LAW OF MOTION OF A DROPLET MOVING WITH VARIABLE VELOCITY IN AIR.

By Raymond B. Abbott.

THE object of this research was to find the law of motion of a droplet moving with variable velocity in a viscous medium, air; and also to find whether the method could be used to determine accurately the value of the electric charge given to the droplet, while falling between charged condenser plates.

Stokes' law[1] has been hitherto supposed to hold only for uniform motion in a straight line. Its limitation is given by Rayleigh[2] to be such that the radius r should be small compared to $\mu/\rho V$, where μ is the coefficient of viscosity of air, ρ is the density of air, and V is the velocity of the droplet. Many of the speeds observed in the experiments herein described are well within this limitation, but the motion is neither uniform nor in a straight line; for the droplet falls through a horizontal electric field, the charge on condenser plates alternating sixty times per second and varying according to the sine law. The droplet therefore oscillates with a horizontal motion while it falls.

The method used was a modified form of the one suggested by E. P. Lewis and used by W. A. Shewhart.[3] The original idea, evidently, was inspired by the work of Millikan on the "Elementary Electrical Charge."[4]

In general, the present method is to photograph the path of the particle as it falls through a periodically changing electric field and to find the law of motion from the form of the curve.

Shewhart showed conclusively that the amplitude of oscillation of the droplet is here proportional to the maximum electric force upon it. But the increase of wave-length which he found, where the field intensity was increased, can be explained by the fact that convection currents are set up in the air where more intense ionization takes place from the sharp edges of the condenser plates, as they were arranged.

He assumed that the path of the droplet was approximately a sine curve but did not find its exact form. In the present work enlargements

[1] G. G. Stokes, Mathematical and Physical Papers, Vol. III., p. 59.
[2] Phil. Magazine, Vol. 36, p. 365, 1893.
[3] Phys. Rev., May, 1917.
[4] Phys. Rev., August, 1913.

of direct photographs of path were made, and the form of the curve was compared with an accurately constructed sine curve. Also, data were obtained showing the angle of lag of the droplet behind the field intensity. Further, the law of motion of spheres of small density and known mass, falling in air, was experimentally determined. Shewhart's apparatus was used, with a feature added, which showed on the photograph the exact position of the droplet at some known value of the field intensity.

Two methods were used for doing this; first, a synchronous motor with a narrow vane attached to the shaft cut off the beam of light[1] when the field intensity was zero in value. This showed as a short break in the photographed path of the droplet (see Fig. 2). A disk of insulating material provided with a metallic ring and two contact makers 180 degrees apart, placed on the shaft of the motor, provided a means for finding the position of the shaft for zero electromotive force of the power circuit. Two brushes connected in series to the power circuit and a voltmeter, shunted with a condenser, provided a circuit which was closed whenever one of the contact makers on the disk hit a brush. By rotating the brushes a position could be found for any desired value of the field intensity.

The apparatus for the second method consisted of a spark gap connected across a large condenser which was charged by the power circuit. The spark occurred twice during each cycle and illuminated the falling droplet, causing a dark spot on the photographed path of the particle. The time of the spark with reference to the zero value of the electromotive force was measured by noting the position of the vane on the shaft of the synchronous motor when the spark occurred. This was done in a dark room where the rotating vane could be seen only when the spark illuminated it. With the apparatus used, the spark occurred approximately five degrees after the zero value of the field intensity was reached (see Fig. 3).

Horizontal and Vertical Components of Air Resistance.

Fig. 1 shows a direct enlargement of the photographed path of a droplet. The points of a sine curve, mathematically constructed upon it, show that the path is a sine curve. The electromotive force which charges the two condenser plates follows the sine law very closely. This is known from oscillographic analyses made in the departments of physics and electrical engineering. The spot of light shows where the droplet is located with reference to the electromotive force of the condenser plates.

One can now definitely express displacement, velocity and acceleration

[1] See Shewhart, Phys. Rev., May, 1918, p. 427.

PHYSICAL REVIEW, VOL. XII., SECOND SERIES,
November, 1918.

