latter cross-section values taken from Fig. 1. Above 4 kev, the charge-transfer cross section becomes too small for this method to be reliable. At the higher energies the relative cross section for electron production was determined by comparing electron signals per unit ion current as a function of ion energy. This high-energy relative cross section was then normalized at 10 kev to the value obtained by comparing the H^-+H electron-production cross section with the $p+H$ charge-transfer cross section, the absolute value for the latter being taken from the preceding paper. At energies of less than 500 ev, the results obtained by comparing the cross sections for electron production and charge transfer in H^-+H collisions were not sufficiently reproducible to warrant their being shown in Fig. 2. However, from these lower-energy data, it appears that the cross section for electron production does not decrease for ion energies down to 50 ev and probably continues to increase. The experimental uncertainties shown in Fig. 2 do not include uncertainties in the charge-transfer cross sections which were used as standards in this measurement.

The degree to which the experimental values should be expected to agree with the results of McDowell and Peach is not entirely clear. In their calculations, only the process $H^-+H \rightarrow H+H+e$ was considered, whereas in our experiments, processes which would result in ionization of the end products of the collisions also contributed to the electron-production signal. Certainly the condition that the experimental values should exceed the cross section for only the simple electron-detachment process is satisfied.

It would be expected that processes leading to ionization of the final collision products would be operative only at the higher ion energies in this experiment, and that such processes cannot be invoked to explain the deviations of the two curves in Fig. 2 as the energy is reduced below 5 kev. It also seems unlikely that the associative detachment process, $H^-+H \rightarrow H_2+e$, can contribute appreciably to the electron-production processes at energies as high as the lower energies of these measurements.[6]

It is interesting to note that McDowell and Peach calculate the energy distribution of the electrons produced in the simple detachment process and find that where their approximation is valid, less than 10% of the ejected electrons should have energies exceeding 13.6 ev. In measuring the cross section for total slow negative particle production, it was found that curves of signal versus vertical magnetic field saturated at about 20 oersteds for all ion energies. Considering the experimental geometry, this implies that a negligible fraction of the electrons had energies in excess of about 20 ev.

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[6] A. Dalgarno (private communication).

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Dissipation in Quantum Mechanics. The Harmonic Oscillator

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The need for a quantum-mechanical formalism for systems with dissipation which is applicable to the radiation field of a cavity is discussed. Two methods that have been used in this connection are described. The first, which starts with the classical Newtonian equation of motion for a damped oscillator and applies the conventional formal quantization techniques, leads to an exact solution; but subsequent discussion shows that this method is invalid, the results being unacceptable from a quantum-mechanical viewpoint. The second method, which considers the interaction of two systems, the lossless oscillator and the loss mechanism, is adopted in the present article. No special model is used for the loss mechanism, but this mechanism is assumed to have a large number of densely-spaced energy states.

The approximations with respect to the loss mechanism that underlie the concept of dissipation are discussed. These approximations are then applied to the analysis, and a differential equation for a coordinate operator of the harmonic oscillator is obtained which has the formal appearance of the Newtonian equation of motion for a driven damped harmonic oscillator, the driving term being an operator referring to the loss mechanism. The presence of the driving term is responsible for the difference between the present theory and that of the first method mentioned above. A solution of the differential equation for the coordinate operator is given explicitly. An examination of the physical significance of the solution shows that the driving term is responsible not only for the thermal fluctuations which are due to the loss mechanism, but also for the proper commutation relationship of the conjugate coordinates of the oscillator and for its zero-point fluctuations.

A generalization of the solution to provide for a classical driving force and coupled atomic systems is given. The results are then restated in a form that refers to the loss mechanism only through the two parameters by which it is usually described—the dissipation constant and the temperature.

INTRODUCTION

MOST quantum-mechanical analyses deal with microscopic phenomena, and since dissipation is a macroscopic concept, there has been little interest, during the historical development of quantum mechanics, in a formalism for systems with dissipation. There is, however, a type of problem, which has acquired considerable interest in recent years, in which dissipa-

tion does enter significantly: the quantum-mechanical analysis of the radiation field in a microwave cavity. A quantum-mechanical treatment of the field is essential in the analysis of phenomena related to spontaneous emission of microscopic systems (such as molecules) which are coupled to the field; furthermore, any treatment of the field, whether classical or quantum mechanical, must consider cavity losses, in general, since they are by no means negligible.

As is well known, the quantum mechanics of the radiation field of a mode in a lossless cavity is the same as the quantum mechanics of a harmonic oscillator. If we consider a single mode of the cavity, with frequency $\omega/2\pi$, and if we express the electric and magnetic fields as

$$\mathbf{E}=-4\pi cP(t)\mathbf{u}(\mathbf{r}), \quad \mathbf{H}=Q(t)\nabla\times\mathbf{u}(\mathbf{r}), \tag{1}$$

where $\mathbf{u}(\mathbf{r})$ is a normalized function describing the spatial dependence of the cavity field, then Q and P are canonically conjugate coordinates satisfying the commutation relationship $[Q,P]=i\hbar$; and the Hamiltonian for the radiation field is

$$H=2\pi c^2P^2+(\omega^2/8\pi c^2)Q^2, \tag{2}$$

the harmonic oscillator Hamiltonian (except for the trivial matter of the constants).[1]

There has been some discussion of dissipation in quantum mechanics in general, and with reference to the harmonic oscillator in particular. The introduction of dissipation into quantum mechanics has followed, broadly speaking, two different methods: One method[2–6] consists of starting with the classical Newtonian equations of motion for a system with dissipation (the dissipation being due to a velocity-dependent force), finding a Lagrangian which leads to these equations of motion, and then proceeding to quantization by conventional formal methods. The other method consists of considering the dissipation as being due to the coupling of two systems,[7–9]—the undamped harmonic oscillator, and the system which produces the damping—and then attacking the problem by an approximation method.

The first method has the attractiveness of a clean-cut formalism leading, in the case of the harmonic oscillator, to exact solutions, which were obtained by Kerner[5] and by Stevens.[6] An exact solution is possible in this instance because the classical equation of motion of the damped harmonic oscillator may be solved exactly, and formal quantization merely has the effect of converting the classical variables into operators, leaving the equation intact. The solution, therefore, remains formally the same, but must now be interpreted as an expression in operators. It will be shown in the present article that this method is unrealistic, and cannot describe a true physical system with quantum-mechanical properties.

The method we shall use is the second, in which we consider a lossless harmonic oscillator coupled to a loss mechanism. Callen and Welton[7] considered the properties of loss mechanisms from a quantum-mechanical viewpoint, but were not interested in a detailed analysis of the properties of the conservative system which is coupled to the loss mechanism. Weber[8] used the results of Callen and Welton to calculate the energy of a damped harmonic oscillator. A quantum-mechanical formalism for the field in a lossy cavity, which permitted a demonstration of its quantum-mechanical consistency as well as provided for the inclusion of driving mechanism and coupled atomic system, was developed in reference 9. However, this development utilized a special model for the loss mechanism, and the formalism made explicit reference to this special model.

