Perturbation Theory Applied to the Nuclear Many-Body Problem*

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Perturbation theory is applied to infinite nuclear matter at the observed density for a well-behaved two-body potential, containing a tensor force. We find that a tensor force can contribute as much as 10 Mev/particle to the binding energy in second order. Perturbation theory is then modified to include the pseudopotential treatment of an infinite repulsive core. We give a detailed derivation of the DeDominicis-Martin and Huang-Yang result for a pure repulsive core. We obtain an expansion jointly in powers of the strength of the attractive potential, and in the range of the core. We find the second-order contributions to the binding energy for several potentials combining an infinite repulsive core with an attractive potential. For each case considered, the second-order terms are large (absolute value about 20 Mev/particle).

I. INTRODUCTION

MANY different workers have applied perturbation theory to obtain the binding energy per particle E/A in infinite nuclear matter for given assumed two-body nuclear forces. In most of their work the emphasis has been on computing E/A as a function of the nuclear density (or corresponding Fermi wave number k_f). The computed equilibrium density and binding energy at that density may then be compared with experimental values.

Perturbation theory calculations made over 20 years ago by Euler[1] showed that, at the equilibrium density, the terms of second order were only several Mev per particle. Swiatecki[2] and Bethe[3] have emphasized that Euler's small ratio of second-order to first-order terms (value about 60 Mev per particle) shows that perturbation theory is converging rapidly for this problem; and that this convergence is due to limitations on possible intermediate states imposed by the Pauli principle. Euler used a well-behaved two-body potential, of exchange character such that nuclear matter will not collapse. The perturbation is the sum of all two-body forces. For the low orders considered here, the Rayleigh-Schrödinger form of perturbation theory applies without any special modifications for the many-body problem.[4] (In higher orders special care must be taken concerning "linked clusters.")

Euler's calculations for a Gaussian two-body potential were repeated by Huby[5] for a Yukawa potential, and by Thouless[6] for an exponential potential. These three potentials of different shapes are for *central* forces with parameters in agreement with the effective range treatment[3] of low-energy properties of the two-particle system. (This implies modernizing Euler's results by inserting modern values for the range of the Gaussian potential.) The numerical results for the second order term are reasonably small and close to each other for the three different shapes of two-body potential. Thus, perturbation theory seems to converge reasonably rapidly for well-behaved central forces.

Several lines of argument, however, have led to the conclusion that the two-body forces are not well-behaved, but instead become extremely large and repulsive at very short distances. Even if this repulsive potential is finite, it is so large that perturbation theory converges poorly.[2] The repulsive potential is usually treated as an infinite repulsive core, of radius c, which gives very similar results to a finite repulsion of somewhat larger range.[7]

While an infinite repulsive core cannot be treated by standard perturbation theory, which involves an expansion in powers of the strength of the potential V_0, an analogous expansion may be made by introduction of a pseudopotential, or closely related t matrix. The expansion parameter becomes k_fc, the product of the Fermi wave number and the core radius. (The infinite repulsive core produces a phase shift of value kc in two-body scattering. This phase shift is averaged over different values of k, the wave number for relative motion. Since the average of k is of order of magnitude k_f, the expansion parameter is expressed as k_fc.) Huang and Yang[8] developed this expansion both for an isolated system of two particles, and for a many-body system with a pure repulsive core force. Independent work by DeDominicis and Martin[9] carried the expansion to the $(k_fc)^3$ term.

Gomes, Walecka, and Weisskopf[10] have considered the problem of a combination of an infinite repulsive core and an attractive potential. Their method of numerical computation involves combining the usual perturbation theory expansion up to terms of the first

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[8] K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957).
[9] C. DeDominicis and P. C. Martin, Phys. Rev. **105**, 1419 (1957).
[10] L. C. Gomes, J. D. Walecka, and V. F. Weisskopf, Ann. Phys. **3**, 241 (1958).

power of the attractive potential V_0 with the De-Dominicis-Martin expansion up to terms of the third power in the parameter $k_f c$. Our paper may be regarded as an extension of their work, in that we spell out the mathematical framework for combining the two expansions, and in that we calculate two terms not included by Gomes et al., i.e., Euler's term from second order perturbation theory for the attractive potential, and a term involving first powers in both the attractive potential and in $k_f c$. (We do not agree with certain features of the Gomes-Walecka-Weisskopf calculation, i.e., their use of the effective mass approximation in calculation of the t matrix; their assumption that the triplet central potential equals the singlet potential; and their choice of core radius for the singlet potential. However, while these details seriously affect the calculations of the binding energy E, they are of less importance for the question considered here of rapidity of convergence of the expansions.)

The mathematical framework for combining the two different expansions was suggested by R. E. Peierls. It is one of several procedures discussed by Tobocman.[11] As emphasized by Tobocman, many different approximation procedures are possible in attacking a many-body problem. In any particular case we must weigh the ease of the procedure against the accuracy achieved. In this paper we adopt a particularly simple procedure, and we study its rapidity of convergence for the problem of infinite nuclear matter. Our purpose is to try to find a relatively simple method that yet converges reasonably rapidly. Such a method, if it exists, would have two main advantages as compared, for example, to Brueckner's method.[12] First, our method would be easier to understand. In particular, it is clear just what approximations are made, and whether the approximations are valid. Second, our method *might* prove directly applicable to other nuclear problems, such as that of the binding energy of a finite nucleus, or the affects of two-body clusters on nuclear reactions.[13] (Brueckner's work on the finite nucleus[14] demands further approximations beyond those made in his treatment of infinite nuclear matter.)

Further, the present treatment whether it converges or not will suggest which terms give a large contribution to the binding energy, and which terms should, therefore, be emphasized in a variation treatment of the nuclear problem.[15] For instance, the important contribution that we find due to tensor forces suggests that a variational wave function should include an adjustable parameter to allow for the corresponding deviation from isotropy of the two-body wave function.