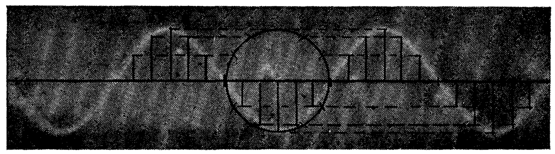

FIG. 1.

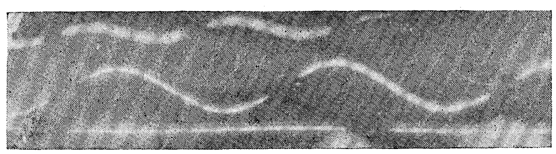

FIG. 2.

FIG. 3.

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in the horizontal direction in terms of the sine law and an unknown function $f(y)$ representing the air resistance; and from the equation of motion, the form of the function $f(y)$ can be ascertained. This will be the first step taken in our effort to discover the law of air resistance in this case.

Accelerated motion of a body in a viscous medium is opposed by two kinds of resistance. There is a viscous or frictional resistance, and also an inertia resistance due to an apparent increase in the mass of the moving body, known as a "fictitious mass." The viscous resistance always opposes motion, at constant or at variable speed, while an inertia resistance opposes only a change of speed.

Let the maximum force on the droplet due to the charge (Ne) and the field intensity X be

$$A = (Ne)X. \tag{1}$$

Let the angle corresponding to the distance fallen by the droplet be x, and the angle of lead of the electromotive force ahead of the droplet be γ; then, since the electromotive force follows the sine law, the electric force on the droplet will be

$$F(y) = A \sin (x + \gamma). \tag{2}$$

The angle $(x + \gamma)$ must increase at a constant rate in order to satisfy the conditions governing the generation of the electromotive force at the dynamo. Two cases are open for investigation, involving the variation of the angle x of the droplet. The angle x either increases at a constant rate and γ is constant, or x may vary in such a manner that the average increase in its value for each quarter cycle is the same for every quarter; then γ must vary in such a manner as will make $(x + \gamma)$ increase at a constant rate at all times. The first case will now be considered, in which γ and dx/dt are constant.

Since the path of the droplet is known from Fig. (1) to be a sine curve, the following quantities can be expressed in terms of the sine law:

$$\begin{aligned} \text{Displacement } y &= a \sin x \\ \text{Angular velocity } \omega &= dx/dt \\ \text{Horizontal velocity } V_y &= a\omega \cos x \\ \text{Horizontal acceleration } p_y &= - a\omega^2 \sin x. \end{aligned} \tag{3}$$

If $f(y)$ represents the expression for the resisting force parallel to Y, and M the mass of the droplet, the completed equation of motion parallel to the Y axis is

$$A \sin (x + \gamma) - f(y) = M(- a\omega^2 \sin x). \tag{4}$$

Solve for $f(y)$, expand $\sin(x+\gamma)$ and collect coefficients of $\sin x$ and $\cos x$; the result is

$$f(y) = (A\cos\gamma + Ma\omega^2)\sin x + (A\sin\gamma)\cos x. \tag{5}$$

Let $\alpha = (A\cos\gamma + Ma\omega^2)$, $\beta = (A\sin\gamma)$ and $c = \sqrt{\alpha^2+\beta^2}$, $\tan j = \alpha/\beta$.

The function $f(y)$ written in terms of these constants is:

$$f(y) = c\cos(x-j). \tag{6}$$

The equation of motion is, therefore,

$$A\sin(x+\gamma) - c\cos(x-j) = M(-a\omega^2\sin x). \tag{7}$$

Equation (5) or (6) may be written $f(y) = \alpha\sin x + \beta\cos x$ which expresses both kinds of resistance, due to viscosity and inertia; for $\alpha\sin x$ is in phase with the acceleration, and $\alpha\cos x$ is in phase with the velocity. Further, $\beta\cos x$ must contain all of the viscous resistance; for the latter must be at its maximum value when the velocity is at its maximum value, but $\alpha\sin x$ is zero at this time.

