsymbol{\omega}_b$.

Now let us group the terms by $v^t\cdots v^w/v^l$, to obtain

$$\sum_l\left[\frac{\partial f_{(l)w\cdots t}}{\partial t}+v\frac{\partial f_{(l-1)v\cdots t}}{\partial x^w}+v^{l-1}a^w\frac{\partial}{\partial v}\left(\frac{f_{(l-1)v\cdots t}}{v^{l-1}}\right)\right.$$
$$+(l+1)\frac{a^if_{(l+1)iw\cdots t}}{v}+le_{iwk}\omega_b{}^kf_{(l)iv\cdots t}$$
$$\left.-C_{(l)w\cdots t}\right]\frac{v^t\cdots v^w}{v^l}=0,$$

changing the e symbol by two transpositions to leave the sign unchanged

$$\sum_l\left[\frac{\partial(f_{(l)w\cdots t})}{\partial t}+v\frac{\partial}{\partial x^w}(f_{(l-1)v\cdots t})+v^{l-1}a^w\frac{\partial}{\partial v}\left(\frac{f_{(l-1)v\cdots t}}{v^{l-1}}\right)\right.$$
$$+\frac{(l+1)a^if_{(l+1)iw\cdots t}}{v}+le_{wki}\omega_b{}^kf_{(l)iv\cdots t}$$
$$\left.-C_{(l)w\cdots t}\right]\frac{v^t\cdots v^w}{v^l}=0. \quad (13)$$

If we write as a compact form of notation

$f_{(l)w\cdots t}=\{\mathbf{f}_l\}$	for the f tensor,
$C_{(l)w\cdots t}=\{\mathbf{C}_l\}$	for the C tensor,
$\partial/\partial x^w(f_{(l-1)v\cdots t})=\nabla_r\{\mathbf{f}_{l-1}\}$	for the space gradient of a symmetric tensor,
$a^if_{(l+1)iw\cdots t}=\mathbf{a}\cdot\{\mathbf{f}_{l+1}\}$	for the scalar product of a vector with a tensor,
$e_{wki}\omega_b{}^kf_{(l)iv\cdots t}=\boldsymbol{\omega}_b\times\{\mathbf{f}_l\}$	for the vector product of a vector with a tensor (unambiguous with a completely symmetric tensor),

we write

$$\sum_l\left(\left[\frac{\partial\{\mathbf{f}_l\}}{\partial t}+v\nabla_r\{\mathbf{f}_{l-1}\}+v^{l-1}\mathbf{a}\frac{\partial}{\partial v}\left(\frac{\{\mathbf{f}_{l-1}\}}{v^{l-1}}\right)\right.\right.$$
$$\left.\left.+l\boldsymbol{\omega}_b\{\mathbf{f}_l\}-\{\mathbf{C}_l\}+(l+1)\frac{\mathbf{a}\cdot\{\mathbf{f}_{l+1}\}}{v}\right]\vdots_l\frac{\{\mathbf{v}^l\}}{v^l}\right)=0. \quad (14)$$