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Our present objective is to examine whether perturbation theory, with certain modifications, converges reasonably well for calculations of the binding energy per particle in nuclear matter. The criteria for convergence are not exact: they amount to looking at the first- and second-order terms and guessing whether the third-order terms will be small enough so that they can either be neglected or estimated by crude approximations. Until we have confidence in the method we shall not try to find the binding energy for comparison with experiment. To emphasize this point of view, we shall not even calculate a first-order term, but merely estimate its order of magnitude as an indication of the rate of convergence of our series. We shall evaluate the second-order terms at the observed nuclear density. The actual two-body forces would presumably give a density of nuclear matter equal to the observed value, so that our expansion should converge reasonably well at the observed density if the expansion is to be of use in evaluation of the equilibrium properties of nuclear matter. If the expansion converges well, it would be straightforward though tedious to extend the second-order (and first-order) calculations as a function of density, and thus determine the equilibrium density and equilibrium properties.[10]

The method used for solving the many-body problem should be chosen in a manner appropriate to the two-body potential assumed. (i) For certain two-body potentials, such as the well-behaved central potentials treated by Euler,[1] ordinary perturbation theory converges well. (ii) For certain potentials including a repulsive core, the modified perturbation theory treated in this paper would converge well. (iii) If our modified perturbation theory converges poorly for a certain potential, it might be possible to modify the theory further to obtain a rapidly converging series for this problem. (iv) None of the three techniques above may converge rapidly; but Brueckner's technique may give a rapidly converging series, i.e., the incoherent scattering terms may be small. (v) For certain potentials, none of the four methods above would give a rapidly converging series. Detailed numerical works must be done for any given two-body potential to determine which category is appropriate.

In the next section we apply perturbation theory to the nuclear many-body problem for the case of a well-behaved two-body potential containing a tensor force, that fits the low-energy measurements. Many examples of such triplet potentials are given by Biedenharn, Blatt, and Kalos[16]: the central and tensor potentials may be chosen to have various shapes; and also for given shapes we may combine a weak central potential (strength parameter $s_c=0.6$) with a strong tensor potential. Alternatively, we may combine a strong central potential ($s_c=1.0$) with a weak tensor potential. We find that the numerical value of the second-order

[16] L. C. Biedenharn, J. M. Blatt, and M. H. Kalos, Nuclear Phys. **6**, 359 (1958).

term is insensitive to the potential shape chosen, but is sensitive to the strengths assumed for the central and tensor potentials. Specifically, the second-order term is almost twice as large for the strong tensor potential, than for the weak tensor potential. (See Tables III and IV). The convergence of perturbation theory is poor for the strong tensor potential, and fair (i.e., not much worse than in Euler's work) for the weak tensor potential.

In Sec. III we develop a modified perturbation theory for treating the infinite repulsive core alone, and we apply this modified perturbation theory to find terms up to the second power of $k_f c$. The $k_f c$ term was derived by Lenz.[17] The $(k_f c)^2$ term result was stated by Huang and Yang[8] and DeDominicis and Martin[9] without a detailed derivation. We shall supply the details.

In Sec. IV we develop our modified perturbation theory for a two-body potential containing both a repulsive core and an attractive potential. In Sec. V we consider two-body potentials taken from Biedenharn, Blatt, and Kalos,[16] and Gammel-Thaler.[18] Our results (Tables V, VI, VII) show that in all cases considered the second order terms are large, i.e., our modified perturbation theory converges poorly for the nuclear many-body problem using an attractive potential combined with a repulsive core of radius about 0.4 fermi.

In the final section we discuss our numerical results, and consider briefly other related methods that are expected to converge better than the methods treated in this paper: Moszkowski's[19] and Lomon's[20] different separation of the potential into two parts, and the use of a velocity-dependent two-body potential[21] instead of a repulsive core.

II. ORDINARY PERTURBATION THEORY

Euler,[1] Huby,[5] and Thouless[6] have applied ordinary perturbation theory to the nuclear many-body problem, for well-behaved central two-body potentials. We shall use a notation similar to Euler's and give only a brief development of the formula for the second order term. The details are given in his paper.

We separate the Hamiltonian H into one part H_0 involving the kinetic energy T alone, and a perturbation v consisting of the sum of well-behaved static two-body potentials. (Of course, many-body forces may be of importance, but are not considered in this paper. Use of velocity-dependent two-body force is mentioned in the last section.)

$$H=T+v. \tag{1}$$

Here and throughout this paper certain summations are implied, but are not stated explicitly: T denotes the sum of the kinetic energies of the individual nucleons, and v denotes a sum of the interaction of all pairs.

Following the standard Rayleigh-Schrödinger perturbation theory, the energy E is given, up to second order by

$$E=\langle\phi_0|T|\phi_0\rangle+\langle\phi_0|v|\phi_0\rangle+\sum_n \frac{\langle\phi_0|v|\phi_0\rangle\langle\phi_n|v|\phi_0\rangle}{E_0-E_n}. \tag{2}$$

Here ϕ_0 and ϕ_n are eigenfunctions for the unperturbed Hamiltonian T for the ground and excited states, respectively, of infinite nuclear matter. Thus, ϕ_0 is a Slater determinant of plane-wave functions for the individual nucleons, with wave numbers from zero up to the value k_f at the Fermi surface. The excited state wave function ϕ_n has a single pair excited, each member of the pair being outside the Fermi sphere. The energies in the denominator E_0 and E_n are the eigenvalues for the unperturbed Hamiltonian; i.e., they are the kinetic energies.

The value of the first term $\langle\phi_0|T|\phi_0\rangle$ is well known[3] for our assumed perfect Fermi gas:

$$(1/A)\langle\phi_0|T|\phi_0\rangle=\tfrac{3}{5}T_f=\tfrac{3}{5}(\hbar^2k_f^2/2M)=24\text{ Mev}. \tag{3}$$

Here T_f is the kinetic energy, of value 40 Mev, at the Fermi surface. Throughout this paper we use $k_f=1.40$ fermi^{-1}, corresponding to the nuclear radius parameter $r_0=1.52/k_f=1.09$ fermis. [For infinite nuclear matter, r_0 is defined by the relation volume per particle $=(4/3)\pi r_0^3$.]

The first order term $\langle\phi_0|v|\phi_0\rangle$ has been calculated by many authors[1,5,6,10] for different assumed potential shapes and exchange mixtures v. At our value for k_f, the literature gives

$$(1/A)\langle\phi_0|v|\phi_0\rangle\cong-60\text{ Mev}. \tag{4}$$

Note that the value of this term differs appreciably among different potentials v that fit the low-energy two-body data. For instance, a weak central force (strength parameter $s_c=0.6$) gives only about 60% of the contribution of a strong central force ($s_c=1.0$). Note that a tensor force gives no contribution to $\langle\phi_0|v|\phi_0\rangle$, since a tensor force averages to zero on angular integration.