In regard to the absolute value of γ, the value of $\cos\gamma$ may be such for every droplet that $a\omega^2M + A\cos\gamma = 0$, that is, γ greater than $\pi/2$ and $M - (A\cos\gamma) = 0$. If α does not equal zero, it contains a fictitious mass term, which for air, may be small enough to neglect. The fictitious mass of a sphere moving in a perfect fluid is discussed in Lamb's Hydrodynamics.[1] It is shown that the increase in the inertia of the sphere is one half the mass of the fluid displaced. In the case here considered the fluid is air which is not a perfect fluid, but the relation gives an idea of the magnitude of the term which appears as an additional inertia. The theoretical value would be approximately one half the ratio of the density of air to the density of water multiplied by the mass of the water droplet.

$$\text{Fictitious mass} = \frac{.00119}{2}M = .0006M.$$

The inertia resistance expressed in terms of the acceleration is

$$\alpha\sin x = \left(\frac{a\omega^2}{a\omega^2}\right)\alpha\sin x = \frac{\alpha}{a\omega^2}(a\omega^2\sin x) = \beta_2 p_y, \tag{7a}$$

where p_y is the horizontal acceleration, and

$$\beta_2 = \frac{-\alpha}{a\omega^2}.$$

This gives β_2 as the fictitious mass term, referred to above.

[1] Lamb, p. 130.

The viscous resisting force, expressed in terms of the velocity, is

$$\beta \cos x = \left(\frac{a\omega}{a\omega}\right)\beta \cos x = \frac{\beta}{a\omega}(a\omega \cos x) = \beta_y V_y. \tag{7b}$$

This indicates that the resisting force is proportional to the velocity; and this is what one would expect, because most of the velocities encountered here are within the range of Stokes' law. With these conditions in mind, one can write the equation of motion in the horizontal direction in a different form from (7):

$$A \sin (x + \gamma) - \beta_y a\omega \cos x = (M + \beta_2)(- a\omega^2 \sin x); \tag{8}$$

and in general,

$$F(y) - \beta_y V_y = (M + \beta_2)p_y. \tag{9}$$

It will now be assumed that a similar form of equation holds for motion in the vertical direction; for experimental results have been found to verify the validity of the assumption.

V_x = vertical velocity at time t,

V_x' = terminal velocity,

m = mass of droplet minus mass of air displaced,

m' = mass of air displaced plus a possible fictitious mass,

$m + m'$ = total mass moving.

The equation of motion is

$$mg - \beta_x V_x = (m + m')\frac{dV_x}{dt} = (M + \beta_2)\frac{dV_x}{dt}. \tag{10}$$

The droplets arrived at terminal velocities before the photographs were taken, so that $(\beta_x V_x') = mg$.

$$\beta_x = mg/V_x, \tag{11}$$

dropping the prime.

Resistance to Motion in the Path of the Droplet.

The expression for the resistance of air, to motion in the path of the droplet is

$$f(t) = f(x) \cos \theta + f(y) \sin \theta = f(x)\frac{V_x}{V_t} + f(y)\frac{V_y}{V_t}, \tag{12}$$

where V_t is the velocity in the path of the droplet, and θ is the angle between the tangent and the X axis, and hence

$$V_x/V_t = \cos \theta, \qquad V_y/V_t = \cos \theta. \tag{13}$$

But $f(x) = \beta_1 V_x$ and $f(y) = \beta_1 V_y$ if $\beta_x = \beta_y = \beta_1$. Therefore

(14) $$f(t) = \beta_1 V_t.$$

The tangential equation of motion is

(15) $$\left[(mg)\frac{V_x}{V_t} + (A \sin x)\frac{V_y}{V_t}\right] - \beta_1 V_t = (M + \beta_2)\left[(-a\omega^2 \sin x)\frac{V_y}{V_t} + \frac{dV_x}{dt}\frac{V_x}{V_t}\right].$$

In general

(16) $$F(t) - \beta_1 V_t = (M + \beta_2)\frac{dV_t}{dt}\Bigg].$$

Theoretical and Experimental Values of the Angle of Lag Compared.

A theoretical expression for the value of γ, the angle of lag, can be deduced, based upon the above assumptions. A verification of this value by experimental results will be a strong argument in favor of these developed relations.

