Asymptotic Conditions and Perturbation Theory

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The formulation of field theories based on few postulates but not using Lagrangian, Hamiltonian, or field equations has been investigated by many authors. In such a theory the coupled integral equations for Green's functions play the role of a substitute for field equations and serve to determine various physical quantities from the theory. In this paper we give a general prescription for the systematic solution of the coupled integral equations in perturbation theory.

We discuss two kinds of Green's functions: (1) the retarded functions, and (2) the time-ordered functions. Especially in the solution of the latter we have to use dispersion relations, and one finds a complete correspondence between the numbers of subtractions in the dispersion relations and the types of interactions in the conventional field theory; furthermore, the so-called renormalized coupling constants can be introduced into our theory through the boundary conditions supplementing the substracted dispersion relations. On the contrary, however, it is not possible even to define the unrenormalized coupling constants in our scheme. In other words, all unobservable divergent quantities are completely eliminated in our formulation, and no divergences occur in the course of the entire calculations.

We apply this method to quantum electrodynamics to illustrate the above statement and also to show how one can get convergent unambiguous solutions in agreement with the conventional renormalized quantum electrodynamics. In discussing quantum electrodynamics, it is necessary to discover how to express the requirement of gauge invariance without referring to Lagrangian, Hamiltonian, or field equations; it is found that a set of equations which is a straightforward generalization of the Ward identity meets this requirement.

I. INTRODUCTION

IN the conventional field theory one starts from a Lagrangian or a Hamiltonian, derives field equations, quantizes the theory, applies perturbation theory, and applies the renormalization procedures to get convergent solutions. Should those solutions correctly describe nature, there should be another way to achieve the same results without encountering any kind of divergences, and the divergences inherent in the conventional theory would suggest that there are physically meaningless and unnecessary concepts involved in the conventional theory. For this reason it is desirable to reformulate the theory by eliminating divergent meaningless quantities from the conventional theory. There is also another reason for requiring the improvement of the present renormalization procedures since they apply only to perturbation theory. We have no ideas at present how to apply the renormalization prescription to approximations other than perturbation theory.[1]

The starting point to reformulate field theory would be to find out and exhaust all kinds of relationships among finite renormalized expressions. A powerful approach along this line of reasoning would be dispersion theory, and in fact it has been suggested by Mandelstam[2] that the combination of the dispersion relations and unitarity of the S matrix would determine the dynamical behavior of the system of few elementary particles. Our approach in this paper is more or less similar to his, but we start from Green's functions rather than from the S matrix.

An important step in this direction is the discovery of the expressions of the S-matrix elements in terms of Heisenberg operators. First, the expression of the S matrix for the pion-nucleon scattering was derived by Low,[3] by Goldberger,[4] and by Nambu,[5] and subsequently it was applied to the Chew-Low theory and dispersion relations. A more general method was proposed by Lehmann, Symanzik, and Zimmermann,[6] who showed the importance of the asymptotic conditions in the derivation of the S matrix elements in terms of Heisenberg operators.

Since then a new approach to quantum field theory has started using the idea that one could determine the dynamics of fields based on only few postulates without reference to a Lagrangian or Hamiltonian.

The postulates which seem to be of the most general validity in the conventional field theory are (1) Lorentz invariance, (2) microscopic causality condition, (3) asymptotic conditions, and (4) irreducibility condition. In such an approach we do not assume even the existence of the field equations, which have only a *korrespondenzmässig* meaning even in the conventional renormalized field theory in the sense that renormalized field equations always involve divergent expressions such as divergent self-energy terms and so on.

In a previous paper[7] to be referred to as I hereafter, an interesting conclusion was drawn that in principle one cannot distinguish composite particles from elementary particles as far as the above mentioned postulates are

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[1] To illustrate the situation, we have not succeeded in renormalizing the Bethe-Salpeter equation for meson-nucleon scattering.

[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958), and subsequent papers. See also the review article by G. F. Chew, University of California Radiation Laboratory Report UCRL-8670 (unpublished).

[3] F. E. Low, Phys. Rev. **97**, 1392 (1955).

[4] M. L. Goldberger, Phys. Rev. **99**, 979 (1955).

[5] Y. Nambu, Phys. Rev. **98**, 803 (1955).

[6] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **1**, 205 (1955).

[7] K. Nishijima, Phys. Rev. **111**, 995 (1958). See also W. Zimmermann, Nuovo cimento **10**, 597 (1958); R. Haag, Phys. Rev. **112**, 669 (1958).

concerned.[8] In the present paper we would like to show the equivalence of our theory to the conventional renormalized theory by solving the integral equations resulting from the asymptotic conditions in perturbation theory.

Since our equations provide us with the renormalized solutions without applying the renormalization procedures, it is expected that they might be applied to the problems of strong interactions to get renormalized solutions without recourse to the perturbation theory.

In Sec. II we shall briefly review the derivation of the coupled integral equations for Green's functions from the asymptotic conditions. Two kinds of Green's functions, the retarded functions and time-ordered functions, are introduced. In Sec. III we shall present a general prescription for solving the integral equations for the retarded functions, and in Sec. IV we discuss the solutions of the integral equations for the time-ordered functions, and introduce dispersion relations for these functions. Next we illustrate the method by applying it to quantum electrodynamics for which perturbation theory is meaningful. In Sec. VA we investigate the problem of how to express the requirement of gauge invariance without reference to Lagrangian, Hamiltonian, or field equations, and we obtain a set of relationships which are equivalent to the gauge invariance. These relationships, as combined with the assumed existence of the Lagrangian, lead to the familar gauge invariant Lagrangian. A closer examination of these relationships shows that they are generalizations of Ward's identity.[9] Finally in Sec. VB we apply the method to quantum electrodynamics.

II. ASYMPTOTIC CONDITIONS AND THE DERIVATION OF INTEGRAL EQUATIONS FOR GREEN'S FUNCTIONS

The asymptotic condition as introduced by LSZ assumes that the matrix elements of field operators converge to those of incoming or outgoing field operators for remote past or future. Based upon this assumption they derived recursion formulas for the matrix elements of the time-ordered products of Heisenberg operators.

The asymptotic condition cannot be proved, but is postulated in their theory. We also postulate this condition in our theory, so that instead of trying to prove it we try to prove the self-consistency of this condition, or the self-consistency of the recursion formulas. In I it has been proved that the self-consistency of this condition is equivalent to the existence of solutions of a set of integral equations derived from the recursion formulas. Since, however, it is not possible to prove the existence theorem for such a complicated set of equations, we shall be content with the general prescription for solving the coupled integral equations in perturbation theory.

Let us first introduce the recursion formulas, which we call the asymptotic condition in this paper, and then derive the coupled integral equations for the Green's functions.

We first postulate the existence of two complete sets of state vectors $\{\Phi_\alpha^{(+)}\}$ and $\{\Phi_\alpha^{(-)}\}$. $\Phi_\alpha^{(+)}$ denotes a state vector for a scattering state satisfying the outgoing wave boundary condition. $\Phi_\alpha^{(-)}$ denotes a similar state but satisfying the incoming wave boundary condition.

We further assume that a state vector may be written in the following form:

$$\Phi_\alpha^{(\pm)}=\Phi_{a_1\cdots a_n}^{(\pm)}\equiv|\alpha,\pm\rangle=|a_1\cdots a_n,\pm\rangle. \quad (2.1)$$

$\Phi_{a_1\cdots a_n}^{(+)}$ means that this stationary scattering state is formed by incoming stable particles a_1, $a_2\cdots$, a_n and the outgoing scattered and/or produced particles. Similarly $\Phi_{a_1\cdots a_n}^{(-)}$ is the superposition of outgoing stable particles a_1, $a_2\cdots$, a_n and the incoming particles. If we assume the neutral scalar theory for simplicity the recursion formulas are given by[10]

$$\begin{aligned}&\langle\beta,-|T[\varphi(x_1)\cdots\varphi(x_n)]|\alpha a,+\rangle\\&\quad=\langle\beta/a,-|T[\varphi(x_1)\cdots\varphi(x_n)]|\alpha,+\rangle\\&\qquad-i\int(dz)\langle0|\varphi(z)|a\rangle\\&\qquad\cdot K_z\langle\beta,-|T[\varphi(x_1)\cdots\varphi(x_n)\varphi(z)]|\alpha,+\rangle, \quad (2.2a)\end{aligned}$$

and

$$\begin{aligned}&\langle\beta a,-|T[\varphi(x_1)\cdots\varphi(x_n)]|\alpha,+\rangle\\&\quad=\langle\beta,-|T[\varphi(x_1)\cdots\varphi(x_n)]|\alpha/a,+\rangle\\&\qquad-i\int(dz)\langle a|\varphi(z)|0\rangle\\&\qquad\cdot K_z\langle\beta,-|T[\varphi(x_1)\cdots\varphi(x_n)\varphi(z)]|\alpha,+\rangle, \quad (2.2b)\end{aligned}$$

where $(dz)=d^4z$, $|\alpha a,\pm\rangle=|a_1\cdots a_n a,\pm\rangle$ and $|\alpha/a,\pm\rangle$ denotes a state obtained by omitting a from $a_1\cdots a_n$ if a is involved, and equal to 0 if a is not involved in $a_1\cdots a_n$. K_z is the Klein-Gordon operator $\Box_z-m$,[2] where m is the rest mass of the quantum of this neutral scalar field. Starting from the above formulas one can derive many different formulas as has been shown in I.

In the above formulas T denotes Wick's time-ordered product,[11] and similarly if we define the anti-time-

[8] That means, for example, that the existence of field equations might be a clue to distinguish elementary particles from composite ones. This is, however, just a guess and not conclusive.

[9] J. C. Ward, Phys. Rev. **77**, 293 (1950); **78**, 182 (1950); Proc. Phys. Soc. (London) **A64**, 54 (1951); H. S. Green, Proc. Phys. Soc. (London) **A66**, 873 (1953); T. D. Lee, Phys. Rev. **95**, 1329 (1954). For the proof of the generalized Ward identity see especially Y. Takahashi, Nuovo cimento **6**, 371 (1957).