In evaluation of these equations some workers[2,3,6] have made a choice of the unperturbed Hamiltonian different from our choice $H_0=T=\hbar^2k^2/2M$. Their argument is that in general a self-consistent effective potential felt by a single particle is velocity-dependent.[22] We might then use $H_0'=T+V(k)$ to find the energies E_0 and E_n in the denominator of the second order term in Eq. (2). In practice, this choice of H_0' complicates the already difficult sum over intermediate states n. These workers then make the effective mass approxi-

[17] W. Lenz, Z. Physik **56**, 778 (1929).
[18] J. Gammel and R. M. Thaler, Phys. Rev. **107**, 1337 (1957). Also see reference 12.
[19] S. A. Moszkowski, Bull. Am. Phys. Soc. **4**, 256 (1959); and University of California at Los Angeles UCLA Technical Report No. 2, O.O.R. (unpublished).
[20] E. L. Lomon, Bull. Am. Phys. Soc. **4**, 48 (1959).
[21] M. Moshinsky, Phys. Rev. **106**, 117 (1957); **109**, 933 (1958).
[22] J. VanVleck, Phys. Rev. **48**, 367 (1935).

mation by selecting the term in $V(k)$ that is quadratic in k, and combining this term with the kinetic energy T to give $H_0''=\hbar^2k^2/2M^*$. Several arguments[10,12,22] show that the effective mass M^* has a value in the range from $0.5M$ to $0.8M$, where M is the true nucleon mass. Since M^* occurs in the denominator in determining the energy denominator of the second order term, the second order term is proportional to the value chosen for M^*. (Note that the eigenfunctions ϕ_0 and ϕ_n are unchanged when we adopt an effective mass.) Thus, the convergence of perturbation theory is improved by the use of an effective mass that is smaller than the true nucleon mass.

It may seem surprising that the value of the second order term should depend on the arbitrary choice of H_0 as H_0'. (One is tempted to make perturbation theory converge extremely rapidly merely by choosing M^* as extremely small.) However, the dependence of perturbation theory results on the choice of H_0 disappears if we go to the third order. Feenberg[23] has shown that the combination $[E^{(2)}]^2/(E^{(2)}-E^{(3)})$ is invariant to a "scale transformation" such as the introduction of an effective mass. [Here $E^{(2)}$ and $E^{(3)}$ are the second and third order contributions to the energy, respectively.]

Since computational problems cause us to stop at the second order perturbation term in this paper, our result does depend on the arbitrary separation into H_0 and v. Neither our use of the true mass M, nor the alternative use of the effective mass M^*, can be regarded as giving an exact answer.

In evaluation of the second-order term of Eq. (2), we neglect the exchange term[1,6] which is much more difficult to evaluate than the ordinary term. For the exchange mixtures in common use such as the Serber force, the exchange term is somewhat smaller than the ordinary term, and has the same sign. Our use of the true mass M, and our neglect of the exchange term are two errors in opposite directions and of comparable magnitude. Our result for the second order term should be in error by an amount less than either of these neglected effects: our calculation may be accurate to 30%. In any case, it should be accurate enough to draw conclusions as to the rapidity of convergence of perturbation theory as applied to the nuclear many-body problem.

Euler[1] has manipulated the second-order term of Eq. (2) into a form convenient for use with different perturbations. The summation over intermediate states n is rather involved: we are concerned with a pair of nucleons, wave-numbers $\mathbf{k}_1$ and $\mathbf{k}_2$ in the ground state, which scatter to an intermediate state n with wave-numbers $\mathbf{k}_1'=\mathbf{k}_1+\mathbf{q}$ and $\mathbf{k}_2'-\mathbf{q}$. (The sum of the momenta is conserved for our present problem of infinite nuclear matter; though it need not be for the finite nucleus problem. All wave numbers are given in units of k_f.) We want the expression

$$E^{(2)}=\sum_n \frac{\langle\phi_0|v|\phi_0\rangle\langle\phi_n|v|\phi_0\rangle}{E_0-E_n}, \tag{5}$$

summed over-all $\mathbf{k}_1$ and $\mathbf{k}_2$ inside the Fermi sphere, and summed over-all n for $\mathbf{k}_1'$ and $\mathbf{k}_2'$ outside this sphere. For the ordinary term (though not for the exchange term), the numerator is only a function of $\mathbf{q}$. The integral of the energy denominator [proportional to $\mathbf{q}\cdot(\mathbf{q}+\mathbf{k}_1-\mathbf{k}_2)$] over $\mathbf{k}_1$ and $\mathbf{k}_2$ in the region satisfying the Pauli principle for $\mathbf{k}_1'$ and $\mathbf{k}_2'$ is done for fixed $\mathbf{q}$. Euler's Appendix gives the result

$$(4\pi^2/15)P(q)/q=\int\int_{u=\frac{1}{2}q}\frac{d^3\mathbf{k}_1 d^3\mathbf{k}_2}{\mathbf{q}\cdot(\mathbf{q}+\mathbf{k}_1-\mathbf{k}_2)}. \tag{6}$$

$$\begin{aligned}0\leq u\leq 1,\ P(u)=P_1(u)=&(4+7\tfrac{1}{2}u-5u^3+1\tfrac{1}{2}u^5)\\&\times\ln(1+u)+29u^2-3u^4\\&+(4-7\tfrac{1}{2}u+5u^3-1\tfrac{1}{2}u^5)\\&\times\ln(1-u)-40u^2\ln 2,\end{aligned} \tag{7}$$

$$\begin{aligned}1\leq u,\ P(u)=P_2(u)=&(4-20u^2-20u^3+4u^5)\\&\times\ln(u+1)+4u^3+22u\\&+(-4+20u^2-20u^3+4u^5)\\&\times\ln(u-1)+(40u^3-8u^5)\ln u.\end{aligned} \tag{8}$$

Numerical values of the function $P(u)$ are given in Table I.

TABLE I. Numerical values of Euler's function $P(u)$.[a]

u	$P(u)$	u	$P(u)$
0	0	1.0	3.82
0.1	0.12	1.2	3.03
0.2	0.48	1.4	2.52
0.3	1.02	1.6	2.18
0.4	1.71	1.8	1.92
0.5	2.47	2.0	1.71
0.6	3.19	3.0	1.11
0.7	3.78	4.0	0.84
0.8	4.15	5.0	0.67
0.9	4.14	6.0	0.56

[a] The function $P(u)$ is given in analytic form in Eqs. (7) and (8) of the text, and in reference 1.

Euler gives convenient and accurate power series expansions:

$$P_1(u)\cong 40(1-\ln 2)u^2-10u^4+(\tfrac{4}{3}u^6)+0.21u^8,$$
$$P_2(u)\cong(10/3)u^{-1}+\tfrac{1}{3}u^{-3}+0.16u^{-5}.$$

The matrix element $\langle\phi_0|v|\phi_n\rangle$ in the numerator of Eq. (5) gives us the Fourier transform of the potential.