The first few terms in this set are

$$\frac{\partial f_0}{\partial t}+\frac{\mathbf{a}\cdot\mathbf{f}_1}{v}-C_0+\left(\frac{\partial\mathbf{f}_1}{\partial t}+v\nabla_rf_0+\mathbf{a}\frac{\partial f_0}{\partial v}+\boldsymbol{\omega}_b\times\mathbf{f}_1\right.$$
$$\left.+2\mathbf{a}\cdot\{\mathbf{f}_2\}-\mathbf{C}_1\right)\cdot\frac{\mathbf{v}}{v}+\left(\frac{\partial\{\mathbf{f}_2\}}{\partial t}+v\nabla_r\mathbf{f}_1+v\mathbf{a}\frac{\partial(\mathbf{f}_1/v)}{\partial v}\right.$$
$$\left.+2\boldsymbol{\omega}_b\{\mathbf{f}_2\}+3\mathbf{a}\cdot\{\mathbf{f}_3\}-\{\mathbf{C}_2\}\right):\frac{\{\mathbf{vv}\}}{v^2}+\left(\frac{\partial\{\mathbf{f}_3\}}{\partial t}+v\nabla_r\{\mathbf{f}_2\}\right.$$
$$\left.+v^2\mathbf{a}\frac{\partial(\{\mathbf{f}_2\}/v^2)}{\partial v}+3\boldsymbol{\omega}_b\times\{\mathbf{f}_3\}+4\mathbf{a}\cdot\{\mathbf{f}_4\}-\{\mathbf{C}_3\}\right)\vdots\frac{\{\mathbf{vvv}\}}{v^3}$$
$$+\left(\frac{\partial\{\mathbf{f}_4\}}{\partial t}+v\nabla_r\{\mathbf{f}_3\}+v^3\mathbf{a}\frac{\partial(\{\mathbf{f}_3\}/v^3)}{\partial v}+4\boldsymbol{\omega}_b\times\{\mathbf{f}_4\}\right.$$
$$\left.+5\mathbf{a}\cdot\{\mathbf{f}_5\}-\{\mathbf{C}_4\}\right)\vdots\frac{\{\mathbf{vvvv}\}}{v^4}$$
$$+\quad\cdots\quad\cdots\quad\cdots\quad=0, \quad (15)$$

where the terms to be retained in a third order expansion have been kept (f_4, f_5 will be omitted). Note that one must consider the fourth order equation to include all the third order terms.

Derivation of Angle-Integrated Transport Equations

The step of interest is to multiply this equation through successively by $\{\mathbf{v}^l\}/v^l=\{(\cos\boldsymbol{\theta}^i)^l\}$ of various orders and to integrate over angle to obtain partially integrated transport equation. This can be done by expressing (15) and $\{\mathbf{v}^l\}/v^l$ as Y_{lms} terms from (9) using the spherical harmonic orthogonality condition and collecting terms, with simplifications from the interdependence of the f_l elements from (5a). The other method is to use (11) for integrating the direction cosine products and using the tensor element expressions in f_{lms} terms to eliminate some of the resulting terms. As a matter of hindsight, it develops that $\sum_i f_{ii}$ and $\sum_i f_{iil}=\sum_i f_{ili}=\sum_i f_{lii}$ are zero for $\{\mathbf{f}_2\}$ and $\{\mathbf{f}_3\}$.[16] One can obtain the tensor expressions without using the spherical expansion beyond the establishing of the symmetry conditions.

Scalar Equation—Density and Energy

The resultant equations, integrated over angle, are as follows:

$$4\pi\left(\frac{\partial f_0}{\partial t}+\frac{\mathbf{a}\cdot\mathbf{f}_1}{v}-C_0\right)+\frac{4\pi}{3}\left(\frac{\partial\{\mathbf{f}_2\}}{\partial t}+v\nabla_r\mathbf{f}_1+v\mathbf{a}\frac{\partial(\mathbf{f}_1/v)}{\partial v}+2\boldsymbol{\omega}_b\times\{\mathbf{f}_2\}+3\frac{\mathbf{a}\cdot\{\mathbf{f}_3\}}{v}-\{\mathbf{C}_2\}\right):\{\mathbf{I}_2\}=0. \quad (16a)$$

As a result of symmetry, etc., this is

$$\frac{\partial f_0}{\partial t}+\frac{\mathbf{a}\cdot\mathbf{f}_1}{v}+\frac{v}{3}\frac{\partial}{\partial v}\left(\frac{\mathbf{a}\cdot\mathbf{f}_1}{v}\right)+\frac{v}{3}\nabla_r\cdot\mathbf{f}_1-C_0=0,$$

or, using

$$\frac{1}{v^p}\frac{\partial v^p f}{\partial v}=\frac{pf}{v}+\frac{\partial f}{\partial v},$$

$$\frac{\partial f_0}{\partial t}+\frac{v}{3}\nabla_r\cdot\mathbf{f}_1+\frac{1}{3v^2}\frac{\partial(v^2\mathbf{a}\cdot\mathbf{f}_1)}{\partial v}-C_0=0. \quad (16b)$$

This equation is just Allis' zero-order equation; it is noteworthy that the higher order ($l>1$) terms do not appear in the final result.