[10] $\Phi_0=|0\rangle$ denotes the vacuum state. For the single particle state $|a\rangle$ we need not distinguish between + and −.

[11] G. C. Wick, Phys. Rev. **80**, 268 (1950).

ordered product $\tilde{T}$, we get

$$T[\varphi(x_1)\cdots\varphi(x_n)]^\dagger=\tilde{T}[\varphi^\dagger(x_n)\cdots\varphi^\dagger(x_1)]$$
$$=\tilde{T}[\varphi(x_n)\cdots\varphi(x_1)]. \quad (2.3)$$

By using the complex conjugate of Eqs. (2.2), and (2.3) one can derive recursion formulas for $\tilde{T}$ products of field operators.

Next the retarded or R product[12,13] of field operators is defined by

$$R[\varphi(x):\varphi(x_1)\cdots\varphi(x_n)]$$
$$=\sum_{\text{Perm.}}(-i)^n\theta(x-x_1')\theta(x_1'-x_2')\cdots\theta(x_{n-1}'-x_n')$$
$$\times[\cdots[\varphi(x),\varphi(x_1')]\varphi(x_2')]\cdots\varphi(x_n')], \quad (2.4)$$

where $x_1', \cdots, x_n'$ is a permutation of $x_1, \cdots, x_n$, and $\theta(x)=1$ if $x_0>0$, and $=0$ if $x_0<0$.

Then combining the recursion formulas for T and $\tilde{T}$ products we get the recursion formulas for R products:

$$\langle\beta,+|R[\varphi(x):\varphi(x_1)\cdots\varphi(x_n)]|\alpha a,+\rangle$$
$$=\langle\beta/a,+|R[\varphi(x):\varphi(x_1)\cdots\varphi(x_n)]|\alpha,+\rangle$$
$$+\int(dz)\langle0|\varphi(z)|a\rangle\cdot K_z\langle\beta,+|R[\varphi(x):\varphi(x_1)$$
$$\cdots\varphi(x_n)\varphi(z)]|\alpha,+\rangle, \quad (2.5a)$$

and

$$\langle\beta a,+|R[\varphi(x):\varphi(x_1)\cdots\varphi(x_n)]|\alpha,+\rangle$$
$$=\langle\beta,+|R[\varphi(x):\varphi(x_1)\cdots\varphi(x_n)]|\alpha/a,+\rangle$$
$$+\int(dz)\langle a|\varphi(z)|0\rangle\cdot K_z\langle\beta,+|R[\varphi(x):\varphi(x_1)$$
$$\cdots\varphi(x_n)\varphi(z)]|\alpha,+\rangle. \quad (2.5b)$$

Now by making use of these formulas we shall derive coupled integral equations for the vacuum expectation values of R and T products of field operators. For simplicity we use the following abbreviations:

$$T[\varphi(x_1)\cdots\varphi(x_n)]=T[x_1\cdots x_n],$$

and

$$R[\varphi(x):\varphi(x_1)\cdots\varphi(x_n)]=R[x:x_1\cdots x_n].$$

First from the definition of the R-product the following equation is obtained:

$$R[x:yx_1\cdots x_n]-R[y:xx_1\cdots x_n]$$
$$+i\sum_{\text{Comb}}[R[x:x_1'\cdots x_k'],$$
$$R[y:x_{k+1}'\cdots x_n']]=0. \quad (2.6)$$

Taking the vacuum expectation value of this equation one writes

$$R(x:yx_1\cdots x_n)-R(y:xx_1\cdots x_n)$$
$$+i\sum_{\text{Comb}}\sum_{\alpha}\langle0|R[x:x_1'\cdots x_k']|\alpha,+\rangle$$
$$\times\langle\alpha,+|R[y:x_{k+1}'\cdots x_n']|0\rangle-(x\rightleftarrows y)=0, \quad (2.7)$$

where

$$R(x:yx_1\cdots x_n)=\langle R[x:yx_1\cdots x_n]\rangle_0. \quad (2.8)$$

The iterated use of the recursion formulas (2.5) enable us to express $\langle0|R|\alpha,+\rangle$ znd $\langle\alpha,+|R|0\rangle$ in terms of the vacuum expectation values of the R products of field operators, and we finally arrive at

$$R(x:yx_1\cdots x_n)-R(y:xx_1\cdots x_n)$$
$$+i\sum_{\text{Comb}}\sum_{l=1}^{\infty}\frac{i^l}{l!}\int(du)(dv)K_{u_1}\cdots K_{u_l}$$
$$\times R(x:x_1'\cdots x_k'u_1\cdots u_l)\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)$$
$$\times K_{v_1}\cdots K_{v_l}R(y:x_{k+1}'\cdots x_n'v_1\cdots v_l)$$
$$-(x\rightleftarrows y)=0, \quad (2.9)$$

where (du) and (dv) denote $d^4u_1\cdots d^4u_l$ and $d^4v_1\cdots d^4v_l$ respectively, and $\Delta^{(+)}$ is defined by

$$i\Delta^{(+)}(x-y)=\sum_a\langle0|\varphi(x)|a\rangle\langle a|\varphi(y)|0\rangle$$
$$=\frac{1}{(2\pi)^3}\int(dp)\theta(p_0)\delta(p^2+m^2)e^{ip(x-y)}.$$

If we introduce the so-called Haag's functions[14] by

$$r(x:x_1\cdots x_n)=K_xK_{x_1}\cdots K_{x_n}R(x:x_1\cdots x_n), \quad (2.10)$$

they satisfy a simpler set of equations than (2.9) as given by

$$r(x:yx_1\cdots x_n)-r(y:xx_1\cdots x_n)$$
$$+i\sum_{\text{Comb}}\sum_{l=1}^{\infty}\frac{i^l}{l!}\int(du)(dv)r(x:x_1'\cdots x_k'u_1\cdots u_l)$$
$$\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)r(y:x_{k+1}'\cdots x_n'v_1\cdots v_l)$$
$$-(x\rightleftarrows y)=0. \quad (2.11)$$

The new set of Eqs. (2.11) is simpler than (2.9) not only in form but also in the method of solution. From (2.10) one can express R functions in terms of r functions as

$$R(x:x_1\cdots x_n)=(-1)^{n+1}\int(dy)(dy_1)\cdots(dy_n)$$
$$\times\Delta_R(x-y)\Delta_R(y_1-x_1)\cdots\Delta_R(y_n-x_n)$$
$$\times r(y:y_1\cdots y_n), \quad (2.12)$$

[12] See I, reference 7; see also K. Nishijima, Progr. Theoret. Phys. (Kyoto) **17**, 765 (1957).

[13] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **6**, 319 (1957).

[14] R. Haag, Kgl. Danske Videnskab, Selskab, Mat.-fys. Medd. **29**, 12 (1955).

where $\Delta_R(x-y)$ is the retarded function satisfying

$$K_x\Delta_R(x-y)=-\delta(x-y). \tag{2.13}$$

These results have already been discussed by Lehmann, Symanzik, and Zimmermann,[13] by Glaser, Lehmann, and Zimmermann,[15] and by the author.[7,12]

The algebraic relation for the time-ordered products corresponding to Eq. (2.6) for the retarded products is given by[7]

$$\sum_{\text{Comb}}(-i)^n(-1)^k\tilde{T}[x_1'\cdots x_k']T[x_{k+1}'\cdots x_n']=0. \tag{2.14}$$

If we introduce $T(x_1\cdots x_n)=(-i)^n\langle T[x_1\cdots x_n]\rangle_0$, we get by taking the vacuum expectation value of (2.14) coupled integral equations for T functions corresponding to (2.9):

$$\begin{aligned}&T(x_1\cdots x_n)+T^*(x_1\cdots x_n)\\&+\sum_{\text{Comb}}{}'\sum_{l=0}^{\infty}\frac{i^l}{l!}\int(du)(dv)K_{u_1}\cdots K_{u_l}\\&\times T(x_1'\cdots x_k'u_1\cdots u_l)\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\&\times K_{v_1}\cdots K_{v_l}T^*(x_{k+1}'\cdots x_n'v_1\cdots v_l)=0,\end{aligned} \tag{2.15}$$

where $\sum_{\text{Comb}}{}'$ means omission of $k=0$ and $k=n$.

In order to simplify (2.15) we introduce

$$\tau(x_1\cdots x_n)=K_{x_1}\cdots K_{x_n}T(x_1\cdots x_n), \tag{2.16}$$

then the τ functions satisfy the following coupled integral equations

$$\begin{aligned}&\tau(x_1\cdots x_n)+\tau^*(x_1\cdots x_n)\\&+\sum_{\text{Comb}}{}'\sum_{l=0}^{\infty}\frac{i^l}{l!}\int(du)(dv)\tau(x_1'\cdots x_k'u_1\cdots u_l)\\&\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\&\times\tau^*(x_{k+1}'\cdots x_n'v_1\cdots v_l)=0.\end{aligned} \tag{2.17}$$

The Fourier transform of this equation represents the unitarity condition of the S matrix when all four-momenta are on the mass shell,[7] so that the above equation represents a generalization of the unitarity condition.

The T functions are expressible in terms of τ functions as

$$\begin{aligned}T(x_1\cdots x_n)=(-i)^n\int(dy_1)\cdots(dy_n)\Delta_F(x_1-y_1)\cdots\\\times\Delta_F(x_n-y_n)\tau(y_1\cdots y_n),\end{aligned} \tag{2.18}$$

where $\Delta_F(x-y)$ is Feynman's Δ function satisfying

$$K_x\Delta_F(x-y)=i\delta(x-y). \tag{2.19}$$

[15] V. Glaser, H. Lehmann, and W. Zimmermann, Nuovo cimento **6**, 1122 (1957).

