Soluble Three-Dimensional Model for Townsend's α*

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A model gas is considered in which all electron-molecule cross sections are isotropic and depend inversely on the velocity v. Collisional energy loss is neglected. The Boltzmann equation for the model is solved for the collision density, where the collision density is the number of collisions that an individual electron makes between v and $v+dv$ over its entire history. The Townsend α is obtained from the collision density, and it is found that α/p is inversely proportional to E/p. It is argued that this model furnishes an upper bound to the true α/p for all E/p; therefore it is concluded that this model demonstrates that at sufficiently high E/p the observed α/p for any real gas must decrease with increasing E/p. The results also shed light on the way electron energy balance or lack of energy balance affects α/p and the drift velocity v_D; it is shown that energy balance is not possible at arbitrarily large E/p. Numerical applications of these results to H_2 are discussed.

I. INTRODUCTION

TOWNSEND'S first coefficient α is defined[1] as the average number of secondary electrons produced in a gas at pressure p by a primary electron which advances one cm along an impressed electric field E. The Boltzmann equation (B.E.) implies that for any one gas the quantity α/p is a function of E/p only. Experimental curves of α/p vs E/p have been obtained in a variety of gases,[1] with E/p ranging from zero to several thousand volts/cm-mm Hg (the conventional units). Typically α/p increases monotonically with increasing E/p throughout the entire observed range of E/p. Experiments exhibiting a maximum in the α/p vs E/p curve exist, but these are atypical and have not gained general acceptance.[2] Reported theoretical derivations[3] of the dependence of α/p on E/p all predict α/p increases monotonically with E/p, in agreement with the typical experimental findings.

On the other hand the following seemingly generally accepted[1,2] simple argument indicates α/p should decrease with increasing E/p at high E/p: At high E/p an electron gains more energy from the field during a mean free path than it can surrender either elastically or inelastically in a collision with a gas molecule, so that at high E/p an electron rapidly accelerates to high energies, where ionization and total cross sections decrease with increasing electron energy[4]; hence with increasingly large E/p α/p (essentially the number of secondary electrons produced per gas molecule encountered in one cm advance along E) decreases because the electron ionization rate decreases while its mean free path (in other words its rate of drift down the field) increases. In fact we have been able to construct several one-dimensional (electrons move only parallel or antiparallel to applied E) models of ionization growth for which the B.E. can be solved exactly,[5] without any further approximation beyond the simplifications originally introduced into the models. These models all showed α/p decreasing at high E/p, as expected from the above simple argument.

The present work is motivated by recent efforts aimed at producing high-energy plasmas for thermonuclear research. To properly understand these experiments the behavior of α/p at very large E/p is required.

II. MODEL

In the present paper we examine a three-dimensional model of ionization growth. In particular, we assume: (i) electrons start out with velocity $v=0$ from a uniform infinite (along x and y) plane cathode at $z=0$, emitting at a constant rate; (ii) the electric field is constant and antiparallel to z, so that between collisions the electrons move with uniform acceleration $a=eE/m$ along positive z; (iii) the electrons collide only with the infinitely massive molecules of a neutral gas, in such a fashion that either they make a purely elastic collision or else they ionize the neutrals—no other types of collisions take place; (iv) all ionized electrons start out with zero velocity, and the incident ionizing electron loses no energy in the process of creating a secondary—thus every electron is created at zero energy and thereafter gains or loses energy only through motion along or against the applied electric field; (v) the total cross section σ and the ionization cross section σ_i for collisions between an electron and a neutral molecule are each spherically symmetric and inversely proportional to the incident electron speed v.

III. DISCUSSION

Our neglect of electron-electron and electron-ion collisions, and of electronic and ionic contributions to the electric field, does not restrict the utility of the above model, since it is implicit in the definition of Town-

* This work was done under a joint General Atomic-Texas Atomic Energy Research Foundation program on controlled thermonuclear reactions.

[1] A. von Engel, *Ionized Gases* (Clarendon Press, Oxford, 1955), pp. 155–157.

[2] L. B. Loeb, *Basic Processes of Gaseous Electronics* (University of California Press, Berkeley, California, 1955), pp. 652 and 668.

[3] T. J. Lewis, Proc. Roy. Soc. (London) **A244**, 166 (1958), who computes α/p vs E/p for H_2, gives references to previous theories of α/p vs E/p.

[4] H. S. W. Massey and E. H. S. Burhop, *Electronic and Ionic Impact Phenomena* (Clarendon Press, Oxford, 1952), pp. 30–188.

[5] E. Gerjuoy and G. W. Stuart (to be published).

send's α that the electron density is small. In other respects our assumptions are very restrictive however, particularly our assumption that the ionization rate $v\sigma_i$ and the total collision rate $v\sigma$ per gas molecule are constant. With this restriction on the energy dependence of σ and σ_i (a restriction which is unnecessary in our one-dimensional models) we find, essentially exactly,

$$\alpha=(s/a\tau^2)(\sigma_i/\sigma), \tag{1}$$

where the constant $\tau=(Nv\sigma)^{-1}$ is the mean free time for collision, with N of course the number of gas molecules per cc. The quantity s is about equal to 3, but is a function of σ_i/σ. Equation (1) implies

$$\alpha/p=19.2s(\sigma_i\sigma W)(E/p)^{-1}, \tag{2}$$

where p is in mm Hg, E is in volts/cm, σ_i and σ are in units of $\pi a_0^2=8.8\times10^{-17}$ cm^2, and W is the energy, in electron volts, at which σ_i and σ are evaluated.