It has been shown that $\beta_1 = (A \sin \gamma)/a\omega$ and

$$+\beta_2 = -\frac{\alpha}{a\omega^2} = \frac{(Ma\omega^2 + A \cos \gamma)}{a\omega^2}$$

where γ is greater than $\pi/2$. This gives

$$\sin \gamma = \frac{\beta_1 a\omega}{A}, \qquad \cos \gamma = -\left(\frac{M + \beta_2}{A}\right)a\omega^2,$$

(17) $$tg\,\gamma = \frac{-\beta_1}{(M + \beta_2)\omega}.$$

But $\beta_1 = mg/V_x$ where V_x is the terminal velocity.

(18) $$tg\,\gamma = \frac{-mg}{(M + \beta_2)V_x\omega}.$$

In the case of a droplet, neglect the fictitious mass β_2 and the mass of the air displaced m', so that $m = M$. Then

(19) $$\tan \gamma = -g/V_x\omega,$$

a known value.

The value of V_x is found from the relation $\lambda/T = V_x$, where λ is the wave-length and T is the known period of oscillation, the frequency of the electromotive force being sixty cycles per second. The value of λ was found by measuring several successive wave-lengths with a pair of fine-pointed dividers and a steel scale.

The angle γ was found by measuring y_1 at the point on the path of the droplet where the photograph indicated the particle to be at the moment of zero field intensity. If the value of y_1 at this point be divided by the amplitude a, $\sin x_1$ is found, since $a \sin x_1 = y_1$.

The value of $(x_1 + \gamma)$ is zero or π at the point where the field intensity is zero.

$$A \sin (x_1 + \gamma) = 0 \text{ for } x + \gamma = 0 \text{ or } n\pi.$$

Since $\cos \gamma$ is negative and $\sin \gamma$ is positive, therefore π is greater than γ, which is greater than $\pi/2$.

Therefore

$$x_1 + \gamma = \pi,$$

$$\gamma = \pi - x_1.$$

Enlargements of the photographs were provided in order to make these measurements, for which a micrometer microscope was used.

In the case of the "spark gap" method of indicating the position of the droplet, the spark occurred five degrees past the position of zero field intensity on the sine curve. A correction of five degrees must be made in this manner:

$$\gamma = 180° - (x_1 - 5).$$

TABLE I.

Water Droplets.

a Cm.	λ Cm.	V_x Cm./Sec.	γ Observed.	γ Calculated.	$(M + \beta_2)/m$.
0.0837	.3410	20.50	171° 12′	172° 45′	0.825
0.2230	.1362	8.16	163° 12′	162° 18′	1.052
0.1305	.1500	9.00	165° 30′	163° 54′	1.043
0.1720	.1525	9.15	166° 30′	164° 10′	1.180
0.0977	.2830	17.00	171° 57′	171° 18′	1.088
0.0887	.1200	7.20	161° 20′	160° 10′	1.037
0.1555	.2490	14.90	168° 38′	170° 12′	0.875
0.0740	.4570	27.40	175° 42′	174° 36′	1.262
0.0337	.4360	26.10	174° 5′	174° 18′	0.965
0.0430	.3970	23.90	174° 7′	173° 48′	1.054
0.0357	.550	33.00	174° 56′	175° 30′	0.885
0.0450	.481	28.80	174° 0′	174° 53′	0.857
				Average.....	1.010

Lycopodium.

0.125	.346	20.7	172° 55′	172° 50′	1.02
0.075	.318	19.1	171° 20′	172° 15′	0.89

The assumptions made, that β_2 is small enough to be neglected, and that the constant $\beta_y = \beta_x = \beta_1$, are entirely justified by these results, within the limits of experimental error. This means that if Stokes' constant ($6\pi\mu r$) holds for the vertical component velocity, it holds for the horizontal and also tangential velocities. This law will be further discussed below, after examining results for the ratio of the fictitious mass to the real mass.

Ratio of the Fictitious Mass to the Real Mass.