It seems that the subject of dissipation in quantum mechanics, with reference to the harmonic oscillator, is of sufficient fundamental interest to justify the development of a general formalism which contains only the experimentally observable parameters of the loss mechanism. It is the purpose of the present article to develop such a formalism without recourse to special models, and, incidentally, to show that the method of formally quantizing the classical equations of motion for a damped system is invalid. For the sake of completeness and intelligibility, a few of the ideas expressed in reference 9, where the question of loss was incidental to other matters, will be restated with somewhat different emphasis. The concept of dissipation is examined in Part I. The problem is formulated in Part II, where explicit expressions are derived which constitute the solution of the problem. In Part III the physical meaning of the solution is investigated and contrasted with results of the formal quantization of the classical equations. In Part IV an extension of the above solution to provide for a driving force and coupled atomic systems is given. Finally, in Part V the results are restated so as to refer only to the loss constant and temperature of the loss mechanism.

I

Before proceeding with any calculations, it will be profitable to discuss the meaning involved in the usual concept of dissipation. When dealing with dissipation in an electrical system specified by a value for the resistance, or dissipation in a mechanical system specified by a value of the coefficient of friction, we are really concerned with the interaction of two types of systems: One is a rather simple system—in the present

[1] For the mechanical oscillator of mass m, replaces $4\pi c^2$ by m^{-1}. No distinction will be made between mechanical and radiation oscillators; the Hamiltonian of Eq. (2) will be used throughout. The language will refer either to the oscillator or the field, as convenient.

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[7] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).
[8] J. Weber, Phys. Rev. **90**, 977 (1953).
[9] I. R. Senitzky, Phys. Rev. **115**, 227 (1959).

case, the harmonic oscillator—which we want to describe completely, and the other is an extremely complicated system—the loss mechanism—which we want to describe very incompletely, as in thermodynamics, for instance. We are interested in the loss mechanism only to the extent that it affects the behavior of the oscillator, and even in this respect the interest is limited. We must concern ourselves only with the lowest order interactions which produce loss, since description of higher order interactions requires a more detailed knowledge of the loss mechanisms than that provided by a single dissipation constant. Underlying the conventional concept of dissipation is the understanding that the loss mechanism is affected only slightly by its interaction with the oscillator; the oscillator, on the other hand, may undergo large changes due to the loss mechanism. Thus, the loss mechanism may be treated by a perturbation-theory approach, but the oscillator may not.

The analysis in Part II will tend to confirm the preceding statements. This analysis may therefore be viewed in two ways: as a derivation of a quantum-mechanical formalism involving dissipation that is based on the above ideas, or as a demonstration of the *necessary approximation* with which we must regard the loss mechanism in order to obtain relationships consistent with the conventional description of dissipation.[10] From either point of view, however, it is important to note that if we find that effects of a given order are necessary to obtain results which amount to dissipation, we must consider *all* effects of that, or lower, order if we are to have a consistent formalism. Thus, we will see that when considering loss, we are also forced to consider an additional phenomenon, which is of great importance from a quantum-mechanical viewpoint, namely fluctuations.

One can also see the necessity of considering fluctuations on physical grounds. The dissipation which the oscillator experiences may be regarded as due to a reaction of the loss-mechanism on the oscillator, the former having been excited by the oscillator in the first place; but if the loss mechanism can react on the oscillator, it can also act on the oscillator because of its own fluctuations, and there is no *a priori* reason why this effect may be ignored in a real physical system. In other words, a coupling mechanism works in either direction. In classical mechanics one might conceptually create a situation in which there are no fluctuations by considering the absolute temperature to be zero, thus divorcing dissipation from fluctuations; but in quantum mechanics there are still the zero-point fluctuations, which cannot be eliminated. Therefore, dissipation and fluctuations must be considered together. Only then can a consistent quantum-mechanical theory be developed.

[10] The mathematical development will lead from the approximations, which we assume to be inherent in the concept of dissipation, to the usual description of dissipation; that is, it will be proven that the approximations are sufficient to obtain conventional dissipation. There will be no mathematical proof that the approximations are necessary, but it will be easy to see that these are the simplest and most reasonable approximations leading to the required result.

II

We consider a harmonic oscillator coupled to a loss mechanism, the combined Hamiltonian being

$$H=H_{\text{osc}}+H_l+\alpha P\Gamma, \tag{3}$$

where H_{osc} is the Hamiltonian of Eq. (2), H_l is the Hamiltonian of the loss mechanism (which we will not specify explicitly), Γ is the coordinate of the loss mechanism through which the coupling to the oscillator takes place, and α is the coupling constant. It does not matter whether the oscillator is coupled to the loss mechanism through the Q or P coordinate (or both), since the two are essentially symmetrical. In the present case, P is chosen because in a cavity the loss mechanism is usually coupled to the electric field. The only essential assumption we make about the loss mechanism is that it has a large number of closely-spaced energy states. We make another assumption which is not essential, but is convenient and holds for most loss mechanisms: The diagonal matrix elements of Γ, when the loss mechanism is free, are zero. This assumption, in classical language, corresponds to the requirement that there be no force exerted on the harmonic oscillator by the loss mechanism which is independent of the time and of the state of the oscillator, or that the loss mechanism in a cavity have no permanent polarization.

The quantum-mechanical formalism to be used is that of the Heisenberg picture, in which the operators contain the time dependence and the wave functions are constant, corresponding to the initial states of the systems. The initial operators are defined as the operators at $t=0$ of the uncoupled systems. (That is, the interaction is turned on at $t=0$.) We consider the initial state of the oscillator in general terms. It may be an energy state or a superposition of energy states. If it is a classical-type state,[11] then $\langle P^2(0)\rangle$ will not differ by much from $\langle P(0)\rangle^2$, and likewise for $Q(0)$. Whenever the results obtained require the use of the initial oscillator state, they will be given in terms of the expectation values of the initial operators. In considering the loss mechanism, however, we want to be more explicit. Obviously, we are not interested in an arbitrary initial state of the loss mechanism, but rather in a thermal type of state which may be described by a temperature. This description is most conveniently given by means of a diagonal density matrix. In the energy representation of the uncoupled loss mechanism, we take this matrix to be

$$\rho_{nm}=\delta_{nm}Ae^{-E_n/kT}, \tag{4}$$

where

$$A^{-1}=\sum_j e^{-E_j/kT}. \tag{5}$$

[11] I. R. Senitzky, Phys. Rev. **95**, 904 (1954).

The expectation value of any matrix O referring to the loss mechanism is given by

$$\langle O\rangle = \text{Trace } O\rho = \sum_n O_{nn}\rho_{nn}. \tag{6}$$

It should be noted that all our operators operate on the state vectors of both the loss mechanism and the oscillator. The representation in the loss-mechanism space is, as specified above, the energy representation of the uncoupled mechanism. The representation in the oscillator space need not be specified for our purposes.