Vector Equation (Momentum, Energy Flow)

$$\frac{4\pi}{3}\left[\frac{\partial\mathbf{f}_1}{\partial t}+v\nabla_r\mathbf{f}_0+\mathbf{a}\frac{\partial f_0}{\partial v}+\boldsymbol{\omega}_b\times\mathbf{f}_1-\mathbf{C}_1+\tfrac{2}{5}\left(\frac{\{\mathbf{I}_2\}:\{\mathbf{f}_3\}}{\partial t}+v\nabla_r\cdot\{\mathbf{f}_2\}+\frac{1}{v^3}\frac{\partial(v^3\mathbf{a}\cdot\{\mathbf{f}_2\})}{\partial v}-\{\mathbf{I}_2\}:\{\mathbf{C}_3\}\right)+\tfrac{1}{5}\left(\frac{\partial\{\mathbf{f}_3\}:\{\mathbf{I}_2\}}{\partial t}+v\nabla_r\{\mathbf{f}_2\}:\{\mathbf{I}_2\}+v^2\mathbf{a}\frac{\partial(\{\mathbf{f}_2\}:\{\mathbf{I}_2\}/v^2)}{\partial v}+3\boldsymbol{\omega}_b\times\{\mathbf{f}_3\}:\{\mathbf{I}_2\}-\{\mathbf{C}_3\}:\{\mathbf{I}_2\}\right)\right]=0, \quad (17a)$$

which reduces to

$$\frac{\partial\mathbf{f}_1}{\partial t}+v\nabla_r f_0+\mathbf{a}\frac{\partial f_0}{\partial v}+\boldsymbol{\omega}_b\times\mathbf{f}_1-\mathbf{C}_1+\tfrac{2}{5}v\nabla_r\cdot\{\mathbf{f}_2\}+\frac{2}{5v^3}\frac{\partial}{\partial v}(v^3\mathbf{a}\cdot\{\mathbf{f}_2\})=0. \quad (17b)$$

The terms on the first line have already been indicated by Allis, while the second line shows the nonisotropic pressure effect ($\nabla_r f_0$ is the isotropic pressure term) and the momentum effect of the electric field and anisotropic pressure. If, from *a priori* considerations one knows that the pressure is isotropic then the $\{\mathbf{f}_2\}$ tensor can be immediately put equal to zero.

Momentum Transport or Pressure Tensor Equation

The final form is, after substituting from (16),

$$\frac{8\pi}{15}\left[\frac{\partial\{\mathbf{f}_2\}}{\partial t}+v(\nabla_r\mathbf{f}_1-\tfrac{1}{3}\nabla_r\cdot\mathbf{f}_1\{\mathbf{I}_2\})+v\frac{\partial}{\partial v}\left(\frac{\mathbf{a}\mathbf{f}_1}{v}-\frac{1}{3}\frac{\mathbf{a}\cdot\mathbf{f}_1}{v}\{\mathbf{I}_2\}\right)+2\boldsymbol{\omega}_b\times\{\mathbf{f}_2\}-\{\mathbf{C}_2\}+\frac{3}{7}\left(v\nabla_r\cdot\{\mathbf{f}_3\}+\frac{1}{v^4}\frac{\partial(v^4\mathbf{a}\cdot\{\mathbf{f}_3\})}{\partial v}\right)\right]_2=0. \quad (18)$$

[Note the use of the $[\quad]_l$ symbol defined after (10).]