[23] E. Feenberg, Ann. Phys. 3, 292 (1958).

For a central potential $-V_0v(r/\beta)$

$$\langle\phi_0|v|\phi_n\rangle=F(q)=-V_0\int e^{ik_f\mathbf{q}\cdot\mathbf{r}}v(r/\beta)d^3r$$

$$=-4\pi V_0\beta^3f(w). \quad (9)$$

Here

$$f(w)=\int_0^\infty(\sin wy/wy)v(y)y^2dy, \quad (10)$$

where $y=r/\beta$ and $w=2ux$ with $x=k_f\beta$. Euler obtains

$$E^{(2)}/A=-(1/2^65\pi^4)(k_f^6/T_f)c_2\int P(u)u[F(u)]^2du. \quad (11)$$

The coefficient c_2 depends on the exchange character of the two-body potential:

$$c_2=3(^1V_+)^2+3(^3V_+)^2+9(^3V_-)^2+1(^1V_-)^2. \quad (12)$$

For the exchange term in second order, Euler gives a coefficient c_2':

$$c_2'=3(^1V_+)^2+3(^3V_+)^2-9(^3V_-)^2-1(^1V_-)^2. \quad (12')$$

The superscript gives the spin multiplicity, and the plus or minus subscript gives the parity of the wave function. (Thus, $^1V_+$ is the value of V_0 for a spin singlet spatially symmetric state.) We assume a Serber force, different (both for V_0 and β) in the spin triplet (even, central) and in the spin singlet states. For each even state $c_2=3(V_0)^2$ with a different V_0 (and β) value depending on spin. Equation (11) becomes

$$E^{(2)}/A=-(3/20\pi^2)x^6(V_0^2/T_f)I \quad (13)$$

$$I=\int P(u)u[f(2ux)]^2du. \quad (14)$$

For the particular case of a Gaussian $v=-V_0\times\exp(-r^2/\beta^2)$, Euler has evaluated the integral I and obtains

$$E^{(2)}/A=-(3/2^75\pi)x^2(V_0^2/T_f)g(x), \quad (15)$$

where $x=k_f\beta$. Table II extends Euler's numerical results for $g(x)$. Euler also gives its asymptotic form for $x\ll1$ and $x\gg1$. [Since Euler's $g(x)=(2^5x^4/\pi)I$, Eqs. (13) and (15) are in agreement.]

TABLE II. Euler's second-order energy.[a]

x	$g(x)$ from Euler
0.1	0.051
0.2	0.107
0.5	0.328
1.0	0.707
1.5	1.034
2.0	1.307
4.5	2.079
8.0	2.479

[a] The second-order energy for a Gaussian potential (without core) is proportional to $g(x)$: See Eq. (15) of the text, and reference 1.

We have reviewed Euler's treatment so that we can extend it to the case of a well-behaved tensor potential $-V_0v_t(r/\beta)S_{12}$. We replace Eq. (9) by[21]

$$\langle\phi_0|v|\phi_n\rangle=F_t(q)=-V_0\int e^{ik_f\mathbf{q}\cdot\mathbf{r}}v_t(r/\beta)S_{12}(\hat{r})d^3r$$

$$=-4\pi V_0\beta^3S_{12}(\hat{q})f_t(w), \quad (16)$$

$$f_t(w)=\int_0^\infty j_2(wy)v_t(y)y^2dy. \quad (17)$$

The spherical Bessel function

$$j_2(z)=-\sin z/z-3\cos z/z^2+3\sin z/z^3. \quad (18)$$

Here $S_{12}(\hat{r})$ is the tensor operator in ordinary space

$$S_{12}(\hat{r})=3(\boldsymbol{\sigma}_1\cdot\hat{r})(\boldsymbol{\sigma}_2\cdot\hat{r})-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2,$$

and

$$S_{12}(\hat{q})=3(\boldsymbol{\sigma}_1\cdot\hat{q})(\boldsymbol{\sigma}_2\cdot\hat{q})-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2,$$

where $\hat{r}$ and $\hat{q}$ are unit vectors. For any given direction $\hat{q}$, the integrals over d^3k_1, d^3k_2, and sum over n can be done as above by Euler giving $P(u)$. In integrating $F_t(\mathbf{q})$ over the direction for $\hat{q}$ we obtain, since for a spin-triplet state,

$$[S_{12}(\hat{q})]^2=8-2S_{12}(\hat{q}),$$

$$\langle4\pi[S_{12}(\hat{q})]^2\rangle_{\rm av}=4\pi(8),$$

or eight times the usual 4π. We now modify Eqs. (13) and (14) using this factor of 8, and using $f_t(2ux)$. For a tensor force of Serber character $c_2=3$ and

$$E^{(2)}/A=-(6/5\pi^2)x^6(V_0^2/T_f)I_t, \quad (20)$$

$$I_t=\int P(u)u[f_t(2ux)]^2du. \quad (21)$$

The factor of 8 in Eq. (19) causes a strong tensor force to give a surprisingly large contribution to the second order energy.

The second-order term $E^{(2)}$ is calculated from Eqs. (15) and (20) for various potentials fitting the low-energy data. Biedenharn, Blatt, and Kalos[16] have provided us with a great variety of triplet potentials. We have the freedom in choice of shape for central and tensor potentials. After the shapes are assumed, there are still four parameters available (central strength and range, and tensor strength and range) to fit only three experimental data (deuteron binding energy and quadrupole moment, and the effective range). Biedenharn et al. use this extra freedom by choosing the central force as weak (strength parameter $s_c=0.6$), with corresponding strong tensor force. Alternatively, the central force can be assumed strong ($s_c=1.0$) with a weak tensor force. We choose a Gaussian shape for the triplet force, and either Gaussian, exponential, or Yukawa shape for

the tensor force. The details of the potentials used are given in the Appendix.

The singlet potential is chosen as a Gaussian shape, with strength parameter 0.95 and effective range 2.5 f.

The central triplet, tensor, and singlet potentials are each assumed to have Serber exchange character. Other cases can easily be treated using Eq. (12).

Certain details involved in using Eq. (20) for a tensor force are also given in the Appendix. These calculations were performed on a desk computer, with an accuracy of several percent. The results are given in Tables III and IV for weak central and strong central potentials, respectively. Table III shows that the strong tensor force associated with the weak central potential gives about -10 Mev contribution to the nuclear binding energy per particle; the value is insensitive to the tensor shape. On the other hand, the weak tensor potential used in Table IV gives about -5 Mev contribution, while the contribution of the central potential remains small, the total second order being about -6.5 Mev.