We proceed now to calculate the time dependence of the operators P and Q. From Eqs. (2) and (3), we obtain as the equations of motion

$$\dot{P} = -(\omega^2/4\pi c^2)Q, \tag{7}$$

$$\dot{Q} = 4\pi c^2 P + \alpha\Gamma, \tag{8}$$

which may be expressed in integral form as[9]

$$P(t) = P^{(0)}(t) - \frac{\omega\alpha}{4\pi c^2}\int_0^t dt_1\,\Gamma(t_1)\sin\omega(t-t_1), \tag{9}$$

$$Q(t) = Q^{(0)}(t) + \alpha\int_0^t dt_1\,\Gamma(t_1)\cos\omega(t-t_1), \tag{10}$$

where $P^{(0)}(t)$ and $Q^{(0)}(t)$ are expressions for the operators of the loss-free oscillator and include the initial operators:

$$P^{(0)}(t) = P(0)\cos\omega t - (\omega/4\pi c^2)Q(0)\sin\omega t, \tag{11}$$

$$Q^{(0)}(t) = Q(0)\cos\omega t + (4\pi c^2/\omega)P(0)\sin\omega t. \tag{12}$$

In order to be able to solve Eqs. (9) and (10) explicitly, we need an expression for $\Gamma(t)$. From Eq. (3), we have

$$\dot{\Gamma}(t) = (i\hbar)^{-1}[\Gamma(t), H_l(t)], \tag{13}$$

and

$$H_l(t) = H_l(0) + \frac{\alpha}{i\hbar}\int_0^t dt_1[H_l(t_1),\Gamma(t_1)]P(t_1). \tag{14}$$

Substituting from Eq. (14) into Eq. (13), we obtain

$$\dot{\Gamma}(t) = (i\hbar)^{-1}[\Gamma(t),H_l(0)] + \frac{\alpha}{\hbar^2}\int_0^t dt_1[\Gamma(t),[\Gamma(t_1),H_l(t_1)]P(t_1)]. \tag{15}$$

An integral form of this equation, as can easily be verified by differentiation, is

$$\Gamma(t) = \Gamma^{(0)}(t) + \frac{\alpha}{\hbar^2}\int_0^t dt_1\int_0^{t_1} dt_2\exp[(i/\hbar)H_l^{(0)}(t-t_1)] \times[\Gamma(t_1),[\Gamma(t_2),H_l(t_2)]P(t_2)] \times\exp[-(i/\hbar)H_l^{(0)}(t-t_1)], \tag{16}$$

where $\Gamma^{(0)}(t)$ is the operator of the uncoupled loss mechanism having matrix elements

$$\Gamma_{ij}^{(0)}(t) = \Gamma_{ij}^{(0)}(0)e^{i\omega_{ij}t} \equiv \tilde{\Gamma}_{ij}e^{i\omega_{ij}t}, \tag{17}$$

with

$$\omega_{ij} \equiv (E_i - E_j)/\hbar.$$

Substituting from Eq. (16) into Eq. (9) ,we obtain

$$P(t) = P^{(0)}(t) - \frac{\omega\alpha}{4\pi c^2}\int_0^t dt_1\,\Gamma^{(0)}(t_1)\sin\omega(t-t_1) - \frac{\omega\alpha^2}{4\pi c^2\hbar^2}\int_0^t dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3\sin\omega(t-t_1) \times\exp[(i/\hbar)H_l^{(0)}(t_1-t_2)] \times[\Gamma(t_2),[\Gamma(t_3),H_l(t_3)]P(t_3)] \times\exp[-(i/\hbar)H_l^{(0)}(t_1-t_2)]. \tag{18}$$

Let us discuss the physical meaning of the last two terms on the right side of Eq. (18). The $\Gamma^{(0)}$ term obviously gives the effect on the oscillator of the fluctuations of the (unperturbed) loss mechanism, which were previously discussed. Although the expectation value of this term vanishes, the expectation value of its square does not; and, as will be shown subsequently, it accounts for the effect of both the thermal and quantum-mechanical fluctuations. The last term contains the reaction of the loss mechanism to the field and is responsible for the loss; but the last term contains much more than that: It contains all the details of the interaction between the oscillator and loss mechanism except the first order effect of the loss-mechanism fluctuations. In accordance with our previous discussion, this exact treatment is not what we want in order to describe dissipation; we therefore perform an approximation on this term which amounts to replacing

$$\exp[(i/\hbar)H_l^{(0)}(t_1-t_2)][\Gamma(t_2),[\Gamma(t_3,H_l(t_3)]P(t_3)] \times\exp[-(i/\hbar)H_l^{(0)}(t_1-t_2)] \tag{19}$$

by

$$\langle[\Gamma^{(0)}(t_2),[\Gamma^{(0)}(t_3),H_l^{(0)}(t_3)]]\rangle P(t_3). \tag{20}$$

We carry out the approximation in two steps. Since, as previously mentioned, there is an assumption underlying the concept of dissipation that the loss mechanism is disturbed only slightly, we can approximate Γ by $\Gamma^{(0)}$ and H_l by $H_l^{(0)}$ under the integral sign in Eqs. (16) and (18).[12] This approximation, incidentally, makes

[12] One might, at first glance, think that it is inconsistent to regard the last term in Eq. (16) as a small perturbation of Γ while not regarding the last term of Eq. (18) as a small perturbation of P; and, indeed, if the magnitude of α were the only relevant consideration, there would be an inconsistency. However, there is another important consideration, namely, the density of states of the loss mechanism, as may be seen from the following qualitative argument: The contribution to the double commutator in Eqs. (16) and (18) comes mainly from the off-diagonal matrix elements of Γ. The effect of the field on these off-diagonal elements is determined, approximately, by the off-diagonal elements of $[\Gamma(t_1),[\Gamma(t_2),H_l(t_2)]]$. The effect of the loss mechanism on the field, however, is determined by the diagonal elements of this

Eq. (18) linear in P, which is necessary in order to obtain linear dissipation, or a dissipation *constant*. Furthermore, we ignore loss-mechanism quantum-mechanical effects of higher order than the second; or, to put it another way, we ignore the quantum-mechanical properties of the loss mechanism in terms of higher order than the second. This means that we can bring P through the commutator bracket and exponential operator out to the extreme right (or left), and replace the remaining commutator by its expectation value times the unit operator.[13] The exponential operators then cancel, and we have (19) changed into (20). Loosely speaking, what we have done in the second step of the approximation is to retain only the classical content (as far as the loss mechanism is concerned) of the last term in Eq. (18); we might say that the second term on the right side gives us all the quantum-mechanical effects we need to describe dissipation, and the last term gives us the classical loss.