Equation (2) shows that our three-dimensional model, like our one-dimensional models but unlike previous theories, makes α/p a decreasing function of E/p at high E/p. In fact α/p is inversely proportional to E/p at all E/p, according to Eq. (2). We make no claim that this simple result, a consequence of the excessive simplicity of our model, represents the variation with E/p of the actual values of α/p in a real discharge. We do claim however that at any E/p Eq. (2) predicts values of α/p exceeding the real values, provided the v^{-1}-proportional σ_i and σ of Eq. (2) are so chosen that they exceed the actual cross sections at all energies, i.e., provided $\sigma_i(W)>\sigma_i'(W)$, $\sigma(W)>\sigma'(W)$ at every W, where the primes denote the actual measured cross sections. More specifically we are claiming that with σ_i and σ chosen as described, our model surely overestimates the rate at which electrons ionize, and equally certainly underestimates the rate at which electrons drift down the applied field. There is no virtue in defending these claims at low E/p, where the values of α/p obtained from (2) are ridiculously large. At high E/p however, where Eq. (2) yields small values of α/p, these claims are more meaningful and we are constrained to defend them. The key fact is that at large v actual cross sections σ' and σ_i' are not isotropic and inversely proportional to v^{-1}, but instead[4] peak forward and decrease as $v^{-2}\ln v$. It follows that in an actual high E/p discharge: (a) electrons reach high energy more readily than our model assumes; (b) a high speed electron moving toward the anode has much less chance than in our model of being turned back toward the chathode, and thereby of being slowed down by motion in the direction of increasing potential energy. These effects (a) and (b) mean that in an actual high E/p discharge electrons drift down the applied field more rapidly than our model predicts, as claimed above, and also that: (c) electrons remain at high energy more readily than our model predicts, at which energies they ionize less readily (again as claimed above) because the actual high energy σ_i' decreases more rapidly than v^{-1}. We recognize that we also have neglected inelastic and elastic energy losses, which tend to keep the electron speeds low, and which therefore in a high E/p high electron energy discharge tend to raise the ionization rate. At high E/p however, when the collisional energy loss per mean free path is much less than the energy gained from the field, collisions should decrease the mean electron energy by only a few percent, and therefore should increase the ionization rate by only an equally small percentage. Consequently we judge this last effect to be unimportant compared to the aforementioned effects (a), (b), and (c). Moreover, as will be discussed in a moment, neglecting collisional energy loss terms in the Boltzmann equation can (and in our model seems to) *decrease* the electron drift velocity down the field, therewith at high E/p further increasing our model's predicted α/p beyond the real values.

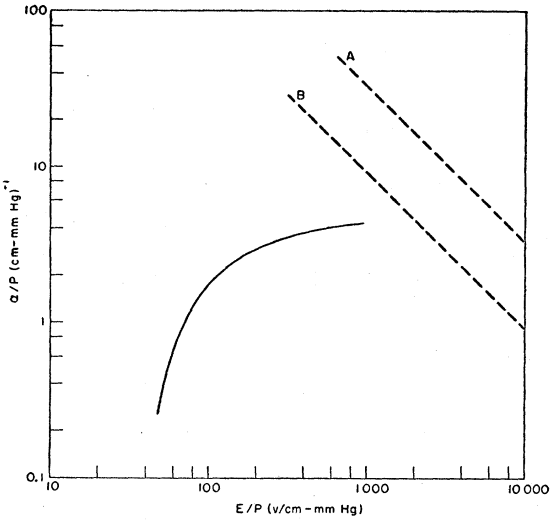

FIG. 1. Comparison of our theoretical values of α/p with experiment. The solid curve is the experimental result. Curve A is a theoretical upper bound on α/p. Curve B is a less extreme upper bound. $s=3.75$ [see discussion under (16)].

We conclude that Eq. (2) demonstrates the actual values of α/p for a real gas must decrease with increasing E/p at sufficiently high E/p, as expected from the argument given in the second paragraph. The import of this conclusion is illustrated in Fig. 1, wherein the solid line is the experimentally observed[6] curve of α/p vs E/p for H_2, and the upper dashed curve A is a plot of Eq. (2), using $\sigma_i=13.6$ and $\sigma=34$ at $W=1$ ev; these values of σ_i and σ satisfy the requirement $\sigma_i(W)>\sigma_i'(W)$, $\sigma(W)>\sigma'(W)$ at all energies. Curve A shows α/p for H_2 must begin to decrease somewhere between $E/p=1000$ and $E/p=7500$; at $E/p=7500$, the height of curve A equals the observed α/p at $E/p=1000$. Actually the discussion of the previous paragraph strongly suggests it is not necessary to have $\sigma_i>\sigma_i'$,

[6] D. J. Rose, Phys. Rev. **104**, 273 (1956).