From (2.18) one can further derive

$$\begin{aligned}\langle T[x_1\cdots x_n]\rangle_0=\int(dy_1)\cdots(dy_n)\Delta_F(x_1-y_1)\cdots\\\times\Delta_F(x_n-y_n)\tau(y_1\cdots y_n).\end{aligned} \tag{2.20}$$

The r and τ functions are closely related to each other and we can easily derive a few relationships between them, for instance,[16]

$$\begin{aligned}&\tau(xx_1\cdots x_n)\\&=-i\sum_{\text{Comb}}\sum_{l=0}^{\infty}\frac{i}{l!}\int(du)(dv)\tau(x_1'\cdots x_k'u_1\cdots u_l)\\&\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\&\times r(x:x_{k+1}'\cdots x_n'v_1\cdots v_l),\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&r(x:x_1\cdots x_n)\\&=i\sum_{\text{Comb}}\sum_{l=0}^{\infty}\frac{i^l}{l!}\int(du)(dv)\tau^*(x_1'\cdots x_k'u_1\cdots u_l)\\&\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\&\times\tau(xx_{k+1}'\cdots x_n'v_1\cdots v_l).\end{aligned} \tag{2.22}$$

III. INTEGRAL EQUATIONS FOR THE r FUNCTIONS

In this section we shall discuss the systematic solution of (2.11) in perturbation theory. Although only the neutral scalar theory is treated here, the method itself is general to cover all other cases. As a matter of fact we apply this method to quantum electrodynamics in Sec. VI.

a. Free Field

There are three boundary conditions to be imposed on the r functions.[7]

(1) $r(x:x_1\cdots x_n)$ is a Lorentz invariant function;

(2) $r(x:x_1\cdots x_n)=0$ unless $x_0>(x_1)_0,\ \cdots,\ (x_n)_0$,

(3) $r(x:x_1\cdots x_n)$ is symmetric in $x_1,\ \cdots,\ x_n$.

Now the condition characterizing the free field is given by

$$r(x:x_1\cdots x_n)=0 \quad \text{for} \quad n>1.$$

In this case it is preferable to discuss the function $R(x:y)=R(x-y)$ rather than the function $r(x:y)=r(x-y)$. The equation for $R(x-y)$ is given by

$$\begin{aligned}R(x-y)-R(y-x)=\int(du)(dv)K_uR(x-u)\\\cdot\Delta(u-v)\cdot K_vR(y-v),\end{aligned} \tag{3.1}$$

[16] One can derive (2.21) from Eq. (B.18) in I. One has to multiply by a factor i the left-hand sides of Eq. (B.18) and the next equation.

where $\Delta(x)=\Delta^{(+)}(x)-\Delta^{(+)}(-x)=\Delta^{(+)}(x)+\Delta^{(-)}(x)$, and the only solution of (3.1) satisfying the three boundary conditions is given by

$$R(x-y)=-\Delta_R(x-y)=\theta(x-y)\Delta(x-y). \quad (3.2)$$

From this solution we get

$$r(x:y)=-K_xK_y\Delta_R(x-y)=K_x\delta(x-y). \quad (3.3)$$

This result is the basis for perturbation theory.

b. Interacting Field[12]

Next let us assume that a weak perturbation is given and that the r functions can be expanded in a power series of a small expansion parameter e.

We further assume the order of magnitude of the r function to be

$$r(x:x_1\cdots x_n)=O(e^{n-1}), \quad (3.4)$$

and in general we may write

$$r(x:x_1\cdots x_n)=\sum_{m=n-1}^{\infty} r_m(x:x_1\cdots x_n), \quad (3.5)$$

where r_m is the term of the order of e^m.

First let us pick up terms of the order of e^m from the r Eq. (2.11):

$$r_m(x:yx_1\cdots x_n)-r_m(y:xx_1\cdots x_n)$$
$$+i\sum_{\text{Comb}}\sum_{s=1}^{m-1}\sum_{l=1}^{\infty}\frac{i^l}{l!}\int(du)(dv)r_s(x:x_1'\cdots x_k'u_1\cdots u_l)$$
$$\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)$$
$$\times r_{m-s}(y:x_{k+1}'\cdots x_n'v_1\cdots v_l)-(x\rightleftarrows y)=0. \quad (3.6)$$

It is worth while to notice that terms corresponding to $s=0$ or m do not occur in the summation,[17] for if we insert $r_0(x-u)=K_x\delta(x-u)$ into the above integral we will get

$$\int(du)K_x\delta(x-u)\cdot\Delta^{(+)}(u-v)=K_x\Delta^{(+)}(x-v)=0.$$

Therefore, if the r functions are known up to the order e^{m-1} the second term in (3.6) is known, and we may write (3.6) as

$$r_m(x:yx_1\cdots x_n)-r_m(y:xx_1\cdots x_n)$$
$$=Q_m(xy:x_1\cdots x_n), \quad (3.7)$$

where $Q_m(xy:x_1\cdots x_n)$ is assumed to be known and is antisymmetric in x and y. Then the next problem is to determine $r_m(x:y\cdots)$ and $r_m(y:x\cdots)$ separately, and we make use of the boundary condition (2) for this purpose. We get

$$r_m(x:yx_1\cdots x_n)=Q_m(xy:x_1\cdots x_n) \quad \text{if} \quad x_0>y_0,$$
$$=0 \quad \text{if} \quad x_0<y_0. \quad (3.8)$$

However, we cannot determine r_m for $x_0=y_0$. The microscopic causality condition requires that the solution of the homogeneous equation

$$r_m(x:yx_1\cdots x_n)-r_m(y:xx_1\cdots x_n)=0$$

should vanish unless $x=y$, and hence the general solution of (3.7) will be given by

$$r_m(x:yx_1\cdots x_n)=\theta(x-y)Q_m(xy:x_1\cdots x_n)$$
$$+\delta(x-y)f(x:x_1\cdots x_n)$$
$$+\sum\text{ derivative of }\delta(x-y)$$
$$\times\text{func. }(x:x_1\cdots x_n). \quad (3.9)$$

This solution satisfies Eq. (3.7) but will not satisfy the symmetry requirement, the third boundary condition, in general. One has to fix the additive undetermined part of (3.9) so as to satisfy the symmetry requirement. Suppose that we have found a special solution $r_m^{(0)}$ that satisfies all boundary conditions, then the general solution satisfying Eq. (3.7) as well as all three boundary conditions is given by

$$r_m(x:yx_1\cdots x_n)$$
$$=r_m^{(0)}(x:yx_1\cdots x_n)+P(\partial/\partial y,\partial/\partial x_1,\cdots,\partial/\partial x_n)$$
$$\times\delta(x-y)\delta(x-x_1)\cdots\delta(x-x_n), \quad (3.10)$$

where P is an invariant symmetric polynomial of differential operators $\partial/\partial y$, $\partial/\partial x_1$, $\cdots$, $\partial/\partial x_n$ and the first few terms will be given by

$$P(\partial/\partial y,\partial/\partial x_1,\cdots\partial/\partial x_n)\delta(x-y)\delta(x-x_1)\cdots\delta(x-x_n)$$
$$=c_0\delta(x-y)\delta(x-x_1)\cdots\delta(x-x_n)$$
$$+c_1\Big(\sum_{k=0}^{n}\Box_{x_k}\Big)\delta(x-y)\delta(x-x_1)\cdots\delta(x-x_n)$$
$$+c_2\Big(\sum_{i\neq k}\frac{\partial}{\partial x_i}\cdot\frac{\partial}{\partial x_k}\Big)\delta(x-y)\delta(x-x_1)\cdots$$
$$\times\delta(x-x_n)+\cdots,$$

where we have used the convention $y=x_0$.

By comparing these results with those of the conventional field theory one sees that the introduction of the additive δ-function terms corresponds to the introduction of interaction terms of the form φ^{n+2}, $\varphi^{n-1}\Box\varphi$, $\varphi^n(\partial\varphi/\partial x)(\partial\varphi/\partial x)$, $\cdots$ in the conventional theory. To see this situation more clearly let us consider the first order perturbation theory; then the r equation in this order is given by

$$r_1(x:yx_1\cdots x_n)-r_1(y:xx_1\cdots x_n)=0. \quad (3.11)$$

[17] This is not the case, however, for the R functions.

The general solution of this equation is given by

$$r_1(x:yx_1\cdots x_n)=P(\partial/\partial y,\partial/\partial x_1,\cdots\partial/\partial x_n)\times\delta(x-y)\delta(x-x_1)\cdots\delta(x-x_n),$$

and one can immediately compare this result with that of the conventional theory and see that the above statement is true. Apart from trivial numerical factors, c_0, c_1, $c_2\cdots$ are the coupling constants for the interactions φ^{n+2}, $\varphi^{n+1}\Box\varphi$, $\varphi^n(\partial\varphi/\partial x)(\partial\varphi/\partial x)$, $\cdots$. Thus we conclude that the constants of integration occurring in the solution of r equations correspond to the coupling constants in the conventional field theory. We will meet a similar situation in the solution of the τ equations.

A difficulty associated with the solution of the r equations is the fact that one cannot write down the formal convergent symmetric solution r_m's in an explicit form. This is a serious difficulty in investigating the general properties of the power series expansion. Nevertheless, we shall illustrate how one can solve the r equations in the application to quantum electrodynamics. This difficulty does not occur, however, in the solution of the τ equations as we shall see in the next section.

IV. INTEGRAL EQUATIONS FOR THE τ FUNCTIONS

In this section we try to solve the τ equations in perturbation theory following the pattern described in the solution of the r equations. The free field is characterized in this case by the propagation function

$$\langle T[x_1,x_2]\rangle_0=\Delta_F(x_1-x_2),$$

and hence

$$\tau(x_1,x_2)=K_{x_1}K_{x_2}T(x_1,x_2)=-iK_{x_1}\delta(x_1-x_2). \quad (4.1)$$

We cannot, however, immediately apply the method described in the previous section to the τ equations, but we first have to prepare for a new set of auxiliary functions.

The Feynman diagrams contributing to the function $\tau(x_1,\cdots,x_n)$ may be divided into two classes. The first one consists of the so-called connected diagrams, and the other consists of disconnected diagrams (Fig. 1 and Fig. 2).

