Depolarization of a Muon by Hyperfine Interaction

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The further depolarization of a muon captured in the 1s Bohr orbit by hyperfine interaction with a nucleus of spin j is calculated. The main result is that the asymmetry parameters of the decay electrons from the $J=j\pm\frac{1}{2}$ states are multiplied by respective asymmetry reduction factors $\frac{1}{3}[1\pm 2/(2j+1)]$.

INTRODUCTION

SUBSEQUENT to the capture of a polarized muon into the 1s Bohr orbit, a process attended by partial depolarization, the muon suffers further depolarization from hyperfine interaction with the nuclear magnetic moment, in the case for nuclear spin $j\neq 0$. The asymmetry of the electron distribution from μ^- decay is thereby reduced. There are independent contributions to the distribution from the states $J=j\pm\frac{1}{2}$, where J is the total angular momentum of the μ-nucleus system, and there is a very rapidly oscillating interference term in the asymmetry, because the analysis assumes the muon "up" at $t=0$. This term will be written, for completeness, but will be "short-time-averaged" to zero in all qualitative discussions. In the simplest case, $j=\frac{1}{2}$, the contribution of electrons from the singlet state is evidently completely isotropic, but for $j>\frac{1}{2}$, both the $j\pm\frac{1}{2}$ states demonstrate a post-hyperfine-interaction residual electron-distribution asymmetry.

The result will be stated first, and then derived. It will then be interpreted with the aid of semiclassical arguments, and finally the application in ascertaining the different lifetimes of the different hyperfine states, possibly in conjunction with more direct measurements, will be briefly indicated.

THE ELECTRONIC DISTRIBUTION

The distribution of the electrons emitted per unit time at time t is

$$A_0(t)+A_1(t)\cos\theta,$$

where θ is the angle from the direction z of original muon polarization to the electron's momentum, and

$$A_0(t)=\alpha_0\left(\frac{j+1}{2j+1}e^{-R_+t}+\frac{j}{2j+1}e^{-R_-t}\right), \tag{1}$$

$$A_1(t)=\tfrac{1}{3}\alpha_1\left(\frac{2j+3}{2j+1}\,\frac{j+1}{2j+1}e^{-R_+t}+\frac{2j-1}{2j+1}\,\frac{j}{2j+1}e^{-R_-t}+8\cos(m_+-m_-)t\times\frac{j(j+1)}{(2j+1)^2}e^{-(R_++R_-)t/2}\right), \tag{2}$$

provided that the electron distribution in the absence of the nuclear magnetic and capture effects would be

$$\alpha_0+\alpha_1\cos\theta. \tag{3}$$

$R_\pm$ are the total rates for the disappearance of a muon in the $J=j\pm\frac{1}{2}$ states, respectively; $m_\pm$ are the respective masses. Note that the nuclear effects drop out at $t=0$, owing to the influence of the rapidly oscillating cosine term, so that the formula makes sense.

DERIVATION

If the nucleus has total angular momentum j and z component of angular momentum m, and if the muon spin is s, with z component σ—we later put $s=\frac{1}{2}$—then we denote our state by

$$\left|\begin{matrix} j & s \\ m & \sigma\end{matrix}\right)=\sum_J C_{m,\sigma}{}^{J,m+\sigma}\left|\begin{matrix} J \\ m+\sigma\end{matrix}\right\rangle, \tag{4}$$

where $C_{m,\sigma}{}^{J,M}$ is a Clebsch-Gordan coefficient. This is taken as the initial state; later, we average over m. The influence of the hyperfine effect and the dependence of μ-capture rates on J is to propagate the different J states differently, so that at time t the state is

$$\left|\begin{matrix} j & s \\ m & \sigma\end{matrix}\right)=\sum_J C_{m,\sigma}{}^{J,m+\sigma}\left|\begin{matrix} J \\ m+\sigma\end{matrix}\right\rangle e^{(im_J-\frac{1}{2}R_J)t}.$$