We proceed now in a straightforward manner. From Eq. (17) we obtain

$$[\Gamma^{(0)}(t_1),[\Gamma^{(0)}(t_2),H_l^{(0)}]]_{ii} = 2\sum_k |\tilde{\Gamma}_{ik}|^2 \hbar\omega_{ik}\cos\omega_{ik}(t_1-t_2), \quad (21)$$

which, with Eqs. (4) and (6) yields

$$\langle[\Gamma^{(0)}(t_1),[\Gamma^{(0)}(t_2),H_l^{(0)}]]\rangle = 2A\sum_{i,k} e^{-E_i/kT}|\tilde{\Gamma}_{ik}|^2\hbar\omega_{ik}\cos\omega_{ik}(t_1-t_2). \quad (22)$$

With the approximation specified by Eqs. (19) and (20), Eq. (18) becomes

$$P(t)=P^{(0)}(t)-\frac{\omega\alpha}{4\pi c^2}\int_0^t dt_1\,\Gamma^{(0)}(t_1)\sin\omega(t-t_1)+L, \quad (23)$$

where

$$L=-\frac{\omega\alpha^2 A}{2\pi c^2\hbar}\sum_{i,k} e^{-E_i/kT}|\tilde{\Gamma}_{ik}|^2\omega_{ik}\int_0^t dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3 \times P(t_3)\sin\omega(t-t_1)\cos\omega_{ik}(t_2-t_3). \quad (24)$$

commutator, since these diagonal elements are very roughly proportional to the density of states ρ (for t_1 sufficiently close to t_2), while the off-diagonal elements do not increase significantly with the density of states. Thus, the last term in Eq. (16) is proportional to α in the off-diagonal elements (and these are the ones which are important in the double commutator), while the last term in Eq. (18) is proportional to $\alpha\rho$. This is essentially the reason that a loss mechanism responsible for the usual type of dissipation must have a large density of states. One can also see from ordinary time-dependent perturbation theory that a large density of states corresponds, for a given energy transfer, to a small transition probability *per state* or to a slight change in the occupation numbers of the energy states. See also Reference 9, where the argument is made much simpler and more quantitative by the consideration of a special model for the loss mechanism.

[13] The replacement of the operator by its expectation value is justified by the fact that all our final results (but not necessarily the intermediate steps) will be expectation values with respect to the loss mechanism; and since the term affected involves only second and higher order interactions, only the higher order quantum-mechanical effects are neglected in the final result.

The t_1 and t_2 integrations may be carried out immediately, yielding

$$L=-\frac{\omega\alpha^2 A}{2\pi c^2\hbar}\int_0^t dt_1\,P(t_1)F(t-t_1), \quad (25)$$

where

$$F(\tau)=\sum_{i,k} e^{-E_i/kT}|\tilde{\Gamma}_{ik}|^2 \times\left[\frac{\cos\tfrac{1}{2}(\omega_{ik}-\omega)\tau\,\sin\tfrac{1}{2}(\omega_{ik}+\omega)\tau}{\omega_{ik}+\omega} - \frac{\cos\tfrac{1}{2}(\omega_{ik}+\omega)\tau\,\sin\tfrac{1}{2}(\omega_{ik}-\omega)\tau}{\omega_{ik}-\omega}\right]. \quad (26)$$

Because the energy levels of the loss mechanism are closely spaced, the summation may be replaced by an integration:

$$\sum_{i,k}\to\int_0^\infty \rho(E_i)dE_i\int_0^\infty \rho(E_k)dE_k, \quad (27)$$

where $\rho(E)$ is the density of states in energy space. Setting

$$E=\tfrac{1}{2}(E_i+E_k), \quad \omega'=\omega_{ik}, \quad (28)$$

we have

$$\int_0^\infty dE_i\int_0^\infty dE_k\to\int_0^\infty \hbar d\omega'\int_{\frac{1}{2}\hbar\omega'}^\infty dE + \int_{-\infty}^0 \hbar d\omega'\int_{-\frac{1}{2}\hbar\omega'}^\infty dE\equiv\int\hbar d\omega'\int dE. \quad (29)$$

Thus,

$$F(\tau)=\int\hbar d\omega'\int dE\,\rho(E+\tfrac{1}{2}\hbar\omega')\rho(E-\tfrac{1}{2}\hbar\omega') \times\exp[-(kT)^{-1}(E+\tfrac{1}{2}\hbar\omega')]\tilde{\Gamma}^2(E+\tfrac{1}{2}\hbar\omega',E-\tfrac{1}{2}\hbar\omega') \times\left[\frac{\cos\tfrac{1}{2}(\omega'-\omega)\tau\,\sin\tfrac{1}{2}(\omega'+\omega)\tau}{\omega'+\omega} - \frac{\cos\tfrac{1}{2}(\omega'+\omega)\tau\,\sin\tfrac{1}{2}(\omega'-\omega)\tau}{\omega'-\omega}\right], \quad (30)$$

where $\tilde{\Gamma}^2(E_i,E_k)$ is a function obtained by averaging $|\tilde{\Gamma}_{ik}|^2$ for all states i and k lying in small intervals about E_i and E_k, respectively. Consider now the integration with respect to E in Eq. (30). We have

$$\int_{\frac{1}{2}\hbar\omega'}^\infty dE\,\rho(E+\tfrac{1}{2}\hbar\omega')\rho(E-\tfrac{1}{2}\hbar\omega') \times\tilde{\Gamma}^2(E+\tfrac{1}{2}\hbar\omega',E-\tfrac{1}{2}\hbar\omega')e^{-E/kT} =\exp(-\hbar\omega'/2kT)B(\omega'), \quad (31)$$

where

$$B(\omega')\equiv\int_0^\infty dE\,\rho(E+\hbar\omega')\rho(E)\tilde{\Gamma}^2(E+\hbar\omega',E)e^{-E/kT}, \tag{32}$$

and likewise

$$\int_{-\frac{1}{2}\hbar\omega'}^\infty dE\,\rho(E+\tfrac{1}{2}\hbar\omega')\rho(E-\tfrac{1}{2}\hbar\omega')\times\tilde{\Gamma}^2(E+\tfrac{1}{2}\hbar\omega',E-\tfrac{1}{2}\hbar\omega')e^{-E/kT}=\exp(\hbar\omega'/2kT)B(-\omega'). \tag{33}$$

In the ω' integration, the contribution to $F(\tau)$ comes mainly from the two neighborhoods about $\omega'=\pm\omega$. We therefore obtain

$$F(\tau)\cong\int_0^\infty \hbar d\omega'[1-\exp(-\hbar\omega'/kT)]\times B(\omega')\frac{\cos\frac{1}{2}(\omega'+\omega)\tau\,\sin\frac{1}{2}(\omega'-\omega)\tau}{\omega'-\omega}$$
$$\cong\tfrac{1}{2}\pi\hbar[1-\exp(-\hbar\omega/kT)]B(\omega)\cos\omega\tau, \tag{34}$$

where we have assumed that $B(\omega')$ is a slowly varying function in the neighborhood of ω, and that $\omega\gg\tau^{-1}$. (For $\tau=t-t_1$, the last assumption can hold only if $t\gg\omega^{-1}$.) Going back to Eq. (25), we have

$$L=-\beta\int_0^t dt_1\,P(t_1)\cos\omega(t-t_1), \tag{35}$$

where

$$\beta\equiv(\omega\alpha^2/4c^3)AB(\omega)[1-\exp(-\hbar\omega/kT)]. \tag{36}$$