If a pure central triplet force were used,[1] then the strength parameter s_c should be about 1.4, and Table IV

TABLE III. Second-order terms for weak central potential (without core).[a]

Singlet Gaussian		-0.7	
Weak central triplet Gaussian		-0.4	
	Gaussian	Exponential	Yukawa
Strong tensor	-11.1	-10.0	-9.7
Total second order	-12.2	-11.1	-10.8

[a] The second-order contribution to the binding energy per particle is given in Mev, computed from Eqs. (15) and (20). The weak central triplet potential has strength parameters $s_c=0.6$; details concerning the potentials used are given in the Appendix.

gives a second-order term of only about -3.3 Mev per particle. Thus, an increase in the strength of the tensor force increases the magnitude of the second order term appreciably from -3.3 for no tensor force to -6.5 for a weak tensor force, to about -11 Mev per particle for a strong tensor force. We do not have at present a precise criterion for the rapidity of convergence of perturbation theory applied to the nuclear many-body problem: a reliable estimate of third-order terms would certainly be very useful. We guess that perturbation theory converges well enough to stop at first order if there were no tensor force; the convergence may be good enough for a weak tensor force, but convergence seems poor for the strong tensor force used in Table III.

III. PURE REPULSIVE CORE

The problem of an infinite repulsive core of radius c has been solved independently by DeDominicis and Martin,[9] and by Huang and Yang[8]; but neither authors give the details of their solution. [*Note added in proof.*—See A. A. Abrikosov and I. M. Khalatinkow, Appendix A1, Repts. Progr. Phys. **22**, 329 (1959).] We shall give

TABLE IV. Second-order terms for strong central potential (without core).[a]

Singlet Gaussian		-0.7	
Strong central triplet Gaussian		-1.3	
	Gaussian	Exponential	Yukawa
Weak tensor	-5.1	-4.5	-4.4
Total second order	-6.8	-6.5	-6.4

[a] The second-order contribution to the binding energy per particle is given in Mev, computed from Eqs. (15) and (20). The strong central triplet potential has strength parameter $s_c=1.0$; details concerning the potentials used are given in the Appendix.

the detailed solution for the binding energy calculated through terms of order $(k_f c)^2$ as part of our more general problem of combining a repulsive core with an attractive potential.

We want to solve for the energy with Hamiltonian $H=T+v_r$:

$$E=\langle\phi_0|T|\psi_0\rangle+\langle\phi_0|v_r|\psi_0\rangle. \tag{22}$$

Here ϕ_0 is as before the eigenfunction for kinetic energy T for infinite nuclear matter. ψ_0 is the eigenfunction for H, and is zero whenever any two nucleons are separated by a distance less than c. Of course, perturbation theory fails for an infinite v_r. This infinite v_r must be replaced by a pseudopotential, or t matrix defined so that

$$\langle\phi_0|v_r|\psi_0\rangle=\langle\phi_0|t|\phi_0\rangle. \tag{23}$$

The pseudopotential satisfies[12,24] the operator equation

$$t=v_r+v_rGt. \tag{24}$$

Here G is the modified Green's function for the kinetic energy operator T, taking account of the Pauli principle in summing over intermediate states n:

$$G(r,r')=\sum_n \frac{\phi_n(r)\phi_n^*(r')}{E_0-E_n}. \tag{25}$$

Bethe and Goldstone[25] show how to solve Eq. (24) in an expansion of powers of $(k_f c)$. This method was followed by DeDominicis and Martin.[9] Instead, we shall follow a method suggested by R. E. Peierls, similar to that used by Tobocman.[11]

Equation (24) can be (formally) solved by iteration for t giving

$$t=v_r+v_rGv_r+v_rGv_rGv_r+\cdots. \tag{26}$$

Also Eq. (24) can be solved first for v_r giving

$$v_r=t(1+Gt)^{-1}=t-tGt+tGtGt+\cdots. \tag{27}$$

If we substitute v_r from Eq. (27) into Eq. (26) we are pursuing a circular argument, and recover the result $t=t$. However, we can write equations similar to (24) and (27) which will be useful in finding an expression

[24] R. J. Eden, *Nuclear Reactions*, edited by P. M. Endt and M. Demeur (Interscience Publishers, Inc., New York, 1959), Vol. 1.
[25] H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) **A238**, 551 (1957).

for t.

$$t_{nx}=v_r+v_rG_{nx}t_{nx}, \tag{28}$$

and

$$v_r=t_{nx}-t_{nx}G_{nx}t_{nx}+t_{nx}G_{nx}t_{nx}G_{nx}t_{nx}+\cdots. \tag{29}$$

Here t_{nx} is the pseudopotential for an isolated system ("nx" standing for "no exclusion principle"). G_{nx} is the Green's function for an isolated system: that is, in Eq. (25) *all* states n are included in the sum. If we substitute v_r from Eq. (29) into Eq. (26) we find an expression for t in terms of t_{nx}:

$$t=t_{nx}+t_{nx}(G-G_{nx})t_{nx}+\text{higher order terms}. \tag{30}$$

This equation is useful since t_{nx} is relatively simple. For a pure repulsive core, t_{nx} is given by

$$t_{nx}=(4\pi\hbar^2c/M)\delta(\mathbf{r}), \tag{31}$$

where we have neglected terms of order c^3 or higher. M is the true nucleon mass.

The energy of the system of fermions is, from Eq. (30),

$$\begin{aligned}E&=\langle\phi_0|T|\phi_0\rangle+\langle\phi_0|t|\phi_0\rangle\\&=\langle\phi_0|T|\phi_0\rangle+\langle\phi_0|t_{nx}|\phi_0\rangle\\&\quad+\langle\phi_0|t_{nx}(G-G_{nx})t_{nx}|\phi_0\rangle\\&\qquad+\text{higher order terms}.\end{aligned} \tag{32}$$

Neglected terms are of order $(k_fc)^3$ or higher. The term $\langle\phi_0|T|\phi_0\rangle$ gives a contribution of $(3/5)T_f$ to E/A, as in Sec. II. The term $\langle\phi_0|t_{nx}|\phi_0\rangle$ gives the Lenz[17] term $(2/\pi)(k_fc)T_f$. The last term $\langle\phi_0|t_{nx}(G-G_{nx})t_{nx}|\phi_0\rangle$ is calculated following Euler's work,[1] outlined in Sec. II. Using Eq. (25),

$$\langle\phi_0|t_{nx}Gt_{nx}|\phi_0\rangle=\sum_n\frac{\langle\phi_0|t_{nx}|\phi_n\rangle\langle\phi_n|t_{nx}|\phi_0\rangle}{E_0-E_n}. \tag{33}$$

This is the usual second-order perturbation theory result of Eq. (5), where now we treat t_{nx} of Eq. (31) as the perturbation. As above, the matrix element from state 0 to state n gives the Fourier transform of the perturbation [See Eq. (9)]:

$$\begin{aligned}\langle\phi_0|t_{nx}|\phi_n\rangle\sim F_c(q)&=\int e^{ik_f\mathbf{q}\cdot\mathbf{r}}(4\pi\hbar^2c/M)\delta(\mathbf{r})d^3r\\&=4\pi\hbar^2c/M.\end{aligned} \tag{34}$$

The sum in Eq. (33) over states n satisfying the Pauli principle again gives us Euler's $P(u)$ defined in Eq. (6), and given explicitly in Eqs. (7) and (8).