We now put

$$\left|\begin{matrix} J \\ m+\sigma\end{matrix}\right\rangle=\sum_{m'',\sigma'} C_{m'',\sigma'}{}^{J,m+\sigma}\left|\begin{matrix} j & s \\ m'' & \sigma'\end{matrix}\right), \tag{5}$$

so that the electron distribution per unit $e+\nu+\bar{\nu}$ phase space and for final nuclear z component of spin m' is

$$\left|\left(e+\nu+\bar{\nu},\ \begin{matrix} j \\ m'\end{matrix}\middle|S\middle|\begin{matrix} j & s \\ m & \sigma\end{matrix}\ t\right)\right|^2=\left|\sum_J C_{m,\sigma}{}^{J,m+\sigma}e^{(im_J-\frac{1}{2}R_J)t}\times\sum_{m'',\sigma'} C_{m'',\sigma'}{}^{J,m+\sigma}\delta_{m',m''}M_{\sigma'}\right|^2, \tag{6}$$

where we have put

$$\left(e+\nu+\bar{\nu},\ \begin{matrix} j \\ m'\end{matrix}\middle|S\middle|\begin{matrix} j & s \\ m'' & \sigma'\end{matrix}\right)=\delta_{m',m''}M_{\sigma'},$$

where

$$M_{\sigma'}=\left(e+\nu+\bar{\nu}\middle|S\middle|\begin{matrix}s\\ \sigma'\end{matrix}\right),$$

because the nucleus takes no part in μ decay. By general arguments an angular distributions, one can show

$$\sum_{\text{helicities}}\int d\psi\,|M_{\sigma'}|^2=\sum_l a_{l,\sigma'}P_l(\cos\theta), \tag{7}$$

where ψ is the angle between the decay plane and the plane determined by the direction of initial μ polarization (z) and the electron's momentum, and is observable only via neutrino or antineutrino counters. Also,

$$a_{l,\sigma'}=(-)^{-\sigma'}C_{\sigma',-\sigma'}{}^{l,0}\sum_{\sigma''}(-)^{\sigma''}C_{\sigma'',-\sigma''}{}^{l,0}a_{\sigma''}, \tag{8}$$

where

$$a_{\sigma''}=\sum_{\text{helicities}}2\pi\left|\left(e+\nu+\bar{\nu}\text{ with }\mathbf{p}_e\text{ parallel to }+z\text{ axis}\middle|S\middle|\begin{matrix}s\\ \sigma''\end{matrix}\right)\right|^2 \tag{9}$$

are $2s+1$ independent positive constants.[1]

After we perform $\sum_{m''}$ in Eq. (6), we note that only one σ' term survives, so that $|M_{\sigma'}|^2$ factors out, whence the integration in (7) is relevant.[2] We therefore substitute Eqs. (7) and (8) into (6), and apply $(2j+1)^{-1}\sum_{m,m'}$, in order to average over the initial and to sum over the final nuclear orientations, yielding, for the electron differential rate from muon-spin z component σ at $t=0$,

$$\begin{aligned}R_\sigma(t)=&\sum_l P_l(\cos\theta)(2j+1)^{-1}\sum_{\sigma''}(-)^{\sigma''}C_{\sigma'',-\sigma''}{}^{l,0}a_{\sigma''}\\ &\times\sum_{\sigma_1'\sigma_2'JJ'mm'MN}C_{m\sigma}{}^{JM}C_{m'\sigma_1'}{}^{JM}C_{m\sigma}{}^{J'N}C_{m'\sigma_2'}{}^{J'N}\\ &\times\cos(m_J-m_{J'})te^{-(R_J+R_{J'})t/2}.\end{aligned} \tag{10}$$

For muon spin $s=\frac{1}{2}$, the first two Clebsch-Gordan coefficients are trivial,[1] and the sums of the products of the remaining four are easily performed directly, to yield Eqs. (1) and (2).

INTERPRETATION

Equation (1) is very simple to interpret. The rate $4\pi A_0(t)$ is composed simply of the separate contributions of different J states with their statistical weights, without any fast oscillation terms, a result which may also be derived for the general muon spin s from Eq. (10).

The P_1 asymmetry term $A_1(t)$, for μ spin $\frac{1}{2}$, may be discussed as follows. Except for the oscillation term, it is similarly constituted of statistically weighted contributions of $\frac{1}{3}[(2j+3)/(2j+1)]\alpha_1$ and of $\frac{1}{3}[(2j-1)/(2j+1)]\alpha_1$, for $J=j\pm\frac{1}{2}$, respectively. The coefficients $\frac{1}{3}(2j+1\pm2)/(2j+1)=\frac{1}{3}[1\pm2/(2j+1)]$ are conveniently termed "asymmetry reduction factors." As j approaches ∞, they approach $\frac{1}{3}$.