If the function $\rho(x_1,\cdots,x_n)$ is defined as the sum of contributions from connected diagrams alone, then it is not difficult to derive the following formula on graphical considerations

$$\tau(xx_1\cdots x_n)=\rho(xx_1\cdots x_n)+\sum_{\text{Comb}}\rho(xx_1'\cdots x_k')\tau(x_{k+1}'\cdots x_n'), \quad (4.2)$$

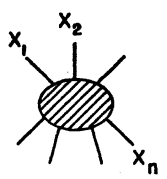


FIG. 1. A connected diagram.

where $x_1',\cdots,x_n'$ is the permutation of $x_1,\cdots,x_n$ and the summation should be taken over all possible combinations that divide $x_1,\cdots,x_n$ into two groups. Eq. (4.2) allows us to express τ functions in terms of ρ functions, and *vice versa*. Therefore one can express the τ equations in terms of the ρ functions to get the ρ equations. For this purpose, however, it is simpler to use the following device: we first pick up those terms from the τ equation which correspond to connected diagrams. First, it is clear from the definition that we may write

$$[\tau(x_1\cdots x_n)+\tau^*(x_1\cdots x_n)]_{\text{conn.}}=\rho(x_1\cdots x_n)+\rho^*(x_1\cdots x_n). \quad (4.3)$$

To pick up terms corresponding to connected diagrams from the nonlinear part of the τ equation, express the τ functions in terms of ρ functions and check if they are connected in any way, either through ρ functions or through $\Delta^{(+)}$ functions (Fig. 3). The ρ equation is, therefore, given by

$$\rho(x_1\cdots x_n)+\rho^*(x_1\cdots x_n)+\sum_{\text{Comb}}\sum_{l=1}^{\infty}\frac{i^l}{l!}\int(du)(dv)\times[\tau(x_1'\cdots x_k'u_1\cdots u_l)\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\times\tau^*(x_{k+1}'\cdots x_n'v_1\cdots v_l)]_{\text{conn.}}=0. \quad (4.4)$$

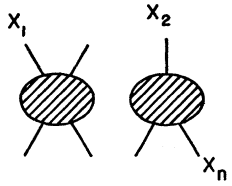


FIG. 2. A disconnected diagram.

The connected part of the nonlinear terms in the above equation may be found by applying the prescription given above. To facilitate understanding of this prescription we shall give an example.

Example.

$$[\tau(x_1x_2u_1u_2)\Delta^{(+)}(u_1-v_1)\Delta^{(+)}(u_2-v_2)\tau^*(x_3x_4v_1v_2)]_{\text{conn.}}.$$

There are three different types of connected diagrams as shown in Figs. 4(a), 4(b), and 4(c). Correspondingly we get three different types of contributions.

$$\rho(x_1x_2u_1u_2)\Delta^{(+)}(u_1-v_1)\Delta^{(+)}(u_2-v_2)\rho^*(x_3x_4v_1v_2), \quad (4.5a)$$

$$\rho(x_1x_2u_1u_2)\Delta^{(+)}(u_1-v_1)\Delta^{(+)}(u_2-v_2)\times[\rho^*(x_3v_1)\rho^*(x_4v_2)+(x_3\rightleftarrows x_4)], \quad (4.5b)$$

and

$$[\rho(x_1u_1)\rho(x_2u_2)+(x_1\rightleftarrows x_2)]\Delta^{(+)}(u_1-v_1)\times\Delta^{(+)}(u_2-v_2)\rho^*(x_3x_4v_1v_2). \quad (4.5c)$$

The free field is characterized in terms of the ρ functions by

$$\rho(x_1x_2)=-iK_{x_1}\delta(x_1-x_2),$$

and

$$\rho(x_1x_2\cdots x_n)=0 \quad \text{for} \quad n>2. \tag{4.6}$$

The functions $\tau(x_1\cdots x_n)$ for the free field, however, do not vanish in general, and this is the reason for preferring the ρ functions over the τ functions in our formulation.

To the ρ equation (4.4) we can apply perturbation theory, and if we know the ρ functions up to the order e^{m-1}, where e is a small expansion parameter as in the previous section, one can immediately find the expression for $\rho_m(x_1\cdots x_n)+\rho_m{}^*(x_1\cdots x_n)$ from the ρ equation quite similarly to the case of τ functions. Then the next task is to separate ρ_m from the sum $\rho_m+\rho_m{}^*$. In the case of the τ functions, the separation was made by using the retarded character of the retarded functions, but in the present case we must discuss this problem in momentum space.

First we write down the CPT theorem:

$$\langle\varphi(x_1)\cdots\varphi(x_n)\rangle_0=\langle\varphi(-x_n)\cdots\varphi(-x_1)\rangle_0, \tag{4.7}$$

which has been proved by Jost[18] from the fundamental postulates alone. This relation leads to the following equation for the τ functions:

$$\tau(x_1,\cdots,x_n)=\tau(-x_1,\cdots,-x_n), \tag{4.8a}$$

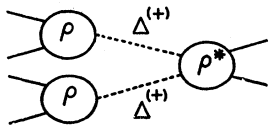


FIG. 3. A typical connected diagram occurring in the nonlinear part of the ρ equation.

and hence

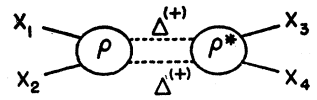
$$\rho(x_1,\cdots,x_n)=\rho(-x_1,\cdots,-x_n). \tag{4.8b}$$

Then we introduce the Fourier transform of ρ by

$$\rho(x_1,\cdots,x_n)=\frac{-i}{(2\pi)^{4(n-1)}}\int(dp_1)\cdots(dp_n)\times\delta(p_1+\cdots+p_n)e^{i(p_1x_1+\cdots+p_nx_n)}\times\mathcal{G}(p_1,\cdots,p_n). \tag{4.9}$$

The CPT theorem as combined with the Lorentz invariance requires that $\mathcal{G}(p_1,\cdots,p_n)$ should be a function of scalar products $p_\alpha p_\beta$ and four-dimensional δ functions $\delta(p_1'+\cdots+p_s')$. Since, however, the ρ functions correspond only to connected diagrams, the Fourier transform of the ρ function will not involve δ functions other than $\delta(p_1+\cdots+p_n)$ expressing the over-all conservation of the energy-momentum. Hence $\mathcal{G}$ will be a function of scalar products $p_\alpha p_\beta$ alone. Now the Fourier transform of $\rho+\rho^*$ is $\mathrm{Im}\mathcal{G}$ so that the separation problem turns out to be the problem of finding $\mathrm{Re}\mathcal{G}$ when $\mathrm{Im}\mathcal{G}$ is known. This kind of relation called a dispersion relation has been extensively

[18] R. Jost, Helv. Phys. Acta **30**, 40 (1957).



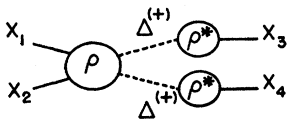
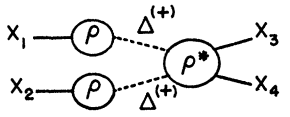


FIG. 4. Connected diagrams corresponding to the expressions (4.5a), (4.5b), and (4.5c).



investigated in recent years, and we shall utilize some results of dispersion theory.

It is known that the Green's function $\mathcal{G}$ may be expressed in the following version in all orders of perturbation theory[19]:

$$\mathcal{G}(p_\alpha p_\beta)=\int\frac{\sigma(c_{\alpha\beta},M^2)dc_{\alpha\beta}dM^2}{(\sum c_{\alpha\beta}p_\alpha p_\beta+M^2-i\epsilon)^N}, \tag{4.10}$$

where σ is a real weight function, M^2 is always positive, and N is a suitable integer. Combining (4.10) with the formulas

$$\int_0^\infty\frac{d\xi'}{\xi'-\xi}\delta(c\xi'+M^2)=\theta(-c)\frac{1}{c\xi+M^2}, \tag{4.11a}$$

and

$$\int_{-\infty}^0\frac{d\xi'}{\xi'-\xi}\delta(c\xi'+M^2)=-\theta(c)\frac{1}{c\xi+M^2}, \tag{4.11b}$$

one can find the dispersion relation for $\mathcal{G}$

$$\mathrm{Re}\mathcal{G}(p_\alpha p_\beta\cdot\xi)=\frac{P}{\pi}\left[\int_0^\infty\frac{d\xi'}{\xi'-\xi}-\int_{-\infty}^0\frac{d\xi'}{\xi'-\xi}\right]\times\mathrm{Im}\mathcal{G}(p_\alpha p_\beta\cdot\xi'), \tag{4.12}$$

where ξ is a scaling parameter to be multiplied into all scalar products. The relation (4.12) can also be expressed as

$$\mathcal{G}(p_\alpha p_\beta\cdot\xi)=\frac{1}{\pi}\left[\int_0^\infty\frac{d\xi'}{\xi'-\xi-i\epsilon}-\int_{-\infty}^0\frac{d\xi'}{\xi'-\xi+i\epsilon}\right]\times\mathrm{Im}\mathcal{G}(p_\alpha p_\beta\cdot\xi'). \tag{4.13}$$

[19] Y. Nambu, Nuovo cimento **9**, 610 (1958).

We have not given the proof of the above dispersion relation, but we know that it is true in all orders of perturbation theory of the conventional theory. Therefore, when we formulate field theories in terms of the time-ordered functions, we may regard the above dispersion relation as a postulate substituting for the Lorentz invariance and microscopic causality conditions. As a matter of fact both conditions are already involved in the dispersion relation. It is very desirable, however, to try to prove this dispersion relation without recourse to the perturbation theory.

An important remark on the dispersion relation (4.13) is that one needs subtractions for some $\mathcal{G}$ functions, because in general one has to add a polynomial of scalar products $p_\alpha p_\beta$ to (4.10). As we shall see below the number of subtractions in the dispersion relation determines the type of interactions introduced in our scheme.