For the term G_{nx} we sum over *all* intermediate states and take the Cauchy principal value, obtaining a new function $P_{nx}(u)$. If $u>1$, all intermediate states will automatically satisfy the Pauli principle, so $P_{nx}(u)=P_2(u)$. For $u<1$, we find that $P_{nx}(u)\neq P_1(u)$; but $P_{nx}(u)$ again equals $P_2(u)$ with the one change that $\ln(u-1)$ in Eq. (8) is rewritten as $\ln|1-u|$.

The term $(G-G_{nx})$ then gives us $[P(u)-P_{nx}(u)]$ which is different from zero only for $u<1$. We keep track of various factors, following Euler. [See our Eq. (11)]. In particular, if we assume a Wigner-character repulsive core and include both ordinary (c_2) and exchange (c_2') terms we have $c_2+c_2'=12$ [See Eqs. (12) and (12′)].

We find

$$\begin{aligned}&(1/A)\langle\phi_0|t_{nx}(G-G_{nx})t_{nx}|\phi_0\rangle\\&\quad=-(1/2^65\pi^4)(k_f{}^6/T_f)(12)(4\pi\hbar^2c/M)^2\\&\qquad\times\int_0^1[P_1(u)-P_{nx}(u)]udu\\&\qquad=(12/35\pi^2)(11-2\ln2)(k_fc)^2T_f.\end{aligned} \tag{35}$$

We have used from Eqs. (7) and (8)

$$\int^1[P_1(u)-P_{nx}(u)]udu=-(11-2\ln2)/7. \tag{36}$$

Combining our first three terms we have the Huang-Yang and DeDominicis-Martin result:

$$\begin{aligned}E/A=T_f[&\tfrac{3}{5}+(2/\pi)(k_fc)\\&+(12/35\pi^2)(11-2\ln2)(k_fc)^2+\cdots].\end{aligned} \tag{37}$$

IV. MODIFIED PERTURBATION THEORY FOR A REALISTIC POTENTIAL

Instead of treating separately a well-behaved attractive potential v or an infinite repulsive core v_r, we shall consider the problem of a potential $w=v+v_r$. This amounts to combining the results of Secs. II and III and adding one extra term for an interference effect due to combining an attractive potential with a repulsive core. Our purpose here is to develop the modified perturbation theory appropriate for this problem in a systematic manner, and to include all terms up to the second power in the strength of the attractive potential, the range of the repulsive core, or a combination of these two parameters.

We write the Hamiltonian $H=T+w$ where $w=v+v_r$. (Note that we define $v=0$ for $r<c$.) As in Eq. (24) we express the t matrix in terms of the perturbation, w, and solve by iteration:

$$t=w+wGt=w+wGw+\cdots. \tag{38}$$

Again G is the Green's function for operator T, taking account of the Pauli principle. We use Eq. (29) for v_r to express w in terms of t_{nx} for a pure repulsive core:

$$w=v+v_r=v+t_{nx}-t_{nx}G_{nx}t_{nx}+\cdots. \tag{39}$$

Substituting Eq. (39) for w in Eq. (38) we obtain, including terms through second order:

$$\begin{aligned}t=&(v+t_{nx}-t_{nx}G_{nx}t_{nx}+\cdots)\\&+(v+t_{nx}+\cdots)G(v+t_{nx}+\cdots)\\&=v+t_{nx}+t_{nx}(G-G_{nx})t_{nx}+vGv+2t_{nx}Gv.\end{aligned} \tag{40}$$

The energy is given by Eq. (32):

$$E=\langle\phi_0|T|\phi_0\rangle+\langle\phi_0|v|\phi_0\rangle+\langle\phi_0|t_{nx}|\phi_0\rangle+\langle\phi_0|t_{nx}(G-G_{nx})t_{nx}|\phi_0\rangle+\langle\phi_0|vGv|\phi_0\rangle+2\langle\phi_0|t_{nx}Gv|\phi_0\rangle+\text{higher order terms.} \quad (41)$$

This expression gives the energy in a combined expansion in powers of t_{nx} (that is the core radius c) and in powers of the attractive potential v.

Gomes, Walecka, and Weisskopf[10] obtain an expansion similar to our Eq. (41); however, they include the DeDominicis-Martin term of third order in t_{nx}, and they omit the vGv and $2t_{nx}Gv$ terms of our equation.

V. APPLICATION OF MODIFIED PERTURBATION THEORY

We shall investigate the rapidity of convergence of our modified perturbation theory expansion Eq. (41) for several different choices of the two-nucleon potential with an infinite repulsive core.

The radius of the infinite core is usually chosen[18] as about 0.4 f, and we have adopted this value here. (We also use 0.4315 f for the potentials taken from Biedenharn's tabulations.[16] The radius of the core is sensitive to the shape chosen for the attractive potential v. If a square-well[26] is chosen for v, then the ^1S phase shifts are best fitted by a core of radius about 0.2 f, or only half the value usually chosen. We do not know which core radius is most appropriate to the Gaussian shape treated below.)

Expressions for the first five terms in Eq. (41) are given in Sec. II or III. We shall develop the expression for the last term, $2\langle\phi_0|t_{nx}Gv|\phi_0\rangle$, and then use these equations to find the various second-order terms, for different choices of the two-body potential. From Eq. (11),

$$2\langle\phi_0|t_{nx}Gv|\phi_0\rangle=2\sum_n\frac{\langle\phi_0|t_{nx}|\phi_n\rangle\langle\phi_n|v|\phi_0\rangle}{E_0-E_n}=2(-1/2^65\pi^4)(k_f{}^6/T_f)c_2\int P(u)uF_c(u)F(u)du. \quad (42)$$

We use Eqs. (9) and (34) for the Fourier transforms $F(q)$ and $F_c(q)$, respectively. As in our treatment of the pure repulsive core, the exchange term integral gives the same result as the ordinary integral. This follows from the argument that $F_c(q)$ is independent of q so an exchange integral

$$\int\int\frac{F_c(q)F(q')}{\mathbf{q}\cdot\mathbf{q}'}d^3k_1d^3k_2=F_c\int\int\frac{F(q')}{\mathbf{q}\cdot\mathbf{q}'}d^3k_1d^3k_2\sim F_cF(q')P(q'). \quad (43)$$

[26] L. T. Bennett, B. L. Eskut, and J. S. Levinger (private communication).