$\sigma > \sigma'$ at all W in order that at high E/p the α/p from (2) assuredly exceed the real α/p. Curve B of Fig. 1, which should be a less extreme upper bound than A to the real high E/p values of α/p in H_2, is obtained using in Eq. (2) more realistic v^{-1} fits to the measured[7,8] cross sections, namely $\sigma_i = 7.2$ and $\sigma = 18$ at $W = 1$ ev. Curves B of Fig. 1 suggests α/p in H_2 begins to decrease somewhere between $E/p = 1000$ and $E/p = 2000$. These estimates from Fig. 1 of the value of E/p where α/p in H_2 begins to decrease agree with estimates based on our one-dimensional models.[5]

IV. ENERGY BALANCE

Confining our attention to steady time-independent dc discharges, with the cathode at $z=0$ and the electron distribution function $f(z,\mathbf{v})$ independent of x, y, we observe that previous theories[3] have computed α from the relation (equivalent to the definition of α at the beginning of this paper)

$$\alpha = \int d\mathbf{v}\, N\sigma_i v f \Big/ \int d\mathbf{v}\, v_z f, \tag{3}$$

together with the assumptions: (i) the electron velocity distribution is independent of z, i.e.,

$$f(z,\mathbf{v}) = n(z)g(\mathbf{v}); \tag{4}$$

(ii) ionization has a negligible effect on f, i.e., ionization terms are dropped from the Boltzmann equation (B.E.) determining f; (iii) $g(\mathbf{v})$ deviates only slightly from spherical symmetry, so that a truncated spherical harmonic expansion can be employed to find g. We have no objections *per se* to Eqs. (3) and (4). As our one-dimensional models explicitly demonstrate, when $\alpha \neq 0$ Eq. (4) must hold at sufficiently large z, in which event it follows rigorously from the v_z^0 moment of the B.E. including ionization that $n(z) = n_0 \exp(\alpha z)$, with α given by (3). On the other hand, as our one-dimensional models also demonstrate, g depends on α and therefore cannot be independent of σ_i, i.e., dropping the ionization terms from the B.E. may greatly alter the electron distribution function. In fact if the electrons gain energy from the electric field more rapidly than they lose energy by collision, as in the case of our three-dimensional model which lacks any collisional energy loss mechanism, then with no ionization ($\alpha = 0$) all electrons originate at $z=0$ and the electron velocity distribution never becomes independent of z. Moreover the assumption that $g(\mathbf{v})$ is nearly spherically symmetric seems particularly dangerous at high E/p, where an electron acquires large momentum along the field between every pair of collisions.

For these reasons we stress that our derivation of (1), given in the following section, makes none of the above assumptions (i)–(iii), and obtains α without using (3) and without explicit determination of f. Especially noteworthy is the importance of not assuming electron energy balance. For our model Eq. (3) yields

$$\alpha = \sigma_i/\sigma\tau v_D, \tag{5}$$

where the drift velocity v_D customarily is supposed to equal $a\tau$ in the case of constant mean free time. Thus Eq. (5) does not seem to agree with (1); rather (1) seems to require $v_D \cong \frac{1}{3}a\tau$. To elucidate this apparent discrepancy, consider the v_z moment[9] of the B.E.

$$\frac{\partial}{\partial z}\int d\mathbf{v}\, v_z^2 f - a\int d\mathbf{v}\, f + \tau^{-1}\int d\mathbf{v}\, v_z f = 0, \tag{6}$$

where we have specialized to constant mean free time and spherically symmetric cross sections; in this circumstance the ionization cross section rigorously disappears from (6). Now when the first term in (6) vanishes, as when there is no ionization and the electrons are in energy balance, Eq. (6) implies the usual $v_D = a\tau$. Neglecting ionization however, at each z all electrons have the same kinetic energy in our model, namely $v^2 = 2az$; consequently, making the further assumption that the distribution function is nearly spherically symmetric, $\langle v_z^2\rangle_{\text{av}} = \frac{1}{3}v^2 = \frac{2}{3}az$, which substituted in (6) yields $v_D = \frac{1}{3}a\tau$. We remark that since the first term in (6) does not vanish when there is ionization growth or when there is no electron energy balance, it is not true that for constant mean free time quantities of physical interest like v_D always can be obtained[10] merely from moments of the B.E.

Since f need not be nearly spherically symmetric, and is z dependent when $\alpha \neq 0$, the simple argument of the preceding paragraph hardly can be the whole

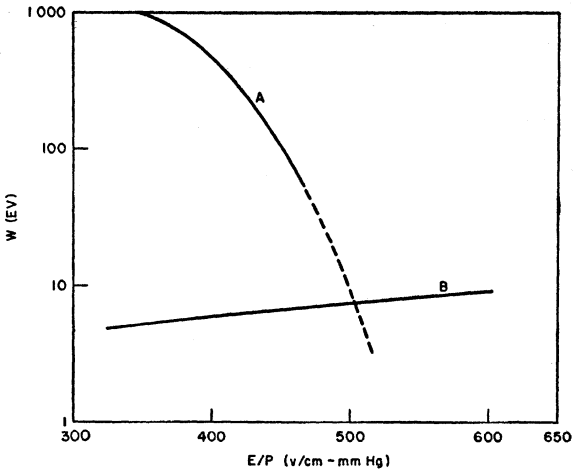

FIG. 2. Electron energy balance. Curve A shows the energy at which the right side of (7) equals eEv_D. Curve B shows the energy at which the right side of (9) equals $\frac{1}{2}$.