(a) Two-Point Green's Function

In the case of the two-point Green's function it is more convenient to discuss $T(x_1,x_2)$ than to deal with $\rho(x_1,x_2)$. In this case we refer to the Källén-Lehmann representation.[20]

$$T(x_1,x_2)=-\langle T[x_1,x_2]\rangle_0 =-\Delta_F(x_1-x_2,m^2)-\int dM^2\,\sigma(M^2) \times\Delta_F(x_1-x_2,M^2), \quad (4.14)$$

where $\sigma(M^2)$ is a positive definite function of the integration variable M^2, and the derivation of $T(x_1,x_2)$ from $\mathrm{Re}T(x_1,x_2)$ is trivial.

(b) n-Point Green's Function ($n>2$)

In general the subtracted dispersion relations are obtained by differentiating (4.12) so that the true dispersion relations will be given by

$$\left(\frac{d}{d\xi}\right)^s \mathrm{Re}\mathcal{G}(p_\alpha p_\beta\cdot\xi) =s!\frac{P}{\pi}\left[\int_0^\infty \frac{d\xi'}{(\xi'-\xi)^{s+1}}-\int_{-\infty}^0 \frac{d\xi'}{(\xi'-\xi)^{s+1}}\right] \times\mathrm{Im}\mathcal{G}(p_\alpha p_\beta\cdot\xi'). \quad (4.15)$$

Let us first prove in perturbation theory that dispersion relations without subtraction for all $n>2$ give the free field as the only solution.

For this purpose we shall pick up terms from the ρ equation which are linear in the coupling constant. Then we get

$$\mathrm{Re}\rho_1(x_1,\cdots,x_n)=0, \quad (4.16)$$

since the nonlinear terms vanish in this order for the same reason as that in the r equation, i.e.,

$$\int (du)\rho_0(xu)\Delta^{(+)}(u-v)=-iK_x\Delta^{(+)}(x-v)=0.$$

From (4.16) we get

$$\mathrm{Im}\mathcal{G}_1(p_1,\cdots,p_n)=0. \quad (4.17)$$

Hence the dispersion relation without subtraction, Eq. (4.12), will require

$$\mathrm{Re}\mathcal{G}_1(p_1,\cdots,p_n)=0, \quad (4.18)$$

or, in other words, no interaction can be introduced. Thus the above statement is verified in perturbation theory.

Next let us discuss the dispersion relations with one subtraction. In the lowest order perturbation theory we again get (4.17) and the dispersion relation with one subtraction gives us

$$(d/d\xi)\,\mathrm{Re}\mathcal{G}_1(p_\alpha p_\beta\cdot\xi)=0. \quad (4.19)$$

The solution of this differential equation is given by

$$\mathrm{Re}\mathcal{G}_1(p_\alpha p_\beta\cdot\xi)=g, \quad \text{const.} \quad (4.20)$$

This result is exactly the same as that given by the conventional field theory with the interaction

$$(g/n!)\varphi(x)^n. \quad (4.21)$$

Thus we see that interactions are introduced through subtractions. In general, when higher order corrections are taken into account, we fix this constant of integration by an appropriate boundary condition. For instance, for the 3-point Green's function we fix this g by a condition like

$$\mathrm{Re}\mathcal{G}(p_1^2=-m^2,\ p_2^2=-m^2,\ p_3^2=-m^2)=g, \quad (4.22)$$

which defines the renormalized coupling constant.

Suppose that we used dispersion relations with two subtractions, then we get instead of (4.19) the equation

$$\frac{d^2}{d\xi^2}\mathrm{Re}\mathcal{G}_1(p_\alpha p_\beta\cdot\xi)=0. \quad (4.23)$$

The solution of this equation is given by

$$\mathrm{Re}\mathcal{G}_1=g+h_1\sum_{\alpha\neq\beta} p_\alpha p_\beta\cdot\xi+h_2\sum_\alpha p_\alpha^2\cdot\xi. \quad (4.24)$$

This solution corresponds, apart from trivial numerical factors, to the following interactions in the conventional field theory:

$$g\varphi^n,\quad h_1\varphi^{n-2}(\partial\varphi/\partial x)^2,\quad \text{and}\quad h_2\varphi^{n-1}\Box\varphi. \quad (4.25)$$

From this result we may draw an interesting conclusion:

The subtractions in the dispersion relations are necessary in order to introduce interactions, and the

[20] G. Källén, Helv. Phys. Acta **25**, 417 (1952); H. Lehmann, Nuovo cimento **11**, 342 (1954); M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

number of subtractions determines the type of interactions in the corresponding conventional field theory.

It is worth while to notice that one cannot introduce an indefinite number of new interactions through subtractions in the dispersion relations, since they will bring about divergences in the higher order calculations of the two-point Green's functions to which one should not apply an arbitrary number of subtractions. See Eq. (4.14).

The τ or ρ equations are much more convenient than the r equations in many respects. Especially in each order of perturbation theory one can give the formal solution of the ρ equation that is Lorentz-invariant, symmetric in all variables,[21] and convergent. This was not possible in the case of the r equation. We shall discuss in this paper, however, the application of the r equations and leave the application of the τ equation to another opportunity.

V. APPLICATION TO QUANTUM ELECTRODYNAMICS

A. Generalization of the Ward Identity

In this and the next sections we shall discuss the application of our method to quantum electrodynamics, since this is the only subject to which one can apply perturbation theory. Usually quantum electrodynamics is characterized by the requirement of gauge invariance. This invariance requirement implies that the Lagrangian, the Hamiltonian, and the field equations should be invariant under gauge transformations. Since, however, we do not have a Lagrangian or Hamiltonian to start with, the question arises of how to express the gauge invariance in our scheme without using such conventional concepts. The solution to this question will be the subject of this section.

In order to solve this problem we start from the Lagrangian and try to find relationships that hold independently of the precise form of the Lagrangian provided that it is gauge invariant.

The Lagrangian expressing the interaction between matter fields and the electromagnetic field is given by

$$\mathfrak{L}_{\text{total}}=\mathfrak{L}_{\text{matter}}(\varphi_a(x),\,[\partial_\mu-ie_aA_\mu(x)]\varphi_a(x))+\mathfrak{L}_{\text{rad}},\quad(5.1)$$

where

$$\mathfrak{L}_{\text{rad}}=-\tfrac{1}{4}F_{\mu\nu}(x)F_{\mu\nu}(x)-\frac{1}{2}\frac{\partial A_\mu(x)}{\partial x_\mu}\cdot\frac{\partial A_\nu(x)}{\partial x_\nu},$$

and

$$F_{\mu\nu}(x)=\frac{\partial A_\nu(x)}{\partial x_\mu}-\frac{\partial A_\mu(x)}{\partial x_\nu}.$$

This is an example of gauge invariant Lagrangians. We assume that there are several fields and the subscript a denotes the species of field. e_a is the charge of the quantum belonging to the field φ_a.

Next we *define* the four current by the following equation:

$$\Box A_\mu(x)=-j_\mu(x).\quad(5.2)$$

Then from the Euler equation this four current is given by

$$j_\mu(x)=\frac{\partial}{\partial A_\mu(x)}\mathfrak{L}_{\text{matter}}\times(\varphi_a(x),\,\partial_\mu\varphi_a(x)-ie_aA_\mu(x)\varphi_a(x)).\quad(5.3)$$

Next if we introduce $D_{a\mu}(x)$ by

$$D_{a\mu}(x)=\partial_\mu\varphi_a(x)-ie_aA_\mu(x)\varphi_a(x),$$

then the four current is expressed as

$$\begin{aligned}j_\mu(x)&=\sum_a\frac{\partial D_{a\nu}(x)}{\partial A_\mu(x)}\cdot\frac{\partial\mathfrak{L}_{\text{matter}}(x)}{\partial D_{a\nu}(x)}\\&=-i\sum_a e_a\varphi_a(x)\frac{\partial\mathfrak{L}_{\text{matter}}(x)}{\partial D_{a\mu}(x)}.\end{aligned}\quad(5.4)$$

In this summation we always include the field operator $\varphi_{a'}(x)=\varphi_a^*(x)$ corresponding to the antiparticle a'. In particular, Eq. (5.4) gives

$$\begin{aligned}j_0(x)=-ij_4(x)&=-\sum_a e_a\varphi_a(x)\frac{\partial\mathfrak{L}_{\text{matter}}(x)}{\partial[\partial_4\varphi_a(a)]}\\&=-i\sum_a e_a\varphi_a\cdot\frac{\partial\mathfrak{L}_{\text{total}}(x)}{\partial\dot{\varphi}_a(x)}\\&=-i\sum_a e_a\varphi_a(x)\pi_a(x),\end{aligned}\quad(5.5)$$

where $\pi_a(x)$ is the canonical conjugate of $\varphi_a(x)$. Then from the canonical commutation relation

$$[\varphi_a(x),\ \pi_b(y)]=i\delta_{ab}\delta^3(x-y)\quad\text{for}\quad x_0=y_0,\quad(5.6)$$

we immediately get

$$[\varphi_a(x),\,j_0(y)]=e_a\varphi_a(x)\delta^3(x-y)\quad\text{for}\quad x_0=y_0.\quad(5.7)$$

This equation is true independently of the spin of the particle a. From this equation we also get the well-known relation

$$[\varphi_a(x),Q]=e_a\varphi_a(x),\quad(5.8)$$

where Q is the total charge of the system defined by

$$Q=\int j_0(x)d^3x.\quad(5.9)$$

Equation (5.7) involves more information than Eq. (5.8) does.

[21] This is because the ρ equation is completely symmetric in all variables, whereas the r equation is not completely symmetric but two variables x and y are distinguished from all others. See Eq. (2.11).