This may be understood very roughly as follows. The averaging of the oscillation term to zero may for pictorial purposes be replaced by a (false) "precession" of the muon around the total $\mathbf{J}$. For large j, we may identify $\mathbf{J}$ and $\mathbf{j}$. Since $\mathbf{j}$ is random, for large j we may roughly take $\mathbf{J}$ to be random. More crudely, take only three possibilities: $\mathbf{J}$ is parallel to either the x, y, or z axes. A muon originally z polarized will lose all polarization from precession around the x or the y axes, but will lose none from precession around the z axis, giving the right answer, $\frac{1}{3}$. More accurately, if an axis makes angle γ with the z axis, and if an electron distribution has asymmetry parameter α relative to the z axis, the distribution we get by spinning the original one about the inclined axis will be of form $A+B'\cos\theta'$, where θ' is now measured from the inclined axis, with asymmetry parameter $\alpha'=B'/A=\alpha\cos\gamma$. Now imagine a cone of evenly distributed axes at angle γ, with such distributions about each axis to be averaged. The result is an $A+B''\cos\theta$ distribution around the original axis, with asymmetry parameter $\alpha''=\alpha'\cos\gamma=\alpha\cos^2\gamma$. Thus the asymmetry reduction factor is $\cos^2\gamma$, and for j large, this is to be averaged over the sphere, giving $\frac{1}{3}$.

But this picture may even be used for general j, if we interpret $\cos^2\gamma$ according to the vector model,

$$\cos^2\gamma=\frac{(\mathbf{s}\cdot\mathbf{J})^2}{\mathbf{s}^2\mathbf{J}^2}=\frac{(\mathbf{s}^2+\mathbf{s}\cdot\mathbf{j})^2}{\mathbf{s}^2\mathbf{J}^2}=\frac{[\mathbf{s}^2+\frac{1}{2}(\mathbf{J}^2-\mathbf{s}^2-\mathbf{j}^2)]^2}{\mathbf{s}^2\mathbf{J}^2};$$

$$\cos^2\gamma=\frac{[J(J+1)-j(j+1)+s(s+1)]^2}{4s(s+1)J(J+1)}.$$

For $s=\frac{1}{2}$, these are indeed the proper asymmetry reduction factors $\frac{1}{3}[1\pm2/(2j+1)]$.

POSSIBLE APPLICATIONS

The fast-oscillation term probably has no applications, because, e.g., for a proton, the frequency is 4.982×10^{13} cps, or 1662 cm^{-1}. It could be a tool for detecting a magnetic moment of 10^{-4} to 10^{-8} nuclear magneton, but it is unlikely that there are any nuclei with such small but nonzero magnetic moments.

However, the fact that different asymmetry reduction factors go with the different total rates $R_\pm$ in Eq.

[1] For the interesting case $s=\frac{1}{2}$, it actually suffices to know $a_{l,-\sigma'}=(-)^l a_{l,\sigma'}$ and that $a_{0,\sigma'}$ is independent of σ', which follow from (8), but are in fact quite obvious directly from (7).

[2] For the case of a muon of general spin s, integration over the azimuthal angle of the electron distribution destroys coherent effects between different initial muon-spin z components σ, so that our discussion is quite general. However, the evidence for $s=\frac{1}{2}$ is very strong see, e.g., Menasha Tausner, dissertation, Columbia University (to be published). This remark is unnecessary for $s=\frac{1}{2}$, because the density matrix for the muon spin can in that case be diagonalized simply by a proper initial choice of z axis.