Equation (24) may therefore be written as

$$P(t)=P^{(0)}(t)-\frac{\omega\alpha}{4\pi c^2}\int_0^t dt_1\,\Gamma^{(0)}(t_1)\sin\omega(t-t_1)-\beta\int_0^t dt_1\,P(t_1)\cos\omega(t-t_1). \tag{37}$$

This is an integral equation for $P(t)$ of the Volterra type, and can be expressed in more familiar form as a differential equation:

$$\ddot{P}+\beta\dot{P}+\omega^2P=-(\omega^2\alpha/4\pi c^2)\Gamma^{(0)}(t), \tag{38}$$

where we have made use of the differential equation for the uncoupled harmonic oscillator,

$$\ddot{P}^{(0)}+\omega^2P^{(0)}=0. \tag{39}$$

Equation (38) is the well-known differential equation for a driven damped harmonic oscillator. Assuming that

$$(\beta/\omega)\ll1, \tag{40}$$

and ignoring the slight frequency shift introduced by the damping,

$$\omega_{\text{damped}}=\omega[1-(\beta/2\omega)^2]^{\frac{1}{2}}, \tag{41}$$

we can write, as the approximate solution of Eq. (38):

$$P=e^{-\frac{1}{2}\beta t}P^{(0)}-\frac{\omega\alpha}{4\pi c^2}\int_0^t dt_1\,\Gamma^{(0)}(t_1)e^{-\frac{1}{2}\beta(t-t_1)}\times\sin\omega(t-t_1). \tag{42}$$

In an entirely analogous manner, we obtain

$$Q=e^{-\frac{1}{2}t\beta}Q^{(0)}+\alpha\int_0^t dt_1\,\Gamma^{(0)}(t_1)e^{-\frac{1}{2}\beta(t-t_1)}\cos\omega(t-t_1). \tag{43}$$

Speaking in terms of operators, the first term of both solutions gives the damped initial oscillation, and the second term gives the driven oscillation. It is to be noted that Eqs. (42)and (43) explicitly exhibit P and Q as operators with respect to both the oscillator and the loss mechanism.[14]

III

We will now examine in detail the physical significance of Eqs. (38), (42), and (43). If we take expectation values of both sides of the equations, the driving term vanishes, since, as mentioned earlier,

$$\langle\Gamma^{(0)}\rangle=0 \tag{44}$$

due to the fact that $\tilde{\Gamma}$ is an off-diagonal matrix. The *expectation value* of P, therefore, satisfies the classical equation for a damped harmonic oscillator, with β as the dissipation constant. In the case of a mechanical oscillator, β is the coefficient of friction (per unit mass), and in the case of the electromagnetic field in a cavity,

$$\beta=\omega/\text{“}Q\text{”}, \tag{45}$$

where “Q” is the quality factor of the cavity. Thus, we have the conventional picture of dissipation for the expectation values of the coordinates. [The expression for β in Eq. (36) is equivalent to the result derived by Callen and Welton.[7]]

Let us now look at the commutation properties of Q and P. We will restrict our interest in $[Q,P]$ to the oscillator alone by taking the expectation value of the commutator with respect to the loss mechanism; that is, we will average the commutator over both the quantum-mechanical and thermodynamic ensembles of the loss mechanism. We denote this by the subscript “osc” to the commutator bracket. Since, of course,

$$[P^{(0)}(t),\Gamma^{(0)}(t')]=[Q^{(0)}(t),\Gamma^{(0)}(t')]=0,\quad[Q^{(0)}(t),P^{(0)}(t)]=i\hbar, \tag{46}$$

we have, from Eqs. (42) and (43),

$$[Q(t),P(t)]_{\text{osc}}=i\hbar e^{-\beta t}+S, \tag{47}$$

[14] When we take expectation values, we can do so with respect to either system only or with respect to both. Unless specified otherwise, the term “expectation value” will imply the latter.

where

$$S\equiv-\frac{\omega\alpha^2}{4\pi c^2}\int_0^t dt_1\int_0^t dt_2\exp[-\beta t+\tfrac{1}{2}\beta(t_1+t_2)]$$
$$\times\langle[\Gamma^{(0)}(t_1),\Gamma^{(0)}(t_2)]\rangle\cos\omega(t-t_1)\sin\omega(t-t_2). \quad (48)$$

From Eq. (17) one easily obtains

$$\langle[\Gamma^{(0)}(t_1),\Gamma^{(0)}(t_2)]\rangle$$
$$=2iA\sum_{j,k}e^{-E_j/kT}|\tilde{\Gamma}_{jk}|^2\sin\omega_{jk}(t_1-t_2), \quad (49)$$

so that

$$S=-\frac{i\omega\alpha^2}{2\pi c^2}A\int_0^t dt_1\int_0^t dt_2\int_0^\infty dE_j\,\rho(E_j)\int_0^\infty dE_k\,\rho(E_k)$$
$$\times e^{-E_j/kT}\tilde{\Gamma}^2(E_j,E_k)\exp[-\beta t+\tfrac{1}{2}\beta(t_1+t_2)]$$
$$\times\cos\omega(t-t_1)\sin\omega(t-t_2)\sin\omega_{jk}(t_1-t_2). \quad (50)$$

This integral is evaluated in Appendix A with the result that

$$S=i\hbar(1-e^{-\beta t}). \quad (51)$$

We therefore have, from Eq. (47),

$$[Q(t),P(t)]_{\rm osc}=i\hbar. \quad (52)$$

We now see part of the significance of the driving term [the $\Gamma^{(0)}$ term] in Eq. (38). If it were absent, then S would be zero, and our commutation relationship would be

$$[Q(t),P(t)]=i\hbar e^{-\beta t}, \quad (53)$$

which means that the Heisenberg uncertainty principle could not apply, and a correct quantum-mechanical formalism would be impossible. This is, in fact, the case in the work of Stevens[6] and of Kerner.[5] They start with the classical equation of motion [Eq. (38) without the $\Gamma^{(0)}$ term] for the *operator* P, and therefore have no choice but to obtain the commutation relationship (53). Thus, we see very clearly that while the classical equations of motion follow from the quantum-mechanical ones by taking expectation values of the operators, we cannot, in the case of systems with dissipation, derive quantum-mechanical operator equations from the classical ones. Incidentally, it should be noted that the operator properties of P and Q in both the oscillator and loss-mechanism spaces are needed in order to obtain the correct commutation relationship.