We have

$$\beta_2 = -\left(M + \frac{A \cos \gamma}{a\omega^2}\right),$$

from which

$$(\beta_2 + M) = -\frac{A \cos \gamma}{a\omega^2} = \left(\frac{-A \cos \gamma}{a\omega}\right)\frac{\sin \gamma}{\omega \sin \gamma} = -\frac{\beta_1}{\omega \, tg\, \gamma},$$

where

$$\beta_1 = \frac{A \sin \gamma}{a\omega}.$$

But

$$\beta_1 = mg/V_x$$

(20)
$$+ (\beta_2 + M) = -\frac{mg}{V_x\omega \, tg\, \gamma}.$$

For the absolute value of tangent $\gamma = -c$,

(21)
$$\left(\frac{\beta_2 + M}{m}\right) = \frac{g}{V_x\omega c}.$$

This ratio is expressed in terms of quantities which can be obtained experimentally. According to all assumptions made, it should be close to unity for the droplets considered. A glance at the tabulated results of Table I. shows that this is true, and our knowledge of the relative sizes of M and m reveals the smallness of β_2. The ratio M/m in terms of densities, for comparative purposes, is

$$M/m = \frac{\sigma}{\sigma - \rho} = 1.002$$

approximately for water.

The wide variation from the average value which occurs for some of the droplets is because of the smallness of the amplitude a, and consequently, because of the difficulty in measuring the tangent of the angle of lag γ, accurately. The droplets with a large amplitude show more consistent results.

If one takes the average value of $(M + \beta_2)/m$ as given in the table, the ratio of β_2/m can be found for a water droplet, where $M/m = 1.002$ by equation (25).

$$\frac{M + \beta_2}{m} = 1.002 + \frac{\beta_2}{m} = 1.01,$$

$$\beta_2/m = 0.008 \text{ or } \beta_2 = .008 \text{ m}.$$

This agrees with the theoretical value calculated before.

Stokes' law is supposed to give the value of all the viscous resistance which is proportional to the velocity.[1] If such be the case, one may write $\beta_1 = 6\pi\mu r$ and make little error; since the equations deduced from the properties of the sine law show that all of the viscous resistance is given in terms of the first power of the velocity. The experimental results also verify this last conclusion. Since the resistance is proportional to the velocity, and since most of the velocities fall within the limitations of Stokes' law (as explained above) one feels justified in making the above assumption ($\beta_1 = 6\pi\mu r$) even for variable velocities.

The ratio of the charge (Ne) to the mass m can be found from these experimental results without the aid of Stokes' law; but the numerical value of β_1, the charge (Ne) and several other quantities, cannot be so determined. If this law be used to get an independent relation between the quantities involved, several unknown quantities may be evaluated in known terms. The law is stated as follows:

The viscous resistance of air for small spheres is equal to $6\pi ur V$. The coefficient β_1 is then equal to $6\pi\mu r$, where μ is the coefficient of viscosity of air, and r is the radius of the sphere. The value of the mass is found from the relation,

(22) $$\beta_1 = mg/V_x = 6\pi\mu r,$$

$m = \pi r^3(\sigma - \rho)$ where σ is the density of the sphere and ρ the density of air. This gives

$$(4/3)\pi r^3(\sigma - \rho)g/V_x = 6\pi\mu r,$$

$$r^2 = \frac{6\mu V_x 3}{4(\sigma - \rho)g} = \frac{9\mu V_x}{2(\sigma - \rho)g},$$

$$r^3 = \left(\frac{(9\mu V_x)}{2g(\sigma - \rho)}\right)^{3/2},$$

$$\text{volume} = (4/3)\pi\left(\frac{9\mu V_x}{2g(\sigma - \rho)}\right)^{3/2},$$

(23) $$m = (4/3)(\sigma - \rho)\left(\frac{9\mu V_x}{2g(\sigma - \rho)}\right)^{3/2},$$

(24) $$M = (4/3)\pi\sigma\left(\frac{9\mu V_x}{2g(\sigma - \rho)}\right)^{3/2},$$

(25) $$M/m = \frac{\sigma}{\sigma - \rho}.$$

[1] G. G. Stokes, Math. and Physical Papers, Vol. III., p. 59.

VALUE OF THE FICTITIOUS MASS β_2.