Pressure Transport or Heat Tensor Equation

$$\frac{8\pi}{35}\left[\frac{\partial\{\mathbf{f}_3\}}{\partial t}+v(\nabla_r\{\mathbf{f}_2\}-\tfrac{2}{5}\nabla_r\cdot\{\mathbf{f}_2\}\{\mathbf{I}_2\})+v^2\frac{\partial}{\partial v}\left(\frac{\mathbf{a}\{\mathbf{f}_2\}}{v^2}-\frac{2}{5}\frac{\mathbf{a}\cdot\{\mathbf{f}_2\}}{v^2}\{\mathbf{I}_2\}\right)+3\boldsymbol{\omega}_b\times\{\mathbf{f}_3\}-\{\mathbf{C}_3\}\right]_3=0. \quad (19)$$

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[16] It seems most probable that this set of relationships applies for all f_l, since they can easily be reduced using powers of the sum of the squares of the direction cosines (which is one), and since both relations produce just $2l+1$ independent tensor elements.

The scalar and vector equations are exactly equivalent to equations for the terms with Y_{000}, Y_{110} (x component) Y_{111} (y component) and Y_{100} (z component), but the tensor equations are in a sense degenerate because the tensor elements are not independent, reflecting the interdependence of the $\cos\theta^{i}$'s. To obtain the equations needed to determine f, we need

$$\sum_{l=0}^{l}(2l+1)=(l+1)+2l(l+1)/2=(l+1)^2$$

equations, in this case sixteen. These equations are obtained by going back to (11) or (12) and expressing the f_l components as f_{lms} terms and the $v^i/v=\cos\theta^i$ terms as spherical harmonics, and then using the orthogonality conditions for each harmonic to obtain sixteen equations for the sixteen f_{lms} terms. The first four equations are exactly equivalent to the f_0 and $\mathbf{f}_1$ component equations.

III. SPHERICAL HARMONIC EQUATIONS FOR OBTAINING COEFFICIENTS

The complete set is as follows. Notice that the fourth order cosine product expansion is needed, as can be seen from (13), and that the spherical harmonic which is the source of each equation is indicated by the subscripts of $\partial f_{lms}/\partial t$ and C_{lms}.

$$\frac{\partial f_{000}}{\partial t}+\frac{v}{3}\left(\frac{\partial f_{110}}{\partial x}+\frac{\partial f_{111}}{\partial y}+\frac{\partial f_{100}}{\partial z}\right)+\frac{1}{3v^2}\frac{\partial}{\partial v}[v^2(a^xf_{110}+a^yf_{111}+a^zf_{110})]-C_{000}=0,$$

$$\frac{\partial f_{110}}{\partial t}+v\frac{\partial f_{000}}{\partial x}+a^x\frac{\partial f_{000}}{\partial v}+(\omega_b{}^yf_{100}-\omega_b{}^zf_{111})-C_{110}+\frac{2v}{5}\left[\frac{\partial}{\partial x}(-\tfrac{1}{2}f_{200}+3f_{220})+\frac{\partial(3f_{221})}{\partial y}+\frac{\partial(\frac{3}{2}f_{210})}{\partial z}\right]$$
$$+\frac{2}{5v^3}\frac{\partial}{\partial v}[v^3[a^x(-\tfrac{1}{2}f_{200}+3f_{220})+a^y(3f_{221})+a^zf_{210}]]=0,$$

$$\frac{\partial f_{111}}{\partial t}+v\frac{\partial f_{000}}{\partial y}+a^y\frac{\partial f_{000}}{\partial v}+(\omega_b{}^zf_{110}-\omega_b{}^xf_{100})-C_{111}+\frac{2v}{5}\left[\frac{\partial}{\partial x}(3f_{221})+\frac{\partial}{\partial y}(-\tfrac{1}{2}f_{200}-3f_{220})+\frac{\partial(\frac{3}{2}f_{211})}{\partial z}\right]$$
$$+\frac{2}{5v^3}\frac{\partial}{\partial v}[v^3[a^x3f_{221}+a^y(-\tfrac{1}{2}f_{200}-3f_{220})+a^z\tfrac{3}{2}f_{211}]]=0,$$