Further insight into this matter is gained by considering fluctuations of P and Q, a significant aspect of which is their (formal) determination of the spontaneous emission of microscopic systems coupled to the cavity field. The fluctuation of an observable corresponding to an operator O is given by $\langle O^2\rangle-\langle O\rangle^2$. In the case of the harmonic oscillator, the fluctuation of the coordinates can therefore be obtained from the expectation value of the energy. We calculate the latter, since the energy has an additional significance of its own. From Eq. (2),

$$\langle H_{\rm osc}\rangle=2\pi c^2\langle P^2\rangle+(\omega^2/8\pi c^2)\langle Q^2\rangle. \quad (54)$$

Equation (42) and (43) can be used to evaluate $\langle P^2\rangle$ and $\langle Q^2\rangle$. Thus

$$\langle P^2\rangle=e^{-\beta t}\langle P^{(0)2}\rangle+U, \quad (55)$$

where

$$T=\frac{\omega^2\alpha^2}{32\pi^2c^4}\int_0^t dt_1\int_0^t dt_2\exp[-\beta t+\tfrac{1}{2}\beta(t_1+t_2)]$$
$$\langle\{\Gamma^{(0)}(t_1),\Gamma^{(0)}(t_2)\}\rangle\sin\omega(t-t_1)\sin\omega(t-t_2),$$

and where the symmetrized product $(\{A,B\}\equiv AB+BA)$ is used for convenience. Since

$$\langle\{\Gamma^{(0)}(t_1),\Gamma^{(0)}(t_2)\}\rangle=2A\sum_{i,k}e^{-E_i/kT}|\tilde{\Gamma}_{ik}|^2$$
$$\times\cos\omega_{ik}(t_1-t_2), \quad (56)$$

$$U=\frac{\omega^2\alpha^2A}{16\pi^2c^4}\int_0^t dt_1\int_1^t dt_2\int_0^\infty dE_i\,\rho(E_i)\int_0^\infty dE_k\,\rho(E_k)$$
$$\times e^{-E_i/kT}\exp[-\beta t+\tfrac{1}{2}\beta(t_1+t_2)]\tilde{\Gamma}^2(E_i,E_k)$$
$$\times\sin\omega(t-t_1)\sin\omega(t-t_2)\cos\omega_{ik}(t_1-t_2). \quad (57)$$

The expression for U is evaluated in Appendix B. The result, averaged over a cycle of the oscillator frequency, is

$$U=\frac{\hbar\omega}{8\pi c^2}\frac{1+\exp(-\hbar\omega/kT)}{1-\exp(-\hbar\omega/kT)}(1-e^{-\beta t}), \quad (58)$$

so that

$$2\pi c^2\langle P^2\rangle=2\pi c^2\langle P^{(0)2}\rangle e^{-\beta t}$$
$$+\tfrac{1}{4}\hbar\omega(1-e^{-\beta t})\frac{1+\exp(-\hbar\omega/kT)}{1-\exp(-\hbar\omega/kT)}. \quad (59)$$

The evaluation of $\langle Q^2\rangle$ is carried out in exactly the same manner, and the term in $(\omega^2/8\pi c^2)\langle Q^2\rangle$ due to $\Gamma^{(0)}$ turns out to be equal to the last term in Eq. (59). (This equality means that the magnetic energy is equal to the electric energy when averaged over a cycle, a result to be expected in view of our approximation $\beta/\omega\ll1$.) We therefore have

$$\langle H_{\rm osc}\rangle=\langle H_{\rm osc}{}^{(0)}\rangle e^{-\beta t}$$
$$+\hbar\omega\{\tfrac{1}{2}+[\exp(\hbar\omega/kT)-1]^{-1}\}(1-e^{-\beta t}). \quad (60)$$

It will be instructive to separate the last expression into several parts and discuss them individually. The energy obtained contains both energy due to the signal which may have been present originally in the cavity (and which is determined by the initial state of the radiation field) and fluctuation energy. Any initial signal energy present is contained in $\langle H_{\rm osc}{}^{(0)}\rangle$ and, as evident from Eq. (60), will be damped out like $e^{-\beta t}$, exactly as it is classically. We thus have a further

illustration that our analysis leads to the conventional type of dissipation. However, $\langle H_{\rm osc}{}^{(0)}\rangle$ contains two parts: the initial signal energy and a second part, $\frac{1}{2}\hbar\omega$. The latter is the zero-point energy, or, as it is commonly called, the vacuum fluctuation energy.[15] Subtracting the signal energy from the total energy, we obtain

$$\langle H_{\rm osc}\rangle_{\rm fluctuation}=\tfrac{1}{2}\hbar\omega e^{-\beta t}+\tfrac{1}{2}\hbar\omega(1-e^{-\beta t}) +\hbar\omega[\exp(\hbar\omega/kT)-1]^{-1}(1-e^{-\beta t}), \quad (61)$$

where we have intentionally written the first two terms on the right side separately, although they add up to $\frac{1}{2}\hbar\omega$.

It is important, for our purpose, to note where the several contributions to the fluctuation energy come from. Referring to the right side of Eq. (61), and retracing its derivation, we see that the first term comes from the initial field and the last two terms come from the loss-mechanism driving term. To put it another way, if the right-hand side of Eq. (38) were zero, the last terms in Eqs. (42) and (43) would be zero, and the only term we would have on the right side of Eq. (61) would be the first one. In that event, the vacuum fluctuation energy of the oscillator would be damped out completely, and, just as we saw from the commutation relationships, we would be left with a classical oscillator—an unfortunate situation indeed, if we want to calculate spontaneous emission of atomic systems in the cavity.The second term on the right side of Eq. (61), which comes from the loss mechanism, supplements the first term so as to maintain a constant value of $\frac{1}{2}\hbar\omega$ for the vacuum fluctuation energy of the field. We can say[9]—and this is purely formal, of course—that as the initial vacuum fluctuations of the field are damped out, the fluctuations of the loss mechanism take over and *drive* the field. We note that, as in the case of the commutation relationships, we utilize the operator properties of P and Q in both the oscillator and loss-mechanism spaces to obtain the correct zero-point fluctuations.

