Energy Gap in Nuclear Matter*

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The magnitude of the energy gap in nuclear matter associated with a highly correlated ground state of of the type believed to be important in the theory of superconductivity has been evaluated theoretically. The integral equation of Cooper, Mills, and Sessler is linearized and transformed into a form suitable for numerical solution. The energy gap, calculated by using an appropriate single-particle potential and the Gammel-Thaler two-body potential, is found to be a very strong function of the density of nuclear matter, and of the effective mass at the Fermi surface. It is concluded that the magnitude of the energy gap for nuclear matter should not be compared directly with experimental values for finite nuclei, although the results suggest that if the theory is extended to apply to finite nuclei it probably would be in agreement with experiment.

I. INTRODUCTION

IT was pointed out by Bohr, Mottelson, and Pines[1] that the spectra of even-even nuclei have an "energy gap" and that the low-lying states of atomic nuclei might have a collective character which could be described in a manner similar to that proposed by Bardeen, Cooper, and Schrieffer[2] in their theory of superconductivity.

If realistic nucleon-nucleon forces are to be used, it is difficult to make a quantitative theoretical evaluation of this suggestion for *finite* nuclei. However, Cooper, Mills, and Sessler[3] have shown how the BCS method could be applied to *infinite* systems of strongly interacting fermions and it has been shown[4] that nuclear matter does in fact have a gap in its energy spectrum.

The object of the present paper is to determine the magnitude of the energy gap for nuclear matter with a view to gaining some insight into the possible results of extending the theory to finite systems.

In the next section, the basic equations of the theory are described, linearized, and rearranged into a form suitable for numerical solution. The results are presented in the form of three figures and discussed in the final section.

II. DESCRIPTION OF THE CALCULATION

1. Basic Equations

The equations from which the energy gap may be calculated are presented in CMS and may be obtained from equations briefly derived in another paper[5] of this issue by letting $\beta \to \infty$.

An infinite system of Fermions has an energy spectrum with a gap, if there is a nontrivial solution of the equation

$$\chi(\mathbf{r}) = -\tfrac{1}{2}\int G(\mathbf{r}-\mathbf{r}')v(\mathbf{r}')\chi(\mathbf{r}')d\mathbf{r}', \tag{1}$$

where

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{[\epsilon^2(k)+\mathfrak{F}^2(\mathbf{k})]^{\frac{1}{2}}}, \tag{2}$$

and

$$\epsilon(k) = e(k) - e(k_F), \tag{3}$$

where $e(k)$ is the effective single-particle energy in nuclear matter. The internucleon pottential is $v(\mathbf{r})$, and $\chi(\mathbf{r})$ the correlation function. The function $\mathfrak{F}(\mathbf{k})$ is the Fourier transform of $\mathfrak{F}(\mathbf{r}) = -v(\mathbf{r})\chi(\mathbf{r})$, so Eq. (1) is nonlinear.

In the numerical computations to be described here, the linear approximation[3] to Eqs. (1) and (2) will be used. It is assumed that $\mathfrak{F}(\mathbf{k})$ is a slowly varying function of $\mathbf{k}$ which may be replaced by a constant ϵ_0 which becomes the eigenvalue of Eq. (1).

2. Rearrangement of the Equations

The simplest way to obtain accurate numerical solutions of Eqs. (1) and (2) is to transform the equations by the method described in ES. The only differences are (i) we take the limit $\beta \to \infty$ and retain ϵ_0 in the equations instead of β_c, and (ii) there are S-state solutions for nuclear matter and they give rise to a true energy gap.

Using Dirac notation, we define

$$|\varphi\rangle = |k_F\rangle - G_0 v|\varphi\rangle, \tag{4}$$

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[1] A. Bohr, B. R. Mottelson, and D. Pines, Phys. Rev. **110**, 936 (1958).

[2] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957), hereafter called BCS.

[3] L. N. Cooper, R. L. Mills, and A. M. Sessler, Phys. Rev. **114**, 1377 (1959), hereafter called CMS.

[4] R. L. Mills, A. M. Sessler, S. A. Moszkowski, and D. G. Shankland, Phys. Rev. Letters **3**, 381 (1959).

[5] V. J. Emery and A. M. Sessler, Phys. Rev. **119**, 43 (1960), hereafter called ES.

where

$$G_0=-\frac{1}{\pi}\int_0^\infty dk\frac{[|k\rangle\langle k|-|k_F\rangle\langle k_F|]}{|\epsilon(k)|}, \tag{5}$$

and

$$\langle r|k_F\rangle=\sin k_F r. \tag{6}$$

Also let

$$L_1(\epsilon_0)=-\frac{1}{\pi}\int_0^\infty \frac{dk}{[\epsilon^2(k)+\epsilon_0^2]^{\frac{1}{2}}}. \tag{7}$$

Then, by the method described in Sec. III. 1 of ES, it is found that

$$L_1(\epsilon_0)=-1/\langle\varphi|v|k_F\rangle. \tag{8}$$

The procedure used is to solve the coordinate space representative of Eq. (4) for $\langle r|\varphi\rangle$ and to use the solution to evaluate $\langle k_F|v|\phi\rangle$. This value is put into Eq. (8) from which ϵ_0 may be determined. The energy gap ΔE is $2\epsilon_0$ (see reference 2).