[7] C. E. Normand, Phys. Rev. **35**, 1217 (1930).
[8] J. T. Tate and P. T. Smith, Phys. Rev. **39**, 270 (1932).
[9] W. P. Allis, *Handbuch der Physik* (Springer-Verlag, Berlin, 1956), Vol. XXI, p. 421.
[10] J. C. Maxwell, *Collected Papers* (Dover, New York, 1952), Vol. II, p. 40.

story, but it seems clear that the factor s in (1) is connected with the tendency for v_D to equal $\sim\frac{1}{3}a\tau$ in our model, instead of $a\tau$. The first term in (6) represents a back pressure from electrons which are diffusing back to lower energy (smaller z) after making collisions at higher energy (larger z); this back diffusion is diminished by collisional energy losses, which prevent the electrons from penetrating to as small z as they would without losses. Evidently in an actual discharge as well as in our idealized model, failure to achieve energy balance tends to make v_D less, and α correspondingly larger, than might otherwise have been expected. Thus it is consequential that the electrons in an actual discharge cannot remain in energy balance as $E/p\rightarrow\infty$, since in this limit v_D and therefore the rate of energy gain eEv_D is increasing, whereas the increasing average electron energy decreases the rate of energy loss. In H_2 for instance, the rate of energy loss by an electron of high energy $W=\frac{1}{2}mv^2$ is[11]

$$\frac{dW}{dt}=v\frac{dW}{dx}\cong v\frac{4\pi e^2N}{W}\ln\frac{2W}{I}, \tag{7}$$

where $I=13.6$ ev, and we have assumed one hydrogen molecule is equivalent to two hydrogen atoms. The solid portion of curve A of Fig. 2 plots vs E/p the energy W_1 in ev at which the right side of (7) equals eEv_D, using the smaller possibility $v_D=\frac{1}{3}a\tau$ to compensate for the fact that the high-energy formula (7) probably considerably underestimates the energy loss at the low energies of interest here; $\tau=(N\sigma v)^{-1}$ was computed using $\sigma=18$ at an energy of 1 ev, as in curve B of Fig. 1. For given E/p, energy balance is possible only for electrons with energies $W\leq W_1$. For $E/p>470$, the form (7) implies the energy gain exceeds the energy loss at all electron energies, but the true curve A presumably continues on to higher E/p as indicated by the dashed extrapolation. The probability that an electron starting at $v=0$ goes a time t without making a collision is[5]

$$Q(t)=\exp\left\{-\int_0^t dt'Nv(t')\sigma[v(t')]\right\}, \tag{8}$$

where $v=at'$. Hence the probability that an electron starting with zero energy attains an energy W in ev before making its first collision is

$$Q\left[\left(\frac{2W}{ma^2}\right)^{\frac{1}{2}}\right]=\exp\left\{-3.1\frac{p}{E}\int_0^W dW'\sigma(W')\right\}, \tag{9}$$

where the units are as in Eq. (2), and we have used $dW=mvdv=mvadt$. Curve B of Fig. 2 plots vs E/p the energy W_2 at which $Q=\frac{1}{2}$, with the integral (9) computed from the observed[7] σ for electrons on H_2, plus a reasonable extrapolation[5] therefrom at energies exceeding 40 ev, the upper energy limit of measurement. For given E/p, an electron starting at $v=0$ has probability equal to $\frac{1}{2}$ of reaching an energy $W\geq W_2$ before making its first collision. Comparison of curves A and B of Fig. 2 indicates that in an H_2 discharge electrons probably are in energy balance at E/p's$<\sim500$, but are unlikely to be in energy balance at higher E/p's.

V. DERIVATION OF α

Let us compute the total number $N(Z)$ of electrons produced on our model in the interval $0\leq z\leq Z$ as a consequence of the emission of a single primary electron from the cathode at $z=0$. Suppose $n(z)dz$ is the number of secondaries produced by the primary in the interval z to $z+dz$. Each secondary is created with speed zero, so that a secondary created at $z=z_1$ never can be found at $z<z_1$; moreover in moving from z_1 to Z the secondary born at z_1 causes the ultimate creation of exactly as many electrons as would the original primary in moving from the cathode to $z=Z-z_1$. Thus

$$N(Z)=\int_0^Z dz\,n(z)+\int_0^Z dz\,n(z)N(Z-z). \tag{10}$$

Suppose now that in the course of its entire history an electron makes $\psi(v)dv$ collisions during the times when its speed was between v and $v+dv$; $\psi(v)$ is termed the collision density. On our model

$$\psi(v)=\frac{v}{(a\tau)^2}\left[3+\int_0^1\frac{dy}{y^2}\frac{e^{-v/a\tau y}}{(1-y\tanh^{-1}y)^2+(\frac{1}{2}\pi y)^2}\right]. \tag{11a}$$

This result is derived below. Eq. (11a) is plotted as curve A in Fig. 3. We plot

$$\psi(v)=\frac{v}{(a\tau)^2}\left[3+\frac{0.738a\tau}{v}\exp[-1.25v^2/(a\tau)^2]\right], \tag{11b}$$

as curve B. This fit to the exact $\psi(v)$ is used in the subsequent analysis. Because there is no energy loss on our model, the primary speed at $z=z_1$, is uniquely determined, so that by the definition of $\psi(v)$

$$n(z)dz=(\sigma_i/\sigma)\psi(v)dv, \tag{12}$$

where $v^2(z)=2az$.

We now solve the integral equation (10). Straightforward application of Laplace transforms[12] presents

$$N(Z)=\frac{1}{2\pi i}\int_c e^{\lambda Z}\frac{\phi(\lambda)d\lambda}{\lambda[1-\phi(\lambda)]}, \tag{13a}$$

where, using (12), the Laplace transform of $n(z)$ is

$$\phi(\lambda)=\int_0^\infty dz\,e^{-\lambda z}n(z)=\frac{\sigma_i}{\sigma}\int_0^\infty dv\exp(-\lambda v^2/2a)\psi(v). \tag{13b}$$

[11] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), p. 252.