The last term in Eq. (61) is obviously due to the thermal fluctuations of the loss mechanism. Since our initial conditions amounted, effectively, to turning on the interaction between loss mechanism and field at time $t=0$, the thermal energy in the field approaches, with the characteristic relaxation time β^{-1}, the steady-state value

$$\hbar\omega[\exp(\hbar\omega/kT)-1]^{-1}, \quad (62)$$

which is in accordance with Planck's radiation law.[16]

We have now verified that our formalism both satisfies quantum-mechanical requirements and leads to the conventional description of dissipation. We have also seen that the driving (or fluctuation) term in Eqs. (38), (42), and (43) is essential from a quantum-mechanical viewpoint, and, incidentally, gives the effect of thermal fluctuations. It is clear that fluctuations may not be separated from dissipation in any quantum-mechanical analysis, and this is, in essence, the reason for the invalidity of the treatments which start from the classical equation of an oscillator with "pure" dissipation.[17]

IV

It is not difficult to extend the formalism to provide for a driving force and for atomic systems coupled to the field. This extension is the central problem in reference 9, is discussed there in detail, and will merely be summarized here in the present context. Going back to the Hamiltonian of Eq. (3), we add whatever terms are needed. A prescribed, classical driving force is obtained by adding a term

$$\alpha_0 P f(t), \quad (63)$$

where $f(t)$ is a prescribed function of the time. An atomic system (molecule) coupled to the field is included by the addition of the terms

$$\alpha_m P\gamma_m+H_m, \quad (64)$$

where H_m is the Hamiltonian of the uncoupled system and γ_m is the coordinate through which this system is coupled to the field. Equations (42) and (43) then become

$$P=e^{-\frac{1}{2}\beta t}P^{(0)}-\frac{\omega}{4\pi c^2}\int_0^t dt_1[\alpha\Gamma^{(0)}(t_1)+\alpha_0 f(t_1) +\alpha_m\gamma_m(t_1)]e^{-\frac{1}{2}\beta(t-t_1)}\sin\omega(t-t_1), \quad (65)$$

$$Q=e^{-\frac{1}{2}\beta t}Q^{(0)}+\int_0^t dt_1[\alpha\Gamma^{(0)}(t_1)+\alpha_0 f(t_1) +\alpha_m\gamma_m(t_1)e^{-\frac{1}{2}\beta(t-t_1)}\cos\omega(t-t_1). \quad (66)$$

The integration of the $f(t_1)$ term may be carried out explicitly. However, $\gamma_m(t_1)$ is one of the unknown operators, just as P and Q are, and Eqs. (65) and (66) become integral forms of two of the equations of motion. The other equations of motion do not involve the loss mechanism explicitly.[9]

V

So far we have used the coupling constant α, the matrix elements $\tilde{\Gamma}_{ik}$, the density of states $\rho(E)$, and the temperature T to specify the behavior of the loss mechanism. The use of these quantities has served to

[15] I. R. Senitzky, Phys. Rev. **111**, 3 (1958), Eq. (18).

[16] It is interesting to note that expression (62) immediately leads to a form of the Johnson-Nyquist noise formula. The bandwidth of the cavity is β, and the power dissipation in the loss mechanism is β times the energy in the cavity. The (noise) power per unit bandwidth coming from the loss-mechanism fluctuations (in the steady state) is therefore equal to the energy in the cavity, which for $\hbar\omega\ll kT$ reduces to kT.

[17] K. W. H. Stevens and B. Josephson, Proc. Phys. Soc. (London) **74**, 561 (1959), mention briefly, in an article based on the results of reference 6, that M. H. L. Pryce has objected to the omission of "some means whereby the oscillator can acquire energy from the fluctuations of the dissipative system."

illuminate the processes involved in the interaction between the oscillator and the loss mechanism; but, just as we do not need a complete description of the behavior of the loss mechanism, we do not need the full set of parameters. Experimentally, our knowledge of the loss mechanism usually consists of information about the loss constant β and the temperature. In principle, one should be able to construct a formalism requiring nothing more. We therefore reformulate the formalism so as to involve only β and T.

The basic equation for the damped harmonic oscillator may be takento be Eq. (38). In this equation the only reference to the loss mechanism is through the loss constant β and the operator $\alpha\Gamma^{(0)}(t)$. There is really not very much we need know about the latter. We must know that it commutes with the operators of the undamped harmonic oscillator; we must know its expectation value; and, finally, we must know the expectation value of a product of two of these operators evaluated at different times, that is, $\alpha^2\langle\Gamma^{(0)}(t_1)\Gamma^{(0)}(t_2)\rangle$. These results are the only ones which we have used in our analysis and which are needed to calculate commutation relationships and expectation values of fields and energies. In accordance with our approximate scheme (which is inherent in the concept of dissipation), it is evident that, generally, no expectation values of higher order products than the second are necessary, because as soon as we obtain a quadratic expression in $\alpha\Gamma^{(0)}$, we replace it by its expectation value with respect to the loss mechanism times the unit operator.

Equation (44) gives us the expectation value of $\alpha\Gamma^{(0)}(t)$. For the expectation value of the product, we have, from Eqs. (4) and (17),

$$\langle\Gamma^{(0)}(t_1)\Gamma^{(0)}(t_2)\rangle=A\sum_{j,k}e^{-E_j/kT}|\tilde{\Gamma}_{jk}|^2e^{i\omega_{jk}(t_1-t_2)}$$

$$=A\int_0^\infty dE_j\rho(E_j)\int_0^\infty dE_k\rho(E_k)\times e^{-E_j/kT}\tilde{\Gamma}^2(E_j,E_k)e^{i\omega_{jk}(t_1-t_2)}. \quad (67)$$

Using Eqs. (28) and (29), we obtain

$$\langle\Gamma^{(0)}(t_1)\Gamma^{(0)}(t_2)\rangle=A\left[\int_0^\infty\hbar d\omega' B(\omega')\exp(-\hbar\omega'/kT)e^{i\omega'(t_1-t_2)}+\int_{-\infty}^0\hbar d\omega' B(-\omega')e^{i\omega'(t_1-t_2)}\right]. \quad (68)$$

Now, in all calculations in which $\langle\Gamma^{(0)}(t_1)\Gamma^{(0)}(t_2)\rangle$ is to be used, there is a subsequent integration over t_1 and t_2; and in these integrations there is a strong weighing factor in favor of $\omega'\sim\omega$, as can be seen from Eqs. (42) and (43) and in the previous calculations involving $\langle[\Gamma^{(0)}(t_1),\Gamma^{(0)}(t_2)]\rangle$ and $\langle\{\Gamma^{(0)}(t_1),\Gamma^{(0)}(t_2)\}\rangle$. We can therefore write

$$\langle\Gamma^{(0)}(t_1)\Gamma^{(0)}(t_2)\rangle\cong\hbar AB(\omega)\left[\exp(-\hbar\omega/kT)\int_0^\infty d\omega' e^{i\omega'(t_1-t_2)}+\int_0^\infty d\omega' e^{-i\omega'(t_1-t_2)}\right]. \quad (69)$$

Using the definition of β given by Eq. (36), we have

$$\alpha^2\langle\Gamma^{(0)}(t_1)\Gamma^{(0)}(t_2)\rangle=\frac{4c^2\hbar\beta}{\omega[1-\exp(-\hbar\omega/kT)]}\left[\exp(-\hbar\omega/kT)\times\int_0^\infty d\omega' e^{i\omega'(t_1-t_2)}+\int_0^\infty d\omega' e^{-i\omega'(t_1-t_2)}\right]. \quad (70)$$

Our purpose has now been accomplished. All the information we need about the operator $\alpha\Gamma^{(0)}(t)$ has been specified in terms of β and T. If we like, we can make use of the relationship[18]

$$\int_0^\infty d\omega\, e^{i\omega\tau}=i\mathcal{P}/\tau+\pi\delta(\tau), \quad (71)$$

where $\mathcal{P}/\tau$ indicates the principal value when an integration is made with respect to τ, to write the right side of Eq. (70) in (formally) closed form.