For γ greater than $\pi/2$

$$-\beta_2 = M - \frac{A \cos\gamma}{a\omega^2}.$$

Transform $(A \cos\gamma)/a\omega^2$,

$$\frac{A \cos\gamma}{a\omega^2}\cdot\frac{\sin\gamma}{\sin\gamma} = \frac{A \sin\gamma}{a\omega}\cdot\frac{\cos\gamma}{\omega\sin\gamma} = \beta_1/\omega\, tg\,\gamma = mg/V_x\, tg\,\gamma.$$

Substitute

(26) $$-\beta_2 = M - \frac{mg}{V_x\omega\, tg\,\gamma}.$$

Let Q equal the volume of a small sphere of radius r, $Q = (4/3)\pi r^3$,

$$-\beta_2 = Q\left(\sigma - \frac{(\sigma-\rho)g}{V_x\omega\, tg\,\gamma}\right).$$

From (24) and (25)

$$(-\beta_2) = (4/3)\pi\left(\frac{9\mu V_x}{2g(\sigma-\rho)}\right)^{3/2}\left(\sigma - \frac{(\sigma-\rho)g}{V_x\omega\tan\gamma}\right).$$

From the value of $\tan\gamma$ expressed in terms of known quantities, it is easy to calculate the ratio of the charge (Ne) upon the droplet to its weight mass m without Stokes' law. From (1), and (7*b*), the values of A and β_1 are $A = (Ne)X$, $\beta_y = \beta_1$,

$$\beta_1 = \frac{A\sin\gamma}{a\omega}.$$

Solve for A

$$A = \frac{\beta_1 a\omega}{\sin\omega}.$$

But

$$\beta_1 = mg/V_x,$$

$$(Ne)X = \frac{mga\omega}{V_x\sin\gamma},$$

(28) $$\frac{(Ne)}{(m)} = \frac{ga\omega}{XV_x\sin\gamma}.$$

The value of $\sin\gamma$ in terms of $tg\,\gamma$ is

$$\sin\gamma = \frac{tg\,\gamma}{\sqrt{1+tg^2\gamma}} = \frac{g}{\sqrt{(V_x\omega)^2+g^2}},$$

(29) $$\frac{(Ne)}{(m)} = \frac{a\omega}{XV_x}\sqrt{(V_x\omega)^2+g^2}.$$

The value of the charge upon the droplet can be calculated from equations (29) and (23), for from (29)

$$(Ne) = \frac{ma\omega}{XV_x} \sqrt{V_x^2\omega^2 + g^2},$$

and from (23)

$$m = (4/3)\pi(\sigma - \rho)\left(\frac{9\mu V_x}{2g(\sigma - \rho)}\right)^{3/2},$$

the value of (Ne) follows:

$$(30) \qquad (Ne) = (4/3)\frac{\pi(\sigma - \rho)a\omega}{XV_x}(g^2 + V_x^2\omega^2)^{1/2}\left(\frac{9\mu V_x}{2g(\sigma - \rho)}\right)^{3/2}.$$

Calculations of $(Ne)/m$, (Ne) and N from the data for two droplets give the following results:

a Cm.	V Cm./Sec.	X E. S. U.	Ne/m.	(Ne).	N.
.1305	9.00	21.	920	7.98×10^{-4}	1033000
.0977	17.00	7.5	1870	4.2×10^{-4}	880000

This indicates that the droplets pick up an enormous number of elementary charges under the conditions of the experiment, so that the method is valueless for the determination of e, the elementary charge.

Measurements of the Density of Lycopodium.

It was necessary to have the mean density of lycopodium for the above calculations. This was found by a porosity method. A formula for the minimum volume occupied by spheres is given in Reports of the U. S. Geological Survey.[1] The ratio of the volume occupied to the whole volume is 0.7405. The powder is composed of small seeds which are fairly regular in size, and are approximate spheres. A mass of 8.12 grams was found to indicate 19.8 c.c. when settled in a graduate. This indicated volume multiplied by the factor .74 gives 14.66 c.c. for the actual volume occupied. This gives an average density of 0.55 gms./c.c.