[12] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, New York, 1953), pp. 972–75.

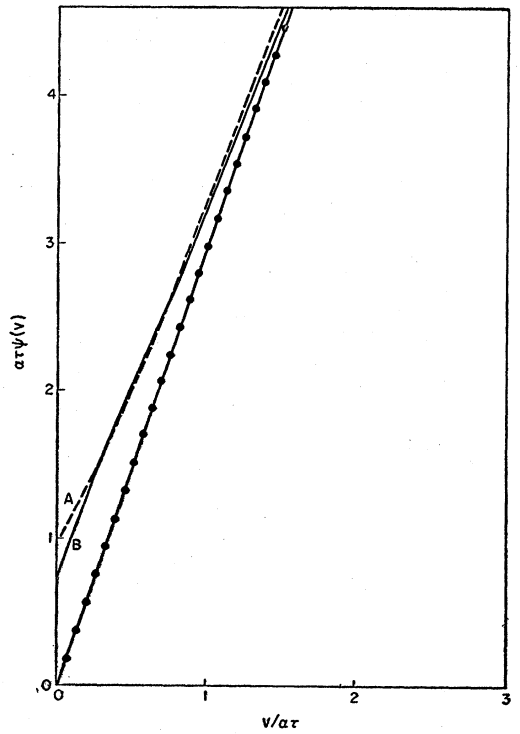

FIG. 3. Electron collision density. Curve A is the exact (11a). Curve B is the form (11b). Also plotted is the asymptote (line through the origin) which $a\tau\psi(v)$ approaches at large $v/a\tau$.

Explicitly, with (11b) in (13b), Eq. (13a) becomes

$$N(Z)=\frac{1}{2\pi i}\int_c \frac{dw}{w}\exp(wZ/a\tau^2)\times\left(\frac{3x(w+2.5)^{\frac{1}{2}}+0.738x(\pi/2)^{\frac{1}{2}}w}{(w-3x)(w+2.5)^{\frac{1}{2}}-0.738x(\pi/2)^{\frac{1}{2}}w}\right), \quad (14)$$

$$x=\sigma_i/\sigma,$$

where the contour of integration is the vertical line Γ shown in Fig. 4 and the real part of $(w+2.5)^{\frac{1}{2}}$ is ≥ 0. The integrand of (14) has a branch point at $w=-2.5$ and poles at $w=0$ and at the three roots of the denominator within the bracket. It can be shown that all three roots are real, but only one is positive. Closing the contour at infinity and around the branch line, the pole due to the single positive root w_1 contributes a term to (14) which increases exponentially with Z. The other singularities contribute terms to (14) which are either constant (the pole at $w=0$), exponentially decreasing (the negative poles), or integrals whose integrands are exponentially decreasing with Z (the branch line integrals). Thus as $Z\to\infty$, the total number of electrons produced in $0\leq z\leq Z$ as a consequence of emission of a single primary from the cathode is

$$N(Z)\cong N_0 e^{\alpha Z}, \quad (15a)$$

where

$$\alpha\equiv w_1/a\tau^2, \quad (15b)$$

and N_0 is a number independent of Z. Suppose next that the cathode has been emitting electrons uniformly for a sufficiently long time that a steady state has been attained in the region $0\leq z\leq Z$. Then the net number of electrons per second crossing the surface $z=Z$ in the direction of increasing z is steady and must equal $N_0e^{\alpha Z}$ times the number of electrons emitted from the cathode per second. In other words, the current density in our model discharge is

$$i(Z)=i_0e^{\alpha Z}, \quad (16)$$

so that α defined by (15b) is Townsend's first coefficient. The w_1 which corresponds to our H_2 values $\sigma_i/\sigma=13.6/34=7.2/18=0.4$, is $w_1=1.5$. For H_2 then, $s=1.5/0.4=3.75$ in (1). When the integral in (11a) is neglected, $s=3$. The same result $s=3$ is obtained in the limit $\sigma_i/\sigma\to 0$.

We do not maintain that replacing (11a) with (11b) yields a Laplace transform of $N(Z)$ [integrand of (14)] with the correct behavior in the entire complex w-plane. The essential point is that the behavior of $N(Z)$ at large Z is determined solely by the largest value of $\lambda\equiv w/a\tau^2$ solving $\phi(\lambda)=1$ in (13a). Because $\psi(v)$ is everywhere positive, on the positive real λ axis the values of $\phi(\lambda)$ from (13b) are only slightly changed by the slight approximation involved in replacing (11a) by (11b), so that the largest positive root of (14) necessarily is very close to the largest root of the exact $\phi(\lambda)=1$. The accuracy of the fit of (11b) to (11a) could be improved by employing a series of several Gaussians, with only an increase in the algebraic complexity of the equation $\phi(\lambda)=1$. However we believe that (11b) reproduces (11a) with sufficient

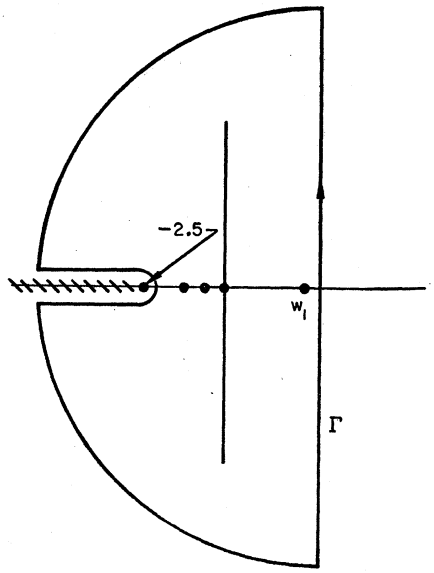

FIG. 4. Contour of integration in w plane for (14).

precision to allow us to term our result for α essentially exact.