The integrand on the right-hand side of Eq. (7) is sharply peaked a the Fermi surface so that it is sufficient to use the effective mass approximation

$$e(k)=\hbar^2k^2/2m^* \tag{9}$$

(m^* being the effective mass at the Fermi surface) and the method of evaluating $L(\beta)$ described by ES to find

$$\Delta E=\frac{8\hbar^2k_F^2}{m^*}\exp\left(\frac{\pi}{2m^*}\frac{\hbar^2k_F}{\langle\varphi|v|k_F\rangle}\right). \tag{10}$$

3. Numerical Procedures

The internucleon potential to be used has a hard core of radius c and an external attraction, for which we make the approximation of ignoring $v(r)\phi(r)$ for $r<c[\phi(r)=\langle r|\phi\rangle]$ while retaining the δ-function contribution on the core.[6] Then, for S states, the coordinate representative of Eq. (4) becomes

$$\varphi(r)=\left[\sin k_F r-\sin k_F c\frac{G_0(r,c)}{G_0(c,c)}\right]-\int_0^\infty dr'\left[G_0(r,r')-\frac{G_0(r,c)G_0(r',c)}{G_0(c,c)}\right]\times v(r')\varphi(r')dr', \tag{11}$$

where

$$G_0(r,r')=\frac{1}{\pi}\int_0^\infty dk\frac{[\sin kr\,\sin kr'-\sin k_F r\,\sin k_F r']}{|e(k)-e(k_F)|}. \tag{12}$$

Now

$$\langle\varphi|v|k_F\rangle=\frac{\sin^2k_Fc}{G_0(c,c)}+\int_0^\infty dr\left[\sin k_Fr-\sin k_Fc\frac{G_0(c,r)}{G_0(c,c)}\right]v(r)\varphi(r). \tag{13}$$

[6] H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) **A238**, 551 (1956).

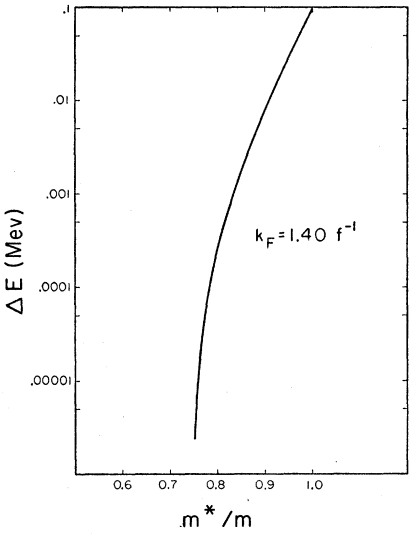

FIG. 1. The energy gap in nuclear matter as a function of the effective mass at the Fermi surface, for "normal density" corresponding to $k_F=1.40$ f^{-1}. The binding energy per nucleon has been taken to be 15.5 Mev in the evaluation of $V(k)$ [Eq. (16) ff.]

Equation (11) has now to be solved numerically and $\langle\phi|v|k_F\rangle$ calculated from Eq. (13). The numerical procedures used were described in Appendix I of ES. Since the singular part of $v(r)\phi(r)$ has been removed explicitly, the number of Gauss points in the quadrature could be reduced without essential loss of accuracy.

4. Potentials

The best available two-nucleon potential has been obtained phenomenologically by Gammel and Thaler.[7] In S states there is probably an energy gap for triplet as well as singlet states. However, the singlet gap is certainly larger and, in view of the complications introduced by tensor forces, the triplet gap has not been calculated here.

The Gammel-Thaler form of $v(r)$ is

$$\begin{aligned} v(r)&=\infty, && \text{for } r<c,\\ &=-{}^1V_s e^{-\mu r}/\mu r, && \text{for } r>c. \end{aligned} \tag{14}$$

For the ^1S state, $^1V_s=425.5$ Mev, $\mu=1.45$ f^{-1} and $c=0.4$ f.

For the single-particle energies, we have taken the form[4]

$$e(k)=\hbar^2/2m+V(k), \tag{15}$$

where

$$V(k)=-V_0/(1+\alpha k^2). \tag{16}$$

V_0 and α are determined by the requirements that $e(k_F)$ shall equal the energy per nucleon in nuclear matter (-15.5 Mev at normal density) and that the effective mass m^* at the Fermi surface, defined by

$$\left.\frac{de(k)}{dk}\right|_{k_F}=\frac{\hbar^2k_F}{m^*} \tag{17}$$

[7] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957).