In John Zeleny's work on the "Fall of Small Spheres in Air,"[2] the density of lycopodium was found by a volumenometer method, which gave 1.175 gms./c.c. He found that Stokes' law was not obeyed by the powder unless this value was about twice too large. It is very interesting to find that the porosity method gives a corrected value for the density consistent with the assumption that the lycopodium powder was following Stokes' law in Zeleny's experiment.

[1] U. S. Geological Survey, Annual Reports, Vol. 19, 2, p. 301.

[2] Zeleny, Phys. Rev., May, 1910.

RESISTANCE TO THE MOTION OF SPHERES OF SMALL DENSITY FALLING IN AIR.

It was thought desirable to find by experiment, the law of air resistance to the motion of spheres falling with velocities much greater than those which are within the range of Stokes' law.

A light rubber balloon was used for the purpose. A chronograph and chronometer measured the time, and electrical means for releasing and stopping the sphere provided an accurate means for recording the time of fall on the chronograph trace. The weight in air was found to the nearest milligram. The size of the sphere was measured and the density of the air was determined from its temperature and pressure.

The data obtained in this way are given in the following table.

TABLE II.

Data for a Light Sphere Falling in Air.

$V' = 243$ cm./sec. $M = 8.941$ gms.
$m = 5.176$ gms. $r = 9.1$ cm.
$m' = 3.765$ " 6 = .00119 gms. per c.c.
Temperature 22″ to 23° C.
$\beta = mg/V' = 21.7$.

Time of Fall.	Distance.	Time of Fall.	Distance.
.0861 sec.	1.10 cm.	.469 sec.	47.0 cm.
.1084	2.3	.520	56.8
.125	3.4	.602	72.0
.138	3.6	.612	73.5
.182	6.4	.710	93.0
.243	11.2	.812	120.1
.246	12.8	.910	139.3
.317	21.7	.911	141.0
.350	24.2	1.082	173.5
.363	27.0	1.181	195.0
.381	31.4	1.389	240.5
.425	39.4	1.715	217.8
.451	42.3	1.950	373.0

From these results a space and time curve was plotted (Fig. 4). This curve is not a parabola, as it would be if the resistance did not exist. One must therefore find a new relation between time and distance fallen through, which will conform to the time-space curve found by experiment. A resistance proportional to the velocity gives the proper relation. The solid curve (Fig. 4) shows the theoretical results according to this assumption and the circles give the experimental values. The broken curve is one for no air resistance. The terminal velocity reached by the sphere was 243 cm. per sec. This is far in excess of any velocity reached by the droplets.

The mathematical development follows:

Vertical velocity at time $t = V_x$.

Terminal velocity $= V'$.

Mass of sphere minus mass of air displaced $= m$.

Mass of air displaced plus a possible fictitious mass $= m'$.

Total mass moving $= m + m' = (M + \beta_3)$.

Acceleration due to weight without air resistance $= g$.

The equation of motion is exactly the same in form as the one deduced for motion in the horizontal direction, based upon the properties of the sine-curve path of a droplet. The equation is

$$mg - \beta_x V_x = (m + m')\frac{dV_x}{dt} = (M + \beta_3)\frac{dV_x}{dt}. \tag{31}$$

The first integral is

$$V_x = \frac{mg}{\beta_x}\left(1 - e^{-\frac{\beta_x t}{m+m'}}\right) \text{ for } V_x = 0, \qquad t = 0, \tag{32}$$

and the second integral is

$$X = \frac{mg}{\beta_x}t + mg\left(\frac{m + m'}{\beta_x^2}\right)\left(e^{-\frac{\beta_x t}{m+m'}} - 1\right). \tag{33}$$

It is well to note that the value of the constant β_x in this case is not given by Stokes' formula $6\,\pi\mu r$. Its value is $\beta_x = mg/V'$. The terminal velocity V' was 243 cm./sec., which gives

$$\beta_x = \frac{5.176 \times 980}{243} = 21.7,$$

while $6\,\pi\mu r = .0313$.

It is an easy matter to deduce a general relation between the variables involved when light spheres are allowed to fall in the air. It is only necessary to eliminate the exponential term in equations (32) and (33).