The ratio σ_i/σ need not be taken constant. Equation (13b) can be evaluated, at least numerically, for σ_i an arbitrary function of v consistent with the $1/v$ nature of σ. The α comes, as before, from the (positive) root of $\phi(\lambda)=1$.

VI. COLLISION DENSITY

There remains the problem of deriving (12). The quantity $\psi(v)$ is defined in terms of the history (in velocity space) of any one electron. This history does not depend on the location (in coordinate space) of the point where the electron is born. Thus for the purposes of finding $\psi(v)$ we imagine a uniformly distributed (from $z=-\infty$ to $z=+\infty$ in coordinate space) initial distribution $f_0(\mathbf{v})\equiv f(\mathbf{v},0)$ of electrons at $t=0$, for which the distribution function $f(\mathbf{v},t)$ at later times, after colliding with the infinitely massive neutral gas molecules, satisfies the B.E.,

$$\frac{\partial f}{\partial t}+a\frac{\partial f}{\partial v_z}=-Nv\sigma(v)f+\int d\mathbf{v}'f(\mathbf{v}')Nv'\sigma(\mathbf{v}'\rightarrow\mathbf{v}), \quad (17)$$

where the total cross section $\sigma(v)$ and the ionization cross section $\sigma_i(v)$ satisfy

$$\sigma(v)=\int d\mathbf{v}'\sigma(\mathbf{v}\rightarrow\mathbf{v}'), \quad (18a)$$

$$\sigma_i(v)=\int d\mathbf{v}'\sigma_i(\mathbf{v}\rightarrow\mathbf{v}'). \quad (18b)$$

Although $\sigma(\mathbf{v}\rightarrow\mathbf{v}')$ includes $\sigma_i(\mathbf{v}\rightarrow\mathbf{v}')$, terms corresponding to creation of electrons by ionization are omitted from the right side of (17), since we are interested in the history of *an* electron, not in the electrons this original electron creates. Hence, substituting (18a) in (17)

$$\frac{d}{dt}\int d\mathbf{v}\, f=0. \quad (19)$$

We remark that electron indistinguishability in the ionization process is inconsequential; it doesn't matter which of the two outgoing electrons is termed the primary and which the secondary, provided that (19) holds and that the integral on the right of (18b) yields the correct total ionization cross section $\sigma_i(v)$.

The definition of $\psi(v)$ implies

$$\psi(v)dv=v^2dv\int_0^\infty dt\int d\mathbf{n}\, Nv\sigma(v)f(v\mathbf{n},t). \quad (20)$$

Equation (20) has the noteworthy feature that $\psi(v)$ for the original discharge, which contains a source at $z=0$ and electron creation processes, is obtained from the solution to (17), which corresponds to a spatially uniform nonionizing ensemble of electrons drifting down the applied electric field. Furthermore the possible utility of (20) depends entirely on whether or not the cross sections, which have not yet been specified in (17), are such that an electron can attain energy balance. If energy balance occurs, an electron spends all its time at finite speeds and (20) is infinite; conversely if energy balance is not attained the integral (20) converges, since then the probability of finding the electron with a finite velocity v becomes vanishingly small as $t\rightarrow\infty$. Hence it is precisely because our model has no collision loss mechanisms, a feature which might ordinarily be regarded as a deficiency, that we are able to introduce usefully the concept of the collision density.

The probable number of electrons with velocity $\mathbf{v}$ at t equals the probable number of electrons furnished at $t=0$ which can reach $\mathbf{v}$ at t without collision, plus the probable number of electrons reaching $\mathbf{v}$, t which made their last collisions at intermediate times T, $0\leq T\leq t$. Therefore, referring to Eq. (8), Eq. (17) is satisfied (as performing the indicated differentiations will verify) by the solution to the integral equation

$$f(\mathbf{v},t)=\int_0^t dT\int d\mathbf{v}'Nv'f(\mathbf{v}',T)\sigma[\mathbf{v}'\rightarrow\mathbf{v}-\mathbf{a}(t-T)]\times\exp\left\{-\int_0^{t-T}dt'Nv(t')\sigma[v(t')]\right\}+f_0(\mathbf{v}-\mathbf{a}t)\times\exp\left\{-\int_0^t dt'Nv(t')\sigma[v(t')]\right\}, \quad (21)$$

where in the exponential under the integral sign

$$\mathbf{v}(t')=\mathbf{v}-\mathbf{a}(t-T-t'), \quad (22a)$$

and in the exponential multiplying f_0

$$\mathbf{v}(t')=\mathbf{v}-\mathbf{a}(t-t'). \quad (22b)$$

We now specialize to our constant mean free time model, for which

$$\sigma(\mathbf{v}\rightarrow\mathbf{v}')=(1/4\pi)(1/v^2)\delta(v-v')\sigma(v), \quad (23a)$$

$$f_0(\mathbf{v})=(1/4\pi v_0^2)\delta(v-v_0), \quad (23b)$$

wherein $\delta(x)$ is the one-dimensional Dirac delta function. Equation (23a) is in agreement with (18a) and the assumption of isotropic collisions without energy loss; in Eq. (23b) v_0 is a small positive quantity which ultimately is permitted to approach zero, in agreement with the assumption that every electron is born with zero speed. Introducing the Laplace transform of f,

$$g(\mathbf{v},\lambda)=\int_0^\infty dt\, e^{-\lambda t}f(\mathbf{v},t). \quad (24)$$