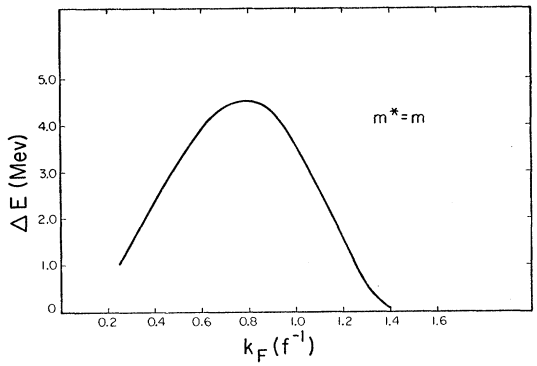

FIG. 2. The energy gap in nuclear matter as a function of density for $m^*=m$, i.e., ignoring the dispersive effects of nuclear matter.

shall have a specified value. The results are to be quoted as a function of m^*/m.

III. RESULTS AND DISCUSSION

The results of the calculations are presented in Figs. 1, 2, and 3 which show the variation of the energy gap with m^* at two densities (normal density, $k_F=1.40$ f^{-1} and a reduced density, $k_F=1.0$ f^{-1}), and the variation with density for $m^*=m$ [i.e., $V(k)=0$].

We have also studied the sensitivity of the quantity

$$T=\hbar^2 k_F/m\langle\varphi|v|k_F\rangle \tag{18}$$

by examining three cases:

(1) *Schrödinger equation*, i.e., $|e(k)-e(k_F)|$ is replaced by $(\hbar^2/2m)(k^2-k_F^2)$, for which $T=-5.06$.

(2) $V(k)=0$, for which $T=-4.02$.

(3) V(k) corresponding to $m^*=0.67$, for which $T=-6.43$.

Thus, T is not very sensitive to the exclusion principle or the dispersive properties of nuclear matter, so that the strong dependence of ΔE on these effects is localized into $L_1(\epsilon_0)$ [see Eq. (8)].

For rough estimates of the magnitude of ΔE, then, it is sufficient to replace T in Eqs. (10) and (18) by $(-\cot\delta)$,[8] where δ is the Schrödinger equation phase shift for relative momentum $\hbar k_F$. Equation (10) becomes

$$\Delta E=\frac{8\hbar^2 k_F^2}{m^*}\exp\left(-\frac{\pi}{2}\frac{m}{m^*}\cot\delta\right), \tag{19}$$

and this approximation improves as k_F decreases.[8]

[8] V. J. Emery, Nuclear Phys. (to be published).

Equation (19) now shows why the energy gap is such a rapidly varying function of density and effective mass and why it is so small at normal density. The ^1S phase shift, δ, is almost zero at a momentum corresponding to normal density, and increases rapidly as the momentum decreases (i.e., as the density decreases). Since both $\cot\delta$ and m^* appear in the exponent in Eq. (19), a small change in either quantity produces a large change in ΔE.

Consequently, it is difficult to draw any quantitative conclusions about the energy gap in finite nuclei from a study of nuclear matter, since at least one very important effect is the variation in density of finite nuclei associated with the nuclear surface.

However, the order of magnitude of the energy gap in finite nuclei might be estimated by averaging the nuclear matter gap over densities from zero to normal density (k_F varying from zero to 1.4 f^{-1}). It can be seen from Figs. 1, 2, and 3 that such an estimate would be consistant with the gaps observed in heavy nuclei[1] so that a theory of finite nuclei analogous to the one presented here might well be expected to be in agreement with experiment.

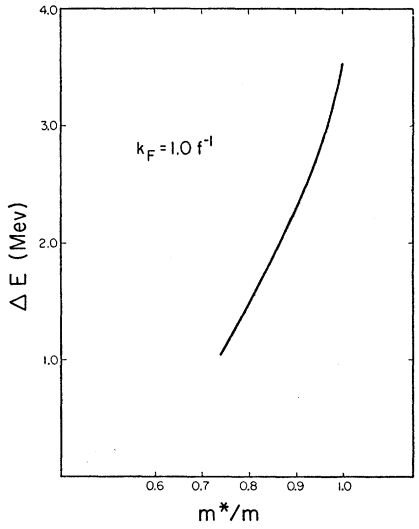

FIG. 3. The energy gap in nuclear matter as a function of the effective mass at the Fermi surface, for a reduced density corresponding to $k_f=1.0$ f^{-1}. The binding energy per nucleon has been taken to be 10.0 Mev in the evaluation of $V(k)$ [Eq. (16) ff.].

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