Then we can verify by a direct calculation the equation

$$\begin{aligned}\Box_x T[A_\mu(x)A_\nu(x')A_\sigma(x'')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]&\\ =-T[j_\mu(x)A_\nu(x')A_\sigma(x'')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]&\\ -\delta(x_0-x_0')T[[\dot{A}_\mu(x),A_\nu(x')],&\\ \times A_\sigma(x'')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]+\cdots&\\ =-T[j_\mu(x)A_\nu(x')A_\sigma(x'')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]&\\ +i\delta^4(x-x')\delta_{\mu\nu}T[A_\sigma(x'')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]&\\ +\cdots,&\end{aligned}$$

where use has been made of the canonical commutation relation for the electromagnetic field $[\dot{A}_\mu(x), A_\nu(x')] = -i\delta_{\mu\nu}\delta^3(x-x')$ for $x_0=x_0'$. The above equation may formally be written as

$$\begin{aligned}\Box_x T[A_\mu(x)\cdots]&\\ =-T[j_\mu(x)\cdots]+i\frac{\delta}{\delta A_\mu(x)}T[\cdots].&\end{aligned}\quad(5.10)$$

This equation is not yet meaningful, however, unless $j_\mu(x)$ is explicitly expressed in terms of field operators, so that we try to eliminate the current operator $j_\mu(x)$ from the above equation. The Euler equation leads to

$$\partial_\mu j_\mu(x)=0. \quad(5.11)$$

We apply $\partial/\partial x_\mu$ to Eq. (5.10) and get

$$\begin{aligned}\Box_x\frac{\partial}{\partial x_\mu}T[A_\mu(x)\cdots]&\\ =\delta(x_0-x_{a0})T[A_\nu(x')A_\sigma(x'')\cdots[\varphi_a(x_a),j_0(x)]&\\ \times\varphi_b(x_b)\cdots]+\cdots+i\delta_{\mu\nu}\frac{\partial}{\partial x_\mu}\delta^4(x-x')&\\ \cdot T[A_\sigma(x'')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]+\cdots&\\ =\left[\sum_a e_a\delta(x-x_a)+i\frac{\partial}{\partial x_\mu}\cdot\frac{\delta}{\delta A_\mu(x)}\right]&\\ \times T[A_\nu(x')A_\sigma(x'')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots],&\end{aligned}$$

or again quite formally we may write

$$\begin{aligned}\Box_x\frac{\partial}{\partial x_\mu}T[A_\mu(x)\cdots]&\\ =\left[\sum_a e_a\delta(x-x_a)+i\frac{\partial}{\partial x_\mu}\cdot\frac{\delta}{\delta A_\mu(x)}\right]T[\cdots].&\end{aligned}\quad(5.12)$$

This equation will be called the Ward-Takahashi equation for the reason to be mentioned later.

The characteristic features of the W-T equations may be summarized as follows:

(1) The W-T equations hold independently of the form of the nonelectromagnetic interactions provided that the total Lagrangian is gauge invariant.

(2) Although we used unrenormalized field operators in the above derivation, the W-T equations hold also for renormalized operators.

(3) The W-T equations are the generalizations of the Ward identity.[9]

(4) In the presence of a Lagrangian that is compatible with the canonical commutation relations, the W-T equations are equivalent to the usual requirement of gauge invariance.

For the above reasons we may regard the W-T equations as the substitute for the requirement of gauge invariance in the absence of a Lagrangian to start with, and in the next section we shall use them as supplementary conditions characterizing quantum electrodynamics.

Among other things the second property is especially important to the formulation of divergence-free field theories, so that we shall give its proof. We start from the following renormalized Lagrangian:

$$\begin{aligned}\mathfrak{L}_{\text{total}}=\mathfrak{L}_{\text{matter}}(\varphi_a(x),[\partial_\mu-ie_aA_\mu(x)]\varphi_a(x))&\\ +\mathfrak{L}_{\text{rad}},&\end{aligned}\quad(5.13)$$

where all field operators as well as e_a represent renormalized ones. In Källén's notation,[22] $\mathfrak{L}_{\text{rad}}$ is given by

$$\mathfrak{L}_{\text{rad}}=-\frac{1-L}{4}F_{\mu\nu}(x)F_{\mu\nu}(x)-\frac{1}{2}\frac{\partial A_\mu(x)}{\partial x_\mu}\cdot\frac{\partial A_\nu(x)}{\partial x_\nu}. \quad(5.14)$$

The current operator $j_\mu(x)$ is defined again by

$$\Box A_\mu(x)=-j_\mu(x). \quad(5.15)$$

The Euler equation gives us the equation

$$j_\mu(x)=j_\mu^{(\text{matter})}-L\left[\Box A_\mu(x)-\frac{\partial^2 A_\nu(x)}{\partial x_\mu\partial x_\nu}\right], \quad(5.16)$$

where $j_\mu^{(\text{matter})}$ denotes the contribution from $\mathfrak{L}_{\text{matter}}$, and the second term on the right-hand side of (5.16) results from the modification of $\mathfrak{L}_{\text{rad}}$. We get, as before, the commutation relation

$$\begin{aligned}[\varphi_a(x),j_0^{(\text{matter})}(y)]=e_a\varphi_a(x)\delta^3(x-y)&\\ \text{for}\quad x_0=y_0.&\end{aligned}\quad(5.17)$$

The commutation relations for the potentials at equal time $x_0=x_0'$ are given by

$$[A_\mu(x),A_\nu(x')]=0, \quad(5.18a)$$

$$\left[\frac{\partial A_\mu(x)}{\partial x_0},A_\nu(x')\right]=\frac{-i}{1-L}\xi_{\mu\nu}\delta^3(x-x'), \quad(5.18b)$$

[22] G. Källén, Helv. Phys. Acta **25**, 417 (1952); **26**, 755 (1953).

and

$$\left[\frac{\partial A_\mu(x)}{\partial x_0},\frac{\partial A_\nu(x')}{\partial x_0'}\right]$$
$$=\frac{L}{1-L}\left(\delta_{\mu 4}\frac{\partial}{\partial x_\nu}+\delta_{\nu 4}\frac{\partial}{\partial x_\mu}\right)\delta^3(x-x'), \quad (5.18c)$$

where $\xi_{\mu\nu}=\delta_{\mu\nu}-L\delta_{\mu 4}\delta_{\nu 4}$. Using these commutation relations we repeat the same calculation as before, and we get

$$\Box_x T[A_\mu(x)A_\nu(x')\cdots]$$
$$=-T[j_\mu(x)A_\nu(x')\cdots]$$
$$+[i\xi_{\mu\nu}/(1-L)]\delta^4(x-x')T[\cdots]+\cdots. \quad (5.19)$$

Next we apply $\partial/\partial x_\mu$ to the above equation, and from the first term we obtain

$$[\sum_a e_a\delta(x-x_a)]T[A_\nu(x')\cdots\varphi_a(x_a)\cdots]$$
$$-\frac{iL}{1-L}(1-\delta_{\nu 4})\frac{\partial}{\partial x_\nu}\delta^4(x-x')$$
$$\cdot T[\cdots\varphi_a(x_a)\cdots]+\cdots. \quad (5.20)$$

The second term in (5.20) results from the noncommutativity between $j_\mu(x)-j_\mu^{(\text{matter})}(x)$ and $A_\nu(x')$. The differentiation of the second term in Eq. (5.19) yields

$$\frac{i(1-L\delta_{\nu 4})}{1-L}\cdot\frac{\partial}{\partial x_\nu}\delta^4(x-x')T[\cdots\varphi_a(x_a)\cdots]+\cdots. \quad (5.21)$$

Summing up terms (5.20) and (5.21) we again find the W-T equations for the renormalized field operators.

In order to verify the third statement we just need to remember Takahashi's proof of the generalized Ward identity. As special cases of the W-T equations we get

$$\Box_x\frac{\partial}{\partial x_\mu}\langle T[A_\mu(x),A_\nu(x')]\rangle_0=i\frac{\partial}{\partial x_\nu}\delta(x-x'), \quad (5.22)$$

and

$$\Box_y\frac{\partial}{\partial y_\mu}\langle T[\psi(x),\bar\psi(x'),A_\mu(y)]\rangle_0$$
$$=-e\{\langle T[\psi(x),\bar\psi(y)]\rangle_0\delta(y-x')$$
$$-\delta(x-y)\langle T[\psi(y),\bar\psi(x')]\rangle_0\}, \quad (5.23)$$

where ψ denotes the operator of the electron field and e the electronic charge. Then we define the vertex operator by

$$\langle T[\psi(x),\bar\psi(x'),A_\mu(y)]\rangle_0$$
$$=-e\int d\xi d\eta d\zeta\, S_F'(x-\xi)\Gamma_\nu(\xi-\eta:\eta-\zeta)$$
$$\times S_F'(\zeta-x')D_{\nu\mu}(\eta-y), \quad (5.24)$$

where

$$S_F'(x-\xi)=\langle T[\psi(x),\bar\psi(\xi)]\rangle_0$$

and

$$D_{\nu\mu}(\eta-y)=\langle T[A_\nu(\eta),A_\mu(y)]\rangle_0.$$

Combination of Eqs. (5.22), (5.23), and (5.24) immediately leads to the generalized Ward identity

$$-i(p-q)_\nu S_F'(p)\Gamma_\nu(p:q)S_F'(q)=S_F'(p)-S_F'(q). \quad (5.25)$$

Hence we may take it for granted that the W-T equations represent the generalizations of the Ward identity.

Finally, if we trace back the calculation leading to the W-T equations assuming suitable commutation relations, we arrive at a partial differential equation for $\mathfrak{L}_{\text{matter}}$ as a function of field operators. The solutions of this equation are given by gauge invariant Lagrangians. Thus the equivalence of the W-T equations to the requirement of gauge invariance is verified.

B. Perturbation Theory

In this section we shall apply perturbation theory to quantum electrodynamics. We solve the r equations subject to the supplementary W-T equations, to illustrate the method.