$$\frac{\partial f_{100}}{\partial t}+v\frac{\partial f_{000}}{\partial z}+a^z\frac{\partial f_{000}}{\partial v}+(\omega_b{}^xf_{111}-\omega_b{}^yf_{110})-C_{100}+\frac{2v}{5}\left[\frac{\partial(\frac{3}{2}f_{210})}{\partial x}+\frac{\partial(\frac{3}{2}f_{211})}{\partial y}+\frac{\partial(f_{200})}{\partial z}\right]$$
$$+\frac{2}{5v^3}\frac{\partial}{\partial v}[v^3(a^x\tfrac{3}{2}f_{210}+a^y\tfrac{3}{2}f_{211}+a^zf_{200})]=0.$$

[These four equations are just (16b) and (17b) because of the simplicity the relations for $l=0$, 1.]

$$\frac{\partial f_{200}}{\partial t}+\frac{v}{3}\left[-\frac{\partial f_{110}}{\partial x}-\frac{\partial f_{111}}{\partial y}+\frac{2\partial f_{100}}{\partial z}\right]+\frac{v}{3}\frac{\partial}{\partial v}\left[\frac{1}{v}(-a^xf_{110}-a^yf_{111}+2a^zf_{100})\right]+3(\omega_b{}^xf_{211}-\omega_b{}^yf_{210})-C_{200}$$
$$+\frac{3v}{7}\left(\frac{2\partial f_{310}}{\partial x}+\frac{2\partial f_{311}}{\partial y}+\frac{\partial f_{300}}{\partial z}\right)+\frac{3}{7v^4}\frac{\partial}{\partial v}[v^4(a^x2f_{310}+a^y2f_{311}+a^zf_{300})]=0,$$

$$\frac{\partial f_{210}}{\partial t}+\frac{v}{3}\left(\frac{\partial f_{100}}{\partial x}+\frac{\partial f_{110}}{\partial z}\right)+\frac{v}{3}\frac{\partial}{\partial v}\left[\frac{1}{v}(a^xf_{100}+a^zf_{110})\right]+\omega_b{}^xf_{221}+\omega_b{}^y(+f_{200}-2f_{220})-\omega_b{}^zf_{211}-C_{210}$$
$$+\frac{2v}{7}\left[\frac{\partial}{\partial x}(-\tfrac{1}{2}f_{300}+5f_{320})+\frac{\partial(5f_{321})}{\partial y}+\frac{\partial(2f_{310})}{\partial z}\right]+\frac{2}{7v^4}\frac{\partial}{\partial v}\{v^4[a^x(-\tfrac{1}{2}f_{300}+5f_{320})+a^y5f_{321}+a^z2f_{310}]\}=0,$$

$$\frac{\partial f_{211}}{\partial t}+\frac{v}{3}\left(\frac{\partial f_{100}}{\partial y}+\frac{\partial f_{111}}{\partial z}\right)+\frac{v}{3}\frac{\partial}{\partial v}\left[\frac{1}{v}(a^yf_{100}+a^zf_{111})\right]-\omega_b{}^x(f_{200}+2f_{220})-\omega_b{}^yf_{221}+\omega_b{}^zf_{210}-C_{211}$$
$$+\frac{2v}{7}\left[\frac{\partial(5f_{321})}{\partial x}+\frac{\partial(-\frac{1}{2}f_{300}-5f_{320})}{\partial y}+\frac{\partial(2f_{311})}{\partial z}\right]+\frac{2}{7v^4}\frac{\partial}{\partial v}[v^4(a^x5f_{321}+a^y(-\tfrac{1}{2}f_{300}-5f_{321})+a^z2f_{311})]=0,$$