Here, as in Eq. (6), $\mathbf{q}=\mathbf{k}_1'-\mathbf{k}_1$=momentum change of the first particle; and $\mathbf{q}'=\mathbf{k}_2'-\mathbf{k}_1$ which equals $-(\mathbf{q}+\mathbf{k}_1-\mathbf{k}_2)$ from momentum conservation. To obtain the last expression on the right of (42), we integrate over $d^3k_1d^3k_2$ for fixed q' in the region such that the Pauli principle is obeyed for $\mathbf{k}_1'$ and $\mathbf{k}_2'$, obtaining $P(q')$. We now integrate the last term over q', and obtain the same result for the exchange term,

$$\int F(q')P(q')q'dq'$$

as for the ordinary term

$$\int F(q)P(q)qdq.$$

We shall include both ordinary and exchange terms and use $c_2+c_2'=6$ for a given spin state (triplet or singlet) for a Serber character attractive potential v. Using the factors in Eq. (11) we have

$$(1/A)2\langle\phi_0|t_{nx}Gv|\phi_0\rangle=(6/5\pi^2)k_fcx^3V_0I_c \quad (44)$$

$$I_c=\int P(u)uf(2ux)du. \quad (45)$$

The function f is defined in Eq. (10). As above, $v=-V_0v(r/\beta)$, where now $v(r/\beta)$ is zero for $r<c$. Also, $x=k_f\beta$.

Note that as in the calculation of $\langle\phi_0|v|\phi_0\rangle$ only a central (singlet or triplet) v contributes in Eq. (44); a tensor potential averages to zero on angular integration.

The first four terms in Eq. (41) are calculated or estimated as similar for any attractive potential v. We choose $k_f=1.4$ f^{-1}, and $c=0.4$ (or 0.43) f. Equation (3) gives us the Fermi energy $T_f=40$ Mev, and

$$(1/A)\langle\phi_0|T|\phi_0\rangle=24 \text{ Mev}.$$

The second term $\langle\phi_0|v|\phi_0\rangle$ is again about -60 Mev per particle and depends on the separation between central (triplet) and tensor force. [See Eq. (4).]

The third term in Eq. (41) is given in Eq. (37):

$$(1/A)\langle\phi_0|t_{nx}|\phi_0\rangle=(2/\pi)(k_fc)T_f=14.3 \text{ Mev} \quad (46)$$

for $c=0.4$ f (use of $c=0.43$ f increases the value to 15.4 Mev).

The fourth term in Eq. (41) is given by Eq. (35):

$$(1/A)\langle\phi_0|t_{nx}(G-G_{nx})t_{nx}|\phi_0\rangle=T_f(12/35\pi^2)(11-2\ln 2)(k_fc)^2=4.1 \text{ Mev}, \quad (47)$$

for $c=0.4$ f. (Use of $c=0.43$ f gives a value of 4.8 Mev.) This second-order term includes both ordinary and exchange integrals, and uses the true nucleon mass.

The last two terms in Eq. (41) are evaluated using

TABLE V. Second-order terms for weak central force with core.[a]

	$(2/A)\langle\phi_0\|t_{nx}Gv\|\phi_0\rangle$	$(1/A)\langle\phi_0\|vGv\|\phi_0\rangle$
Singlet	+ 5	− 3
Central triplet	+10	− 4
Strong tensor	0	−16 (Gaussian) −16.6 (exponential)
Total	+15	−23

[a] See Eqs. (13), (20), and (44) for the calculations, and see the Appendix and reference 16 for the potentials used. The energies are given in Mev.

Eqs. (13), (20), and (44). [Since G is given by Eq. (25),

$$\langle\phi_0|vGv|\phi_0\rangle = \left\langle\phi_0\left|v\sum_n \frac{\phi_n(r)\phi_n^*(r')}{E_0-E_n}v\right|\phi_0\right\rangle$$

$$=\sum_n \frac{\langle\phi_0|v|\phi_n\rangle\langle\phi_n|v|\phi_0\rangle}{E_0-E_n}=E^{(2)},$$

where $E^{(2)}$ is the standard second-order term of Eq. (5).] Note that for $\langle\phi_0|vGv|\phi_0\rangle$ we calculate only the ordinary term; while for $2\langle\phi_0|t_{nx}Gv|\phi_0\rangle$ we include both ordinary and exchange terms.

We shall take our attractive potentials mostly from Biedenharn, Blatt, and Kalos[16] (their Table 2.3). From their work we estimate the singlet intrinsic range b_s=effective range $-2c=1.6$ f; and we choose the shape as shifted Gaussian. Their values for the central (triplet) and tensor forces are quoted in the Appendix. We shall use their cases of shifted Gaussian shape for the central triplet, combined with shifted Gaussian or exponential shape for the tensor force. As in Sec. II, we treat separately the case of weak central ($s_c=0.6$) combined with strong tensor, and strong central ($s_c=1.0$) combined with weak tensor.

Our results for the strong central potential are given in Table V, and for the weak central potential in Table VI.

We treat the Gammel-Thaler[18] potential in the same manner, obtaining the second-order terms given in Table VII. Note that we have not included their two-body spin-orbit force.

We see from Tables V, VI, and VII that the total second-order terms are large in each of the five cases treated. The $(2/A)\langle\phi_0|t_{nx}Gv|\phi_0\rangle$ term varies from 15 to 23 Mev, and the usual second-order term $(1/A)\langle\phi_0|vGv|\phi_0\rangle$ varies from −20 to −33 Mev. There appears to be some slight hope of convergence, since if we add these two terms together, and also use the result given in Eq. (47), we obtain −3 Mev, +8 Mev, and −8 Mev, respectively, for the three tables. These numerical results *cannot* be taken seriously in our present calculation, since our $\langle\phi_0|vGv|\phi_0\rangle$ term is for the ordinary term only, and the other two terms include the exchange term as well. Also, the three terms will likely be affected differently by use of a velocity-dependent "shell-model" potential $V(k)$ in the unperturbed Hamiltonian. Finally, the cancellation may work well for the second-order terms, but poorly for third-order terms.