From (32)

$$1 - e^{-\frac{\beta_1 t}{m+m'}} = \frac{\beta_x V_x}{mg}.$$

Substitute in (33)

$$X = \frac{mg}{\beta_x}t - \frac{mg}{\beta_x^2}(m + m')\frac{\beta_x V_x}{mg}.$$

Simplify

$$X = \left(\frac{mg}{\beta_x}\right)t - \left(\frac{m + m'}{\beta_x}\right)V_x.$$

Let

$$V' = \frac{mg}{\beta_x}$$

and

$$T = \frac{m + m'}{\beta_x},$$

$$X = V't - TV_x. \tag{34}$$

The initial conditions are $x = 0$, $t = 0$, $V_x = 0$. V' is the terminal velocity, and T is some function of the time. This is true for velocities where the resistance is proportional to the velocity. Fig. 4 shows this is true for terminal velocities as large as 243 cm. per sec.

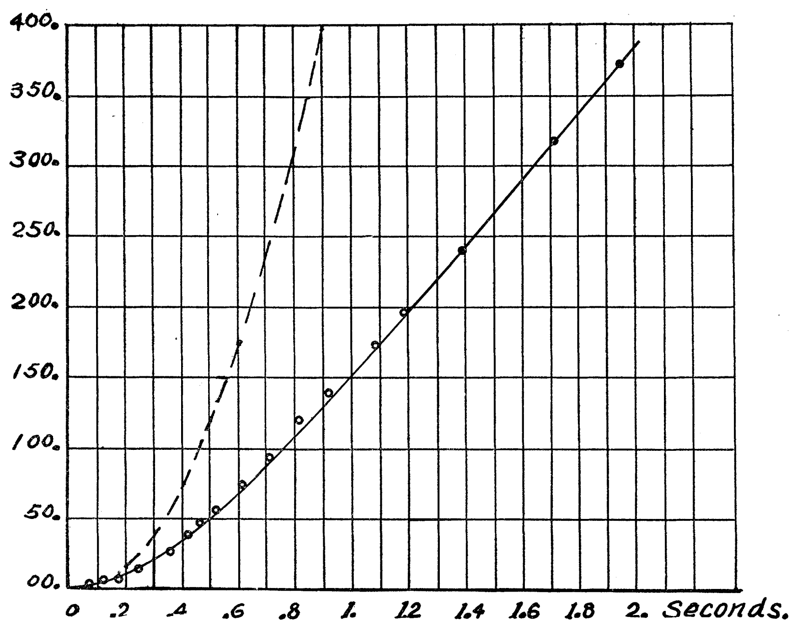

Fig. 4.

After V_x reaches terminal velocity V', TV' becomes a constant, which can be called S, representing some distance to be subtracted from $V't$, or

$$X + S = V't.$$

Conclusions.

The assumption that the vertical component of the velocity of a falling droplet oscillating horizontally in an alternating field is constant throughout the complete cycle, is verified within the limits of experimental error. The resisting force due to air is proportional to the speed in the path of the droplet.

No reason has been discovered why Stokes' constant $6\pi\mu r$ does not apply to this case of accelerated motion, as well as to uniform motion in a straight line.

The inertia of the droplet may be increased by a fictitious mass to a slight degree, due to its motion in the air.

The droplets pick up a very large number of elementary charges, so that the experiment is useless for finding the value of e.

For much larger speeds than are within the limits of Stokes' law, the resisting force is proportional to the velocity.

In conclusion, I wish to thank Prof. E. P. Lewis under whose direction the work has been carried on, for his many suggestions.

University of California,
April 15, 1918.

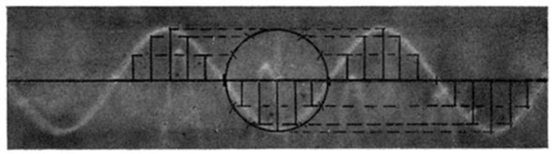

Fig. 1.

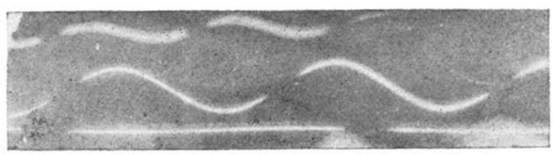

Fig. 2.

Fig. 3.