$$\frac{\partial f_{220}}{\partial t}+\frac{v}{6}\left(\frac{\partial f_{110}}{\partial x}-\frac{\partial f_{111}}{\partial y}\right)+\frac{v}{6}\frac{\partial}{\partial v}\left[\frac{1}{v}(a^x f_{110}-a^y f_{111})\right]+\tfrac{1}{2}[\omega_b{}^x f_{211}+\omega_b{}^y f_{210}-4\omega_b{}^z f_{221}]-C_{220}$$
$$+\frac{v}{7}\left[\frac{\partial(-\frac{1}{2}f_{310}+15f_{330})}{\partial x}+\frac{\partial(\frac{1}{2}f_{311}+15f_{331})}{\partial y}+\frac{\partial(5f_{320})}{\partial z}\right]+\frac{1}{7v^4}\frac{\partial}{\partial v}[v^4(a^x(-\tfrac{1}{2}f_{310}+15f_{330})+a^y(\tfrac{1}{2}f_{311}+15f_{331})+a^y f_{320})]=0,$$

$$\frac{\partial f_{221}}{\partial t}+\frac{v}{6}\left(\frac{\partial f_{111}}{\partial x}+\frac{\partial f_{110}}{\partial y}\right)+\frac{v}{6}\frac{\partial}{\partial v}\left[\frac{1}{v}(a^x f_{111}+a^y f_{110})\right]+\tfrac{1}{2}[-\omega_b{}^x f_{210}+\omega_b{}^y f_{211}+4\omega_b{}^z f_{220}]-C_{221}$$
$$+\frac{v}{7}\left[\frac{\partial(-\frac{1}{2}f_{311}+15f_{331})}{\partial x}+\frac{\partial(-\frac{1}{2}f_{310}-15f_{330})}{\partial y}+\frac{\partial(5f_{321})}{\partial z}\right]+\frac{1}{7v^4}\frac{\partial}{\partial v}[v^4(a^x(-\tfrac{1}{2}f_{311}+15f_{331})+a^y(-\tfrac{1}{2}f_{310}-15f_{330})+a^z5f_{321})]=0,$$

$$\frac{\partial f_{300}}{\partial t}+\frac{3v}{5}\left[-\frac{\partial f_{210}}{\partial x}-\frac{\partial f_{211}}{\partial y}+\frac{\partial f_{200}}{\partial z}\right]+\frac{3v^2}{5}\frac{\partial}{\partial v}\left[\frac{1}{v^2}(-a^x f_{210}-a^y f_{211}+a^z f_{200})\right]+6(\omega_b{}^x f_{311}-\omega_b{}^y f_{310})-C_{300}=0,$$

$$\frac{\partial f_{310}}{\partial t}+\frac{v}{5}\left[\frac{\partial(f_{200}-f_{220})}{\partial x}-\frac{\partial f_{221}}{\partial y}+\frac{\partial 2f_{210}}{\partial z}\right]+\frac{v^2}{5}\frac{\partial}{\partial v}\left[\frac{1}{v^2}(a^x(f_{200}-f_{220})-a^y f_{221}+a^z 2f_{210})\right]+\omega_b{}^x 5f_{321}+\omega_b{}^y(f_{300}-5f_{320})-\omega_b{}^z f_{311}-C_{310}=0,$$

$$\frac{\partial f_{311}}{\partial t}+\frac{v}{5}\left[\frac{\partial(-f_{221})}{\partial x}+\frac{\partial(f_{200}-f_{220})}{\partial y}+\frac{\partial 2f_{211}}{\partial z}\right]+\frac{v^2}{5}\frac{\partial}{\partial v}\left[\frac{1}{v^2}(a^x(-f_{221})+a^y(f_{200}-f_{220})+a^z 2f_{221})\right]+\omega_b{}^x(f_{300}-5f_{320})-\omega_b{}^y f_{321}+\omega_b{}^z f_{310}-C_{311}=0,$$