We first have to define the retarded product for Fermion operators, and for this purpose we introduce a fictitious auxiliary field $\eta(x)$. The operator $\eta(x)$ is assumed to anticommute with all Fermion operators and also with itself, i.e., $\{\eta(x),\eta(x')\}=0$. For the sake of convenience we shall review here the connection between Wick's T product and Dyson's P product. If we let $\psi(x)$ be the field operator of a Fermion field, then the connection is given by

$$P[\eta(x_1)\psi(x_1),\eta(x_2)\psi(x_2),\cdots,\eta(x_n)\psi(x_n)]$$
$$=\eta(x_n)\cdots\eta(x_2)\eta(x_1)T[\psi(x_1)\psi(x_2)\cdots\psi(x_n)]. \quad (6.1)$$

This suggests the way of defining the retarded product for Fermion operators. We define the retarded product by

$$R[\eta(x)\psi(x):\eta(x_1)\psi(x_1),\cdots,\eta(x_n)\psi(x_n)]$$
$$=\eta(x_n)\cdots\eta(x_1)\eta(x)R[\psi(x):\psi(x_1)\cdots\psi(x_n)]. \quad (6.2)$$

The generalization of this definition to other cases is evident. Next in the case of Boson fields we have the following equation connecting R with T and $\tilde T$ products:

$$R[x:x_1\cdots x_n]=\sum_{\text{Comb}} i^n(-1)^k\tilde T[x_{k+1}'\cdots x_n']$$
$$\times T[xx_1'\cdots x_k']. \quad (6.3)$$

In the presence of Fermion fields this equation is modified as follows:

(1) When x is the argument of a Fermion operator, we get

$$\sum_{\text{Comb}} i^n(-1)^k(-1)^{n'(k'+1)}(-1)^P\tilde T[x_{k+1}'\cdots x_n']$$
$$\times T[xx_1'\cdots x_k'], \quad (6.4)$$

where n' is the number of Fermion operators in $x_1'\cdots x_n'$, and k' the number in $x_1'\cdots x_n'$. P denotes the permutation $(x_1\cdots x_n)\to(x_1'\cdots x_n')$, and $(-1)^P$ is equal to $+1$ or -1 according to whether P is even or odd.

(2) When x is the argument of a Boson operator, then we have

$$\sum_{\text{Comb}} i^n(-1)^k(-1)^{k'(n'+1)}(-1)^P\tilde{T}[x_{k+1}'\cdots x_n']\times T[xx_1'\cdots x_k']. \quad (6.5)$$

These relationships will be used in later calculations.

Next we have to give the asymptotic conditions for the electron and electromagnetic fields. First one has to be careful about the order of subscripts designating a state vector. When a_i and a_j denote the incoming or outgoing electrons, the Pauli principle requires the relation

$$|a_1\cdots a_i\cdots a_j\cdots a_m,\pm\rangle = -|a_1\cdots a_j\cdots a_i\cdots a_m,\pm\rangle. \quad (6.6)$$

Also we have to define the state $|\alpha/a,\pm\rangle$ by

$$a_1\cdots a_m/a,\pm\rangle=\sum_{i=1}^{m}(-1)^{i-m}\delta(aa_i)\times|a_1\cdots a_{i-1}a_{i+1}\cdots a_m,\pm\rangle. \quad (6.7)$$

The asymptotic conditions are then given by

$$\text{(1a)}\quad \langle\beta a,+|R[:]|\alpha,+\rangle-\langle\beta,+|R[:]|\alpha/a,+\rangle = \int(dz)\langle a|A_\mu(z)|0\rangle\times\Box_z\langle\beta,+|R[:A_\mu(z)]|\alpha,+\rangle, \quad (6.8)$$

and its complex conjugate equation. $|a\rangle$ denotes a one photon state.

$$\text{(2a)}\quad \mp\langle\beta a,+|R[:]|\alpha,+\rangle+\langle\beta,+|R[:]|\alpha/a,+\rangle = \int(dz)\langle a|\bar{\psi}(z)|0\rangle\times D_z\langle\beta,+|R[:\psi(z)]|\alpha,+\rangle, \quad (6.9a)$$

$$\text{(2b)}\quad \langle\beta,+|R[:]|\alpha a,+\rangle\mp\langle\beta/a,+|R[:]|\alpha,+\rangle = \int(dz)\langle 0|\psi(z)|a\rangle\times\tilde{D}_z\langle\beta,+|R[:\bar{\psi}(z)]|\alpha,+\rangle, \quad (6.9b)$$

where $|a\rangle$ denotes a one electron state, and one has to take a $-$ or $+$ sign depending on whether the number of suppressed Fermion operators in the R-product is even or odd. D's are defined by

$$D_z=\gamma_\mu(\partial/\partial z_\mu)+m=\gamma\partial_z+m,$$

and

$$\tilde{D}_z=\gamma_\mu{}^T(\partial/\partial z_\mu)-m=\gamma^T\partial_z-m.$$

By applying the operation of charge conjugation to (2a) and (2b) one finds the asymptotic conditions for the positron.

$$\text{(3a)}\quad \mp\langle\beta a,+|R[:]|\alpha,+\rangle+\langle\beta,+|R[:]|\alpha/a,+\rangle = \int(dz)\langle a|\psi(z)|0\rangle\times\tilde{D}_z\langle\beta,+|R[:\bar{\psi}(z)]|\alpha,+\rangle, \quad (6.10a)$$

$$\text{(3b)}\quad \langle\beta,+|R[:]|\alpha a,+\rangle\mp\langle\beta/a,+|R[:]|\alpha,+\rangle = \int(dz)\langle 0|\bar{\psi}(z)|a\rangle\times D_z\langle\beta,+|R[:\psi(z)]|\alpha,+\rangle, \quad (6.10b)$$

where $|a\rangle$ denotes a one positron state.

In practical calculations we need contraction functions, which are defined and given by[23]

$$\sum_{\text{one electron}}\langle 0|\psi_\alpha(x)|a\rangle\langle a|\bar{\psi}_\beta(y)|0\rangle = -iS_{\alpha\beta}^{(+)}(x-y)=-i(\gamma\partial_x-m)_{\alpha\beta}\Delta^{(+)}(x-y),$$

$$\sum_{\text{one positron}}\langle 0|\bar{\psi}_\alpha(x)|a\rangle\langle a|\psi_\beta(y)|0\rangle = -i\tilde{S}_{\alpha\beta}^{(+)}(x-y)=-i(\gamma^T\partial_x+m)_{\alpha\beta}\Delta^{(+)}(x-y), \quad (6.11)$$

$$\sum_{\text{one photon}}\langle 0|A_\mu(x)|a\rangle\langle a|A_\nu(y)|0\rangle = i(\delta_{\mu\nu}+2M\partial^2/\partial x_\mu\partial x_\mu)D^{(+)}(x-y).$$

It is convenient to introduce the following functions:

$$S_R(x)=(\gamma\partial-m)\Delta_R(x),\quad \tilde{S}_R(x)=(\gamma^T\partial+m)\Delta_R(x), \quad (6.12)$$

where

$$\Delta_R(x)=-\theta(x_0)\Delta(x),\quad \text{and}\quad K_x\Delta_R(x)=-\delta(x). \quad (6.13)$$

The corresponding D functions may be obtained from Δ's by putting the rest mass equal to zero.

With these preliminaries we shall solve the r equations in quantum electrodynamics.

Vertex Function

In developing perturbation theory it is of fundamental importance to determine the lowest order vertex function, since it corresponds to the determination of the interaction Hamiltonian. We have to solve the r equation to determine $\langle R[A_\mu(x):\psi(y)\bar{\psi}(z)]\rangle_0$.

The arguments given in Sec. III render the equation

$$\Box_x D_y\tilde{D}_z\langle R[A_\mu(x):\psi(y)\bar{\psi}(z)]\rangle_0 - \Box_x D_y\tilde{D}_z\langle R[\psi(y):A_\mu(x)\bar{\psi}(z)]\rangle_0=0 \quad (6.14)$$

in the first order approximation. The retarded property

[23] This equation requires a special choice of the gauge. M is a gauge dependent constant.

of the R product assures us of the following solution:

$$\Box_x D_y \tilde{D}_z \langle R[A_\mu(x):\psi(y)\bar{\psi}(z)]\rangle_0 = \mathcal{O}_\mu \delta(x-y)\delta(x-z), \quad (6.15)$$

where $\mathcal{O}_\mu$ is a vector formed of γ matrices and differential operators. In order to determine $\mathcal{O}_\mu$ more precisely we shall refer to the W-T equations. First, from the relation

$$R[A:\psi\bar{\psi}] = -T[A\psi\bar{\psi}] - \tilde{T}[\psi\bar{\psi}]A - \bar{\psi}T[A\psi] + \psi T[A\bar{\psi}], \quad (6.16)$$

which follows from (6.4), we get with the aid of the W-T equation

$$\Box_x \frac{\partial}{\partial x_\mu} \langle R[A_\mu(x):\psi_\alpha(y)\bar{\psi}_\beta(z)]\rangle_0 = -e[\delta(x-y)\theta(y-z) + \delta(x-z)\theta(z-y)] \times \langle\{\psi_\alpha(y), \bar{\psi}_\beta(z)\}\rangle_0. \quad (6.17)$$

To the first order in e we can approximate this expression by

$$-ie[\delta(x-y)S_R(y-z)_{\alpha\beta} + \delta(x-z)\tilde{S}_R(z-y)_{\beta\alpha}].$$

Hence we get in the e^1 approximation the equation

$$\Box_x \frac{\partial}{\partial x_\mu} D_y \tilde{D}_z \langle R[A_\mu(x):\psi(y)\bar{\psi}(z)]\rangle_0 = -ie\gamma\partial_x \delta(x-y)\delta(x-z). \quad (6.18)$$

Comparison of Eqs. (6.15) and (6.18) yields the relation

$$-ie\gamma_\mu \frac{\partial}{\partial x_\mu}\delta(x-y)\delta(x-z) = \mathcal{O}_\mu \frac{\partial}{\partial x_\mu}\delta(x-y)\delta(x-z). \quad (6.19)$$

The general solution of this equation is given by

$$\mathcal{O}_\mu = -ie\gamma_\mu + iP(\partial)\sigma_{\mu\nu}(\partial/\partial x_\nu), \quad (6.20)$$

where $P(\partial)$ denotes an invariant polynomial of differential operators as appeared in Sec. III. The first term in (6.20) represents the Dirac type interaction and the second term gives the Pauli type interaction or the derivatives thereof, i.e., if we pick up the constant term from $P(\partial)$ we are left with a term of the form $i\lambda\sigma_{\mu\nu} \times (\partial/\partial x_\nu)$ which is known by the name of Pauli interaction. Anyhow the solution (6.20) gives gauge invariant interactions as was predicted.