We summarize our results. The operator equation for a damped harmonic oscillator in which the damping time is much larger than a period may be written as

$$\ddot{P}(t)+\beta\dot{P}(t)+\omega^2P(t)=D(t), \quad (72)$$

where $D(t)$ is an operator referring to the loss mechanism. It commutes with the operators of the undamped oscillator and is defined by the following two properties (no others are needed):

$$\langle D(t)\rangle=0, \quad (73)$$

$$\langle D(t_1)D(t_2)\rangle=\frac{\omega^3\hbar\beta}{4\pi^2c^2}\left\{i\frac{\mathcal{P}}{t_1-t_2}+2\pi\delta(t_1-t_2)\times\left[\frac{1}{2}+\frac{1}{\exp(\hbar\omega/kT)-1}\right]\right\}, \quad (74)$$

β being the dissipation constant and T the temperature

[18] See, for instance, W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1954), third edition.

of the loss mechanism.[19] The solution of Eq. (72) is

$$P(t)=P^{(0)}e^{-\frac{1}{2}\beta t}+\frac{1}{\omega}\int_0^t dt_1 D(t_1)e^{-\frac{1}{2}\beta(t-t_1)}\sin\omega(t-t_1), \quad (75)$$

and the corresponding expression for Q is

$$Q(t)=Q^{(0)}e^{-\frac{1}{2}\beta t}-\frac{4\pi c^2}{\omega^2}\int_0^t dt_1 D(t_1)e^{-\frac{1}{2}\beta(t-t_1)}\times\cos\omega(t-t_1), \quad (76)$$

where we have made approximations based on $\beta/\omega\ll 1$. If the oscillator is driven by a "force" and is coupled to other systems in the manner indicated by Eqs. (63) and (64), we replace $D(t)$ in Eqs. (72), (75), and (76) by

$$D(t)-\frac{\omega^2}{4\pi c^2}\alpha_0 f(t)-\frac{\omega^2}{4\pi c^2}\sum_m \alpha_m\gamma_m(t).$$

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APPENDIX A

We evaluate the expressions for for S in Eq. (50). Introducing a change of variables specified by

$$\xi=t_1+t_2, \quad \eta=t_1-t_2, \quad (1A)$$

we have

$$\int_0^t dt_1\int_0^t dt_2 \rightarrow \tfrac{1}{2}\left(\int_0^t d\xi\int_{-\xi}^{\xi}d\eta+\int_t^{2t}d\xi\int_{-(2t-\xi)}^{2t-\xi}d\eta\right)\equiv\tfrac{1}{2}\int d\xi\int d\eta. \quad (2A)$$

Making use of Eqs. (28) and (29), we obtain

$$S=-\frac{i\omega\alpha^2 A}{8\pi c^2}\int d\xi\int d\eta\int\hbar d\omega'\int dE\,\rho(E+\tfrac{1}{2}\hbar\omega')\times\rho(E-\tfrac{1}{2}\hbar\omega')\bar{\Gamma}^2(E+\tfrac{1}{2}\hbar\omega', E-\tfrac{1}{2}\hbar\omega')\times\exp[-(E+\tfrac{1}{2}\hbar\omega')/kT]e^{-\beta(t-\frac{1}{2}\xi)}\times[\sin\omega(2t-\xi)+\sin\omega\eta]\sin\omega'\eta. \quad (3A)$$

[19] It is interesting to note that the first term in the curly bracket of Eq. (74) is antisymmetric in t_1-t_2, while the second term is symmetric. Thus, only the first term contributes to the commutator of $D(t_1)$ and $D(t_2)$, while only the second term contributes to the symmetrized product. $\langle D(t_1)D(t_2)\rangle$ may be considered as a correlation function for the fluctuations of the loss mechanism. The quantum-mechanical fluctuations can be separated from the thermal fluctuations by setting $T=0$.

Because of the symmetric limits of integration with respect to η, the first term in the square bracket of Eq. (3A) drops out. Carrying out the E integration, we have

$$S=-\frac{i\omega\alpha^2 A}{8\pi c^2}\int d\xi\int d\eta\int_0^\infty\hbar d\omega' B(\omega')e^{-\beta(t-\frac{1}{2}\xi)}\times\sin\omega'\eta\sin\omega\eta[\exp(-\hbar\omega'/kT)-1]. \quad (4A)$$

Noting that the main contribution to the integral comes from $\omega'\sim\omega$, and using the definition of β given by Eq. (36), we obtain

$$S\cong\frac{i\hbar\beta}{2\pi}\int d\xi\int d\eta\, e^{-\beta(t-\frac{1}{2}\xi)}\frac{\mathcal{P}}{\eta}\sin\omega\eta=\tfrac{1}{2}i\hbar\beta e^{-\beta t}\int_0^{2t}d\xi\, e^{\frac{1}{2}\beta\xi}=i\hbar(1-e^{-\beta t}). \quad (5A)$$

APPENDIX B

We now evaluate the expression for U in Eq. (57). We make the same change of variables as that in Appendix A, average over a cycle of the oscillator frequency (which eliminates an oscillatory term in t), and integrate over E. The result is

$$U=\frac{\omega^2\alpha^2 A}{128\pi^2c^4}\int d\xi\int d\eta\int_0^\infty\hbar d\omega' B(\omega')e^{-\beta(t-\frac{1}{2}\xi)}\times[1+\exp(-\hbar\omega'/kT)]\times[\cos(\omega-\omega')\eta+\cos(\omega+\omega')\eta]. \quad (6A)$$

Noting, as in Appendix A, that the main contribution to the integral comes from $\omega'\sim\omega$, we have

$$U\cong\frac{\hbar\omega\beta}{32\pi^2c^2}\left(\frac{1+\exp(-\hbar\omega/kT)}{1-\exp(-\hbar\omega/kT)}\right)\int d\xi\int d\eta\, e^{-\beta(t-\frac{1}{2}\xi)}\times\int_0^\infty d\omega'[\sin\omega\eta\sin\omega'\eta+\cos\omega\eta\cos\omega'\eta]. \quad (7A)$$

The ω' integration over the square bracket in Eq. (7A) yields[18]

$$(\mathcal{P}/\eta)\sin\omega\eta+\pi\delta(\eta)\cos\omega\eta, \quad (8A)$$

and the subsequent straightforward integration over η and ξ gives

$$U=\frac{\hbar\omega}{8\pi c^2}\frac{1+\exp(-\hbar\omega/kT)}{1-\exp(-\hbar\omega/kT)}(1-e^{-\beta t}). \quad (9A)$$