$$\frac{\partial f_{320}}{\partial t}+\frac{v}{10}\left[\frac{\partial f_{210}}{\partial x}-\frac{\partial f_{211}}{\partial y}+\frac{\partial 2f_{220}}{\partial z}\right]+\frac{v^2}{10}\frac{\partial}{\partial v}\left[\frac{1}{v^2}(a^x f_{210}-a^y f_{211}+a^z 2f_{220})\right]+\omega_b{}^x(\tfrac{1}{2}f_{311}+3f_{331})+\omega_b{}^y(\tfrac{1}{2}f_{310}-3f_{330})+\omega_b{}^z 2f_{321}-C_{320}=0,$$

$$\frac{\partial f_{321}}{\partial t}+\frac{v}{10}\left[\frac{\partial f_{211}}{\partial x}+\frac{\partial f_{210}}{\partial y}+\frac{\partial 2f_{221}}{\partial z}\right]+\frac{v^2}{10}\frac{\partial}{\partial v}\left[\frac{1}{v^2}(a^x f_{211}+a^y f_{210}+a^z 2f_{221})\right]+\omega_b{}^x(-\tfrac{1}{2}f_{310}-3f_{330})+\omega_b{}^y(\tfrac{1}{2}f_{311}+3f_{331})+\omega_b{}^z 2f_{320}-C_{321}=0,$$

$$\frac{\partial f_{330}}{\partial t}+\frac{v}{10}\left[\frac{\partial f_{220}}{\partial x}+\frac{\partial f_{221}}{\partial y}\right]+\frac{v^2}{10}\frac{\partial}{\partial v}\left[\frac{1}{v^2}(a^x f_{220}+a^y f_{221})\right]+\tfrac{1}{2}(\omega_b{}^x f_{321}+\omega_b{}^y f_{320}+\omega_b{}^z 6f_{331})-C_{330}=0,$$

$$\frac{\partial f_{331}}{\partial t}+\frac{v}{10}\left[\frac{\partial f_{221}}{\partial x}+\frac{\partial f_{220}}{\partial y}\right]+\frac{v^2}{10}\frac{\partial}{\partial v}\left[\frac{1}{v^2}(a^x f_{221}+a^y f_{220})\right]+\tfrac{1}{2}(\omega_b{}^x f_{320}+\omega_b{}^y f_{321}+\omega_b{}^z 6f_{330})-C_{331}=0,$$

IV. DISCUSSION

Thus, by using the tensor expansion, it has been possible to side-step the difficulties involved in trying to cast Boltzmann's equation in spherical harmonic form. It can now be seen that the difficulty lies in the complications of the higher order terms; it may be possible, now that a correct expansion has been given for terms up to $l=3$, to deduce what recursion-type relations are needed to give the same result and then to prove their truth analytically. The result would be somewhat academic, since the present equations go as far as one could reasonably wish, but would provide an independent check on the tensor derivation.

The next step is to see whether, by a suitable choice of reference frames, one can simplify the equations. There are two immediate preferred directions that come to mind, the direction of the magnetic field and of the electric field. Since the $\boldsymbol{\omega}_b\times\cdots$ terms are somewhat more awkward, a sensible choice seems to be to take $\boldsymbol{\omega}_b$ along the z or polar axis, making $\omega_b{}^x=\omega_b{}^y=0$. If the magnetic field is zero and if the electric field direction is fixed (as it usually is in the zero magnetic field case), then the obvious choice is to take $\mathbf{a}$ along the polar axis. If $\mathbf{a}$ varies in direction with time (e.g., elliptic polarization) then we cannot choose $\mathbf{a}$ to be along a coordinate axis because this introduces a moving frame and hence virtual accelerations. The ac electric field perpendicular to the dc magnetic field imposed on a plasma is always elliptically polarized, so it seems that taking $\boldsymbol{\omega}_b$ along the polar axis is as far as one can go in simplification by choice of axes in the presence of a magnetic field.