Equations (21)–(24) yield

$$g(\mathbf{v},\lambda)=\int_0^\infty dt\, e^{-\lambda t}\int_0^\infty dT\int d\mathbf{v}'\frac{f(\mathbf{v}',T)}{4\pi v'^2\tau}\times\delta[v'-|\mathbf{v}-\mathbf{a}(t-T)|]e^{-(t-T)/\tau}+\int_0^\infty dt\, e^{-\lambda t}\frac{\delta[|\mathbf{v}-\mathbf{a}t|-v_0]}{4\pi v_0^2}e^{-t/\tau}. \quad (25)$$

In the multiple integration of (25) first interchange the order of integration of t and T, so that t is integrated between T and ∞; then introduce the variable $u=t-T$. Thus we obtain an integral equation for g, namely

$$g(\mathbf{v},\lambda)=\int_0^\infty du\int d\mathbf{v}'\, e^{-\lambda u}e^{-u/\tau}\frac{g(\mathbf{v}',\lambda)}{4\pi v'^2\tau}\delta[v'-|\mathbf{v}-\mathbf{a}u|]+\int_0^\infty du\, e^{-\lambda u}e^{-u/\tau}\frac{\delta[|\mathbf{v}-\mathbf{a}u|-v_0]}{4\pi v_0^2}. \quad (26)$$

Next we integrate (26) over all directions $\mathbf{n}$ of $\mathbf{v}=v\mathbf{n}$. This integration is performed introducing the new variable

$$y=|\mathbf{v}-\mathbf{a}u|=(v^2+a^2u^2-2avu\mu)^{\frac{1}{2}},\qquad dy=-(avu/y)d\mu, \quad (27)$$

in place of $\mu\equiv\cos\theta$, where of course $d\mathbf{n}=\sin\theta d\theta d\phi=d\mu d\phi$. We find

$$\int d\mathbf{n}\, g(\mathbf{v},\lambda)=\frac{1}{\tau}\int_0^\infty du\, e^{-\lambda u}e^{-u/\tau}\int_0^\infty dv'\times\int d\mathbf{n}'g(\mathbf{v}',\lambda)\frac{2\pi}{4\pi}\int_{|v-au|}^{v+au}dy\frac{y}{avu}\delta(v'-y)+\int_0^\infty du\, e^{-\lambda u}e^{-u/\tau}\frac{2\pi}{4\pi v_0^2}\times\int_{|v-au|}^{v+au}dy\frac{y}{avu}\delta(y-v_0). \quad (28)$$

In Eq. (28) v and a are no longer vectors, i.e., $|v-au|$ is simply the absolute magnitude of the number $v-au$, with v, a, u all intrinsically positive.

With the definition

$$H(v,\lambda)=\frac{v}{\tau}\int d\mathbf{n}\, g(\mathbf{v},\lambda), \quad (29)$$

Eq. (29) can be put in the form

$$H(v,\lambda)=\int_0^\infty dv'K(v,v',\lambda)\left[H(v',\lambda)+\frac{1}{v'}\delta(v'-v_0)\right], \quad (30)$$

where the kernel

$$K(v,v',\lambda)=\frac{1}{2\tau a}\int_0^\infty du\frac{e^{-u/\tau}e^{-\lambda u}}{u}\int_{|v-au|}^{v+au}dy\,\delta(y-v'). \quad (31)$$

Since the integral over the δ function in (31) vanishes unless

$$|v-au|<v'<v+au, \quad (32)$$

it follows that, for any positive v and v'

$$K(v,v',\lambda)=\frac{1}{2a\tau}\int_{|v-v'|/a}^{(v+v')/a}\frac{du}{u}e^{-u/\tau}e^{-\lambda u}=\frac{1}{2a\tau}\left\{E_1\left[\frac{(\lambda\tau+1)|v-v'|}{a\tau}\right]-E_1\left[\frac{(\lambda\tau+1)(v+v')}{a\tau}\right]\right\}, \quad (33)$$

with E_1 the exponential integral

$$E_1(x)=\int_1^\infty du\frac{e^{-xu}}{u}=\int_x^\infty du\frac{e^{-u}}{u}. \quad (34)$$

VII. CONNECTION WITH NEUTRON TRANSPORT THEORY

It now can be seen that Eq. (30) is identical with an integral equation which occurs in a much-studied neutron transport problem, namely the problem of the neutron density in a homogeneous isotropic scattering medium containing an isotropic point source of neutrons; in fact the results for the neutron transport problem immediately yield the solution to Eq. (30), and thereby yield the collision density $\psi(v)$, since from Eqs. (20), (24), and (29),

$$\psi(v)=vH(v,0). \quad (35)$$

Case, de Hoffmann and Placzek[13] solve the integral equation

$$\rho(\mathbf{r})=\int d\mathbf{r}'\frac{e^{-|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|^2}\left[c\rho(\mathbf{r}')+\frac{1}{4\pi r'^2}\delta(r')\right]. \quad (36)$$