TABLE VI. Second-order terms for strong central potential with core.[a]

	$(2/A)\langle\phi_0\|t_{nx}Gv\|\phi_0\rangle$	$(1/A)\langle\phi_0\|vGv\|\phi_0\rangle$
Singlet	+ 5	− 3
Central triplet	+18	−13
Weak tensor	0	− 5 (Gaussian) − 4 (exponential)
Total	+23	−20

[a] See Eqs. (13), (20), and (44) for the calculations, and see the Appendix and reference 16 for the potentials used. The energies are given in Mev.

VI. DISCUSSION

Tables V, VI, and VII show that various second-order terms for an attractive potential with an infinite repulsive core are of the order of 20 Mev per particle. We conclude that our present modification of perturbation theory gives a poorly converging series for calculation of the binding energy of infinite nuclear matter, at the empirical density, with the potentials chosen.

One reason for the large second order terms is that the use of a repulsive core necessitates use of a deeper and narrower attractive potential. Thus, the $\langle\phi_0|vGv|\phi_0\rangle$ terms in Tables V and VI, are much larger than corresponding terms in Tables III and IV.

It seems possible that certain changes either in the potential used or in the modified perturbation theory would substantially improve our convergence.

First, one might use a velocity-dependent two-body potential[21] instead of the static potential with repulsive core chosen in this paper. This *might* give second order terms similar to those in Tables III and IV for a static potential without core.

Second, the potential with core could be chosen with a smaller core radius,[26] for a different assumed shape of the attractive potential v. Use of a core radius of only 0.2 f would cause a substantially smaller absolute value for the $\langle\phi_0|t_{nx}Gv|\phi_0\rangle$ and $\langle\phi_0|vGv|\phi_0\rangle$ terms.

Third, one might follow the method[19,20] of a different separation between large and short range potentials. We can repeat the derivation of Eq. (41) for the binding energy starting with the perturbation $w=v_L+v_s$ instead of our previous $w=v+v_r$. v_s is a short-range potential including the core and part of the attractive potential outside the core. For $r<b$, $v_s=w$. v_L is the remaining large-range attractive potential; for $r>b$, $v_L=w$. The parameter b can be chosen[21] as 0.8 or 0.9 f, so that v_s gives zero phase shift for low-energy scattering of the isolated two-body system; or b can be taken as an adjustable parameter.

Equation (28) is now rewritten using v_s instead of the repulsive core v_r:

$$t_{nx}'=v_s+v_sG_{nx}t_{nx}'. \quad (48)$$

Then t_{nx}' and v_L are used in the derivation, to replace t_{nx} and v, respectively. We obtain instead of Eq. (41):

$$E=\langle\phi_0|T|\phi_0\rangle+\langle\phi_0|v_L|\phi_0\rangle+\langle\phi_0|t_{nx}'|\phi_0\rangle+\langle\phi_0|t_{nx}'(G-G_{nx})t_{nx}'|\phi_0\rangle+\langle\phi_0|v_LGv_L|\phi_0\rangle+2\langle\phi_0|t_{nx}'Gv_L|\phi_0\rangle+\text{higher order terms.} \quad (49)$$

This series must converge as well or better than Eq. (41) used above, since we can choose $b=c$ and obtain again our results in Tables V, VI, VII. Moszkowski[19] gives persuasive arguments that his Eq. (49) should give a rapidly converging series.

Of course, there is no assurance that any of these methods will be successful. The alternate method of Brueckner[12] is to find t from the analog of Eq. (24) for the complete perturbation w:

$$t=w+wGt. \quad (50)$$

The solution is particularly difficult since Brueckner and Gammel use the Green's function G for an unperturbed Hamiltonian H_0 including a velocity-dependent potential $V(k)$. They then need to find a self-consistent solution of Eq. (50) by an iterative procedure. As Brueckner has pointed out,[27] our second-order term in Table VII due to the Gammel-Thaler tensor potential is about three times the -6 Mev per particle that the tensor potential contributes in the Brueckner-Gammel treatment.

TABLE VII. Second-order terms for Gammel-Thaler potential.[a]

	$(2/A)\langle\phi_0\|t_{nx}Gv\|\phi_0\rangle$	$(1/A)\langle\phi_0\|vGv\|\phi_0\rangle$
Singlet	+10	− 7
Central triplet	+11	− 6
Tensor	0	−20
Total	+21	−33

[a] See Eqs. (13), (20), and (44) for the calculations, and see the Appendix for the potentials used. Note that the Gammel-Thaler spin-orbit term has not been included. The energies are given in Mev.

APPENDIX

The attractive two-body potentials used were taken from Biedenharn, Blatt, and Kalos,[16] and from Gammel and Thaler[18] (as quoted by Brueckner and Gammel,[12] Table III). We give the potentials without core in Table VIII, first the weak central potential and associated strong tensor potentials, and then the strong central potential and weak tensor potentials.

The integrals $f_t(w)$ needed in Eq. (17) were evaluated analytically for Gaussian, exponential, and Yukawa potentials, without core.

$$v_G(y)=\exp(-y^2)$$

gives

$$f_t^G(w)=\frac{3\pi}{2w^3}\operatorname{erf}(\tfrac{1}{2}w)-\frac{\sqrt{\pi}}{4}\left(\frac{6}{w^2}+1\right)\exp(-w^2/4),$$

where

$$\operatorname{erf}(t)=\frac{2}{\sqrt{\pi}}\int_0^t\exp(-x^2)dx, \quad (A1)$$

$$v_E(y)=e^{-y}$$

gives

$$f_t^E(w)=\frac{3}{w^3}\tan^{-1}w-\frac{3}{w^2}\frac{1}{w^2+1}-\frac{2}{(w^2+1)^2}, \quad (A2)$$

$$v_Y(y)=e^{-y}/y$$

gives

$$f_t^Y(w)=-\frac{3}{w^3}\tan^{-1}w+\frac{3}{w^2}-\frac{1}{1+w^2}. \quad (A3)$$

The integral I_t of Eq. (21) was evaluated numerically, using the different functions f_t above, and the function $P(u)$ from Table I. The resulting I_t depends on the value of $x=k_f\beta$. Values for x and I_t are given in Table VIII, and are used in Eq. (20) to give the second-order term $E^{(2)}/A$ of Tables III and IV.

The attractive potentials with core are defined to be zero inside the core radius. They are given in Table IX. The shifted Gaussian central potential has the form

$$v_G(y)=\exp[-(y-c')^2], \quad \text{where } c'=c/\beta.$$

TABLE VIII. Potentials used