If we now re-examine the statement in Eq. (1) we can see that it applies only to the one-dimensional case, but that, suitably generalized to Eq. (2), the expansion can be applied to Boltzmann's equation in very general terms indeed to yield Eqs. (10), (14), (16), (17), (18), (19), and (20) which probably go as far as one would like to take the expansion.

This approach has also demonstrated the correspondence between the order of the spherical harmonic or the tensor expansion and the order of tensor transport quantities. Usually these quantities are in terms of the particular or peculiar velocity, the velocity referred to an average velocity frame of reference ($\mathbf{v}-\langle\mathbf{v}\rangle$ rather than $\mathbf{v}$). Since this frame of reference changes with time and place it is not suited to the spherical harmonic expansion unless the average velocity has at least a large constant part. Usually the peculiar velocity terms will have to be calculated from the rest frame velocity expansion, clumsy though it may seem.

The assumption of a scalar pressure requires only the zero and first order spherical harmonic terms, while an anisotropic pressure will necessitate second order terms and a pressure transport tensor must imply third order terms.

Even if this clarification of the spherical harmonic expansion does not produce a rewarding attack on plasma problems it should lead to a clearer understanding of basic expansions of the distribution function and the relation of this approach to others.

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Nuclear Spin Relaxation in Liquid Helium 3†

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The longitudinal relaxation time of liquid He^3 has been measured as a function of temperature above 1°K and of magnetic field below 13 kgauss in a number of sample containers. At a temperature of 2.0°K and in a magnetic field of 10 kgauss the longitudinal relaxation time, T_1, varied with the sample container from 60 seconds to 400 seconds. The transverse relaxation time, T_2, was measured by a new method and was approximately 30 seconds at a field of 10 kgauss in all sample containers. T_1 was determined as a function of magnetic field at 2.0°K in a single sample container; the values increased from less than 50 seconds in approximately zero field to 400 seconds at 13 kgauss. An impurity relaxation model is proposed to explain the T_1 results. By assuming both wall relaxation and a bulk relaxation given by the Bloembergen, Purcell, and Pound theory, the dependence of T_1 on pressure and temperature can be quantitatively understood. The low values of T_2 are inconsistent with the Bloembergen, Purcell, and Pound theory and may be due to the presence of paramagnetic impurities in suspension in the bulk liquid.

INTRODUCTION

USING adiabatic fast passage techniques, we have measured the longitudinal and transverse nuclear magnetic relaxation times, T_1 and T_2, in liquid He^3 contained in sample chambers of different sizes and materials. The T_1 measurements supplement recently reported values obtained at three different laboratories.[1–3] Our values of T_1 versus magnetic field do not agree with some recent measurements of Romer[2] which gave a T_1 independent of field. The T_2 measurements, the first reported for liquid He^3, were obtained by a new method which makes possible the measurement of long transverse relaxation times. The measured values of T_2 are an order of magnitude less than T_1.

Nuclear relaxation in liquid He^3 has been analyzed by various workers[1–4] in terms of the Bloembergen, Purcell, and Pound theory for classical liquids.[5] It would be surprising if the intrinsic relaxation in liquid He^3 were completely described by this theory since it does not take account of any quantum statistical effects. Nevertheless, we can show that most of the data above 1°K are compatible with this theory if impurity effects are considered. To explain the various T_1 and T_2 results we propose an impurity relaxation model based on a wall relaxation in parallel with the bulk relaxation. A reason for the inequality of T_1 and T_2 will be suggested.

The experimental technique will be described in detail and the method of measuring T_2 will be discussed. During the course of these measurements nuclear maser effects were observed.

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