In the following calculations we retain only the first term in (6.20) and put

$$\mathcal{O}_\mu = -ie\gamma_\mu, \quad (6.21)$$

for as we shall see in the next calculation the other terms give rise to divergent results corresponding to the fact that they are unrenormalizable. If we put (6.21) into (6.15) we find

$$\langle R[A_\mu(x):\psi_\alpha(y)\bar{\psi}_\beta(z)]\rangle_0 = ie\int (du) D_R(x-u)[S_R(u-y)\gamma_\mu S_R(u-z)]_{\alpha\beta}, \quad (6.22)$$

and similarly

$$\langle R[A_\mu(x):\bar{\psi}_\alpha(y)\psi_\beta(z)]\rangle_0 = -ie\int (du) D_R(x-u) \times [S_R{}^T(u-y)\gamma_\mu{}^T S_R{}^T(u-z)]_{\alpha\beta}. \quad (6.23)$$

Photon Propagator

A typical example to illustrate our method is the calculation of the photon propagator $\langle R[A_\mu(x): A_\nu(y)]\rangle_0$. Although this problem was discussed previously[12] it seems to be worth while to reproduce the argument here in connection with the W-T equations.

The W-T equation for the photon propagator provides us with

$$\Box_x \frac{\partial}{\partial x_\mu}\langle R[A_\mu(x): A_\nu(y)]\rangle_0 = \frac{\partial}{\partial x_\nu}\delta(x-y). \quad (6.24)$$

Combining this equation with the Källén-Lehmann representation we find

$$\langle R[A_\mu(x): A_\nu(y)]\rangle_0 = -\delta_{\mu\nu}D_R(x-y) + \mathcal{G}_{\mu\nu}(x-y) \quad (6.25)$$

in the gauge used to write down Eq. (6.11), and $\mathcal{G}_{\mu\nu}$ is given by

$$\mathcal{G}_{\mu\nu}(x) = \int d\kappa^2 \sigma(\kappa^2) \times \frac{1}{(2\pi)^4}\int_C (dk) e^{ikx} \times \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2+\kappa^2}, \quad (6.26)$$

where the contour of integration C over k_0 should be chosen infinitesimally above the real axis. Thus our problem is reduced to the determination of $\sigma(\kappa^2)$.

The integral equation for the photon propagator, which is correct to the order e^2, is given by

$$\langle R[A_\mu(x): A_\nu(y)]\rangle_0 - \langle R[A_\nu(y): A_\mu(x)]\rangle_0 + i[\sum_{\text{one photon}} \langle 0|A_\mu(x)|\gamma\rangle\langle\gamma|A_\nu(y)|0\rangle + \sum_{\text{one pair}} \langle 0|A_\mu(x)|a\tilde{b},+\rangle\langle a\tilde{b},+|A_\nu(y)|0\rangle - (x \rightleftarrows y, \mu \rightleftarrows \nu)] = 0. \quad (6.27)$$

Retaining only terms of the order e^2, we get

$$\mathcal{G}_{\mu\nu}^{(2)}(x-y)-\mathcal{G}_{\nu\mu}^{(2)}(y-x)-2M^{(2)}(\partial^2/\partial x_\mu\partial x_\nu)$$
$$\times D(x-y)+\int(du)(dv)D_{u1}\tilde{D}_{u2}$$
$$\times\langle R[A_\mu(x):\psi(u_1)\bar{\psi}(u_2)]\rangle_0\tilde{S}^{(+)}(u_1-v_1)$$
$$\times S^{(+)}(u_2-v_2)\tilde{D}_{v1}D_{v2}$$
$$\times\langle R[A_\nu(y):\bar{\psi}(v_1)\psi(v_2)]\rangle_0-(x\rightleftarrows y)=0. \quad (6.28)$$

Inserting the expressions (6.22) and (6.23) one arrives at

$$\mathcal{G}_{\mu\nu}^{(2)}(x-y)-\mathcal{G}_{\mu\nu}^{(2)}(y-x)-2M^{(2)}(\partial^2/\partial x_\mu\partial x_\nu)D(x-y)$$
$$=-i\int(du)(dv)D_R(x-u)D_R(y-v)Q_{\mu\nu}(u,v), \quad (6.29)$$

where $Q_{\mu\nu}$ is given by

$$Q_{\mu\nu}(u,v)$$
$$=e^2Sp[\gamma_\mu\cdot S^{(+)}(u-v)\cdot\gamma_\nu\cdot\tilde{S}^{(+)}(u-v)^T]-(u\rightleftarrows v)$$
$$=-\frac{e^2}{(2\pi)^6}\int_{4m^2}^{\infty}d\kappa^2\frac{\pi}{3\kappa}(\kappa^2-4m^2)^{\frac{1}{2}}\left(2+\frac{4m^2}{\kappa^2}\right)$$
$$\times\int(dk)e^{ik(u-v)}(\delta_{\mu\nu}k^2-k_\mu k_\nu)\epsilon(k_0)\delta(k^2+\kappa^2). \quad (6.30)$$

Inserting Eq. (6.26) into Eq. (6.29) one finds

$$\sigma^{(2)}(\kappa^2)=-\frac{e^2}{3(2\pi)^2}(\kappa^2-4m^2)^{\frac{1}{2}}\frac{\kappa^2+2m^2}{\kappa^5}\theta(\kappa^2-4m^2), \quad (6.31)\dagger$$

and thus we obtained an unambiguous gauge invariant photon propagator to the order e^2. This result agrees with the calculation in the conventional renormalization theory.

In the above calculation we chose $\mathcal{O}_\mu=-ie\gamma_\mu$, but we could take $\mathcal{O}_\mu=-ie\gamma_\mu+i\lambda\sigma_{\mu\nu}(\partial/\partial x_\nu)$ by including the Pauli term. Then, however, $\sigma^{(2)}(\kappa^2)$ would tend to a constant for $\kappa^2\to\infty$ and the κ^2 integral in the Källén-Lehmann representation of $\mathcal{G}_{\mu\nu}(p)$ would diverge logarithmically. For this reason (6.21) is the only solution we can employ as far as perturbation theory is concerned, and all Green's functions in quantum electrodynamics are uniquely determined in perturbation theory.

† *Note added in proof.*—Eq. (6.31) gives the correct solution of (6.29) provided that $M^{(2)}$ is given by

$$2M^{(2)}=-\int_0^\infty d\kappa^2\frac{\sigma^{(2)}(\kappa^2)}{\kappa^2}.$$

The author thanks Prof. G. Källén for suggesting him the use of this special gauge.

Compton Scattering

We have shown that our method is more convenient than the Feynman-Dyson theory to calculate the photon propagator, since our theory leads to an unambiguous gauge invariant result. It is, however, not the case for the calculation of simple matrix elements such as the lowest order Compton scattering. Nevertheless we think it instructive to show how to solve this problem in our scheme.

The matrix element that we want to know is given by

$$\langle p_f|R[A_\mu(x):A_\nu(y)]|p_i\rangle. \quad (6.32)$$

The r equation for this matrix element is easily written down

$$\langle p_f|R[A_\mu(x):A_\nu(y)]|p_i\rangle-\langle p_f|R[A_\nu(y):A_\mu(x)]|p_i\rangle$$
$$+i\sum_\alpha[\langle p_f|A_\mu(x)|\alpha,+\rangle\langle\alpha,+|A_\nu(y)|p_i\rangle$$
$$-(x\rightleftarrows y,\mu\rightleftarrows\nu)]=0. \quad (6.33)$$

In the e^2 approximation we can readily evaluate the following quantity:

$$i\sum_\alpha\langle p_f|\Box_x A_\mu(x)|\alpha,+\rangle\langle\alpha,+|\Box_y A_\nu(y)|p_i\rangle$$
$$=i\sum_p\langle p_f|\Box_x A_\mu(x)|p\rangle\langle p|\Box_y A_\nu(y)|p_i\rangle$$
$$+i\sum_{\tilde{p}}\langle p_f|\Box_x A_\mu(x)|p_f p_i\tilde{p},+\rangle$$
$$\times\langle p_f p_i\tilde{p},+|\Box_y A_\nu(y)|p_i\rangle$$
$$=i\sum_p\langle p_f|\Box_x A_\mu(x)|p\rangle\langle p|\Box_y A_\nu(y)|p_i\rangle$$
$$-i\sum_{\tilde{p}}\langle 0|\Box_x A_\mu(x)|p_i\tilde{p},+\rangle$$
$$\times\langle p_f\tilde{p},+|\Box_y A_\nu(y)|0\rangle,$$

where use has been made of the asymptotic condition. We already know in the e^1 approximation all the matrix elements occurring in the above expression, and we are ultimately left with the following result:

$$\Box_x\Box_y\langle p_f|R[A_\mu(x):A_\nu(y)]|p_i\rangle$$
$$=-e^2\langle p_f|\bar{\psi}(x)|0\rangle\gamma_\mu[(\gamma\partial_x-m)\Delta_R(x-y)]$$
$$\times\gamma_\nu\langle 0|\psi(y)|p_i\rangle-e^2\langle p_f|\bar{\psi}(y)|0\rangle$$
$$\times\gamma_\nu[(\gamma\partial_x+m)\Delta_R(x-y)]\gamma_\mu\langle 0|\psi(x)|p_i\rangle. \quad (6.34)$$

This solution is certainly compatible with the W-T equation

$$\Box_x\frac{\partial}{\partial x_\mu}R[A_\mu(x):A_\nu(y)]=\frac{\partial}{\partial x_\nu}\delta(x-y).$$

The general solution of (6.33) is obtained by adding terms involving $\delta(x-y)$ and derivatives thereof. They are not taken as solutions for the same reason as we have excluded the Pauli term, namely, they lead to divergences in the higher order calculations.

We have shown in this section how one can apply our method to quantum electrodynamics and learned how it works especially when the conventional theory gives ambiguous results. Although we used the r equations here, the τ equations are more convenient for most purposes and simpler to formulate, and we hope to discuss various topics on the τ equations at another opportunity.