They show ρ is spherically symmetric, and when $c<1$ is given by

$$\rho(r,c)=\frac{1}{4\pi r}\left[-\frac{\partial K_0^2}{\partial c}e^{-K_0r}+\int_0^1\frac{d\mu}{\mu^2}e^{-r/\mu}\frac{1}{(1-c\mu\tanh^{-1}\mu)^2+(\frac{1}{2}\pi c\mu)^2}\right], \quad (37)$$

[13] K. M. Case, F. de Hoffmann, and G. Placzek, *Introduction to the Theory of Neutron Diffusion* (Los Alamos Scientific Laboratory, Los Alamos, New Mexico, 1953), Vol. I, Chap. 4, esp. pp. 55–58, 67–71.

where K_0 is given by

$$K_0/\tanh^{-1}K_0=c. \tag{38}$$

In the limit $c=1$, $K_0=0$ and $\partial K_0^2/\partial c=-3$ in (37).

In Eq. (36), recalling ρ is spherically symmetric, the integral over all directions $\mathbf{n}'$ of $\mathbf{r}'=r'\mathbf{n}'$ can be performed by the same device as in (27), i.e., by introducing the new variable

$$y=|\mathbf{r}-\mathbf{r}'|=(r^2+r'^2-2rr'\mu)^{\frac{1}{2}}, \tag{39}$$

in place of $\mu=\mathbf{r}\cdot\mathbf{r}'/rr'$. Therewith we discover Eq. (36) is equivalent to

$$P(r,c)=\frac{1}{2}\int_0^\infty dr'[E_1(|r-r'|)-E_1(r+r')]\times\left[cP(r,c)+\frac{1}{r'}\delta(r')\right], \tag{40}$$

where

$$P(r,c)=4\pi r\rho(r,c). \tag{41}$$

On the other hand in the limit $v_0\to 0$, the substitution

$$r=(a\tau)^{-1}(\lambda\tau+1)v, \tag{42}$$

in Eqs. (30) and (33) converts (30) to

$$Q(r,\lambda)=\frac{1}{2}\int_0^\infty dr'[E_1(|r-r'|)-E_1(r+r')]\times\left[\frac{Q(r',\lambda)}{\lambda\tau+1}+\frac{(\lambda\tau+1)}{(a\tau)^2}\frac{\delta(r')}{r'}\right], \tag{43}$$

with

$$Q(r,\lambda)=H\left(\frac{a\tau r}{\lambda\tau+1},\lambda\right). \tag{44}$$

Equations (40) and (43) mean

$$Q(r,\lambda)=\frac{\lambda\tau+1}{(a\tau)^2}P\left(r,\frac{1}{\lambda\tau+1}\right), \tag{45}$$

because the source in (43) is $(a\tau)^{-2}(\lambda\tau+1)$ times the source in (40). Thus, recalling Eqs. (24) and (29), we have obtained an explicit expression for the spherically symmetric part of the Laplace transform of $f(\mathbf{v},t)$. In particular for $\lambda=0$

$$H(r,0)=\frac{1}{(a\tau)^2}P\left(\frac{r}{a\tau},1\right)=\frac{1}{(a\tau)^2}\left[3+\int_0^1\frac{d\mu}{\mu^2}e^{-r/a\tau\mu}\times\frac{1}{(1-\mu\tanh^{-1}\mu)^2+(\frac{1}{2}\pi\mu)^2}\right]. \tag{46}$$

Equations (35) and (46) yield the previously quoted (12).

We comment finally that the integral equation for $H(v,0)$, or equivalently for $\psi(v)$, can be formulated from first principles, without recourse to the B.E. (17). In the course of its entire history the number of collisions made by an electron when its speed is v equals the integral over all v' of the number of times the electron receives speed v' (by collision or in the process of creation) multiplied by the probability $K(v|v')$ that an electron starting with speed v' makes its next collision at speed v. Hence

$$\psi(v)=\int_0^\infty dv'K(v|v')[\psi(v')+\delta(v)], \tag{47}$$

where the δ function expresses the fact that the electron is created at zero speed. Recalling (8), the probability $K(v|\mathbf{v}')$ that a particle starting with velocity $\mathbf{v}'$ makes its next collision at speed v is

$$K(v|\mathbf{v}')=\int_0^\infty dt\,N\sigma[v(t)]v(t)\times\exp\left\{-\int_0^t dt'Nv(t')\sigma[v(t')]\right\}\delta[v(t)-v], \tag{48}$$

where $\mathbf{v}(t)=\mathbf{v}'+\mathbf{a}t$ and the δ function in (48) picks out these instants at which the particle attains speed v after starting with velocity $\mathbf{v}'$. When the scattering cross section is isotropic, the initial direction $\mathbf{n}'$ of $\mathbf{v}'=v'\mathbf{n}'$ is random, and

$$K(v|v')=\frac{1}{4\pi}\int d\mathbf{n}'K(v/v'\mathbf{n}'). \tag{49}$$

Direct integration of (48) and (49) for our constant mean free time model yields

$$K(v|v')=\frac{1}{2}\frac{1}{a\tau}\frac{v}{v'}\left[E_1\left(\frac{|v-v'|}{a\tau}\right)-E_1\left(\frac{v+v'}{a\tau}\right)\right]. \tag{50}$$

Using (50), Eq. (47) is seen to be equivalent to Eq. (30) for $H(v,0)$ and indeed $K(v,v',0)$ of Eq. (33) now is interpretable from the relation

$$K(v,v',0)=(v'/v)K(v|v'). \tag{51}$$

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