(2) may be of aid in the experimental determination of $R_\pm$. As is well known, the ratio of these rates is of considerable interest. If one believes in a simple nuclear shell model, then the ratio of the two rates provides important information on the specific μ-capture interaction,[3] and conversely, if one uses a definite μ-capture theory, one may gain information about the nuclear state. For example, if a nucleus of high spin had that spin formed from proton spins only, the effect would be enhanced over that from the favored state of shell models, where only one nucleon spin is aligned, and the remaining contribution is provided by nucleon orbital angular momentum. A possibly more plausible reason for expecting an enhancement of the lifetime difference effect is that square-well calculations[4] indicate an enormous dominance of the uppermost protons in μ capture, which is probably in part due to the defectiveness of a naive single-particle model. One way to look at the relative importance of μ capture in upper shells is to note what the actual fluctuation of μ capture behavior is relative to systematic dependence on A and Z, but another way is to look at the R_+/R_- ratio, which comes from the effect of upper-shell μ capture in a shell model, and which one might expect to be enhanced if one wishes to attach some general validity to the numerical results of references 4 for closed shells of a square well.

The direct method to obtain R_+ and R_- is to analyze the time distribution of the electrons or of the neutrons from the disappearance of muons.[3] That the asymmetry of the electrons $\alpha(t)=A_1(t)/A_0(t)$ has the form predicted by Eqs. (1) and (2) may provide important redundant information. Further, by integrating (1) and (2) over time, and dropping the negligible contribution of the oscillation, one obtains for the asymmetry parameter α of the time-integrated distribution,

$$\alpha=-\frac{1}{3}\frac{\alpha_1}{\alpha_0}\frac{[(2j+3)/(2j+1)][(j+1)/(2j+1)]\tau_+ + [(2j-1)/(2j+1)][j/(2j+1)]\tau_-}{[(j+1)/(2j+1)]\tau_+ + [j/(2j+1)]\tau_-}, \tag{11}$$

where α_1/α_0 is the asymmetry parameter from the partially depolarized muon in the $1s$ state prior to the influence of the hyperfine interaction and μ capture, and $\tau_\pm=1/R_\pm$ are the respective lifetimes. This is a relation between the nuclear spin j, the parameter α_1/α_0, and the parameter R_+/R_-, if α is measured. By taking a ratio with the "α" from a spin-zero isotope, the parameter α_1/α_0 may be eliminated, or it may be estimated.

The character of the entire problem may be complicated by the induction of hyperfine transitions by the electrons.[5] Of course, the picture of the simple asymmetry reduction factors $\frac{1}{3}[1\pm 2/(2j+1)]$ would then be spoilt. In the presence of cases possibly complicated by the action of electrons, it may be important to confirm a simple analysis of a simple case by redundant measurements.

Note that although the asymmetry reduction factors $\frac{1}{3}[1\pm 2/(2j+1)]$ approach the common value $\frac{1}{3}$ as $j\rightarrow\infty$, they do so slowly; e.g., for $j=\frac{5}{2}$, their ratio is still 2.

The considerations here are applicable to a rate either differential in the electron energy, or fused experimentally over electron energies. The effect of small components of the muon wave function for high Z may be included in the parameter α_1/α_0, and does not alter the way the nuclear spin enters into the problem. Noncoplanarity of the final leptonic momenta due to nuclear recoil also does not alter the results; one need only specify a particular rigid configuration of final momenta, and replace the "decay plane" above by, e.g., the plane of the final electron and neutrino momenta, to appropriately generalize the argument.[6]

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[3] J. Bernstein, T. D. Lee, C. N. Yang, and H. Primakoff, Phys. Rev. **111**, 313 (1958).

[4] E. I. Dolinsky and L. D. Blokhintsev, Nuclear Phys. **10**, 527 (1959), and Elihu Lubkin, dissertation, Columbia University (to be published).

[5] This possibility may be indicated by certain experimental results of V. L. Telegdi, orally communicated to me by Robert D. Sard, Washington University, St. Louis, Missouri, 1959.

[6] Dr. S. Weinberg has brought to my attention a similar calculation by H. Überall, Phys. Rev. **114**, 1640 (1959), where the total rates $R_\pm$ are estimated on the basis of the usual μ-capture interaction and a Schmidt model for the nucleus.

[7] Experimental results for some $j\neq 0$ nuclei have been reported in a Russian preprint by L. B. Egorov, A. E. Ignatenko, and D. Chulten, of the Joint Institute for Nuclear Research. A previous publication on $j=0$ cases, by A. E. Ignatenko, L. B. Egorov, B. Khalupa, and D. Chulten appears in J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 1131 (1958) [translation: Soviet Phys.—JETP **35**(8), 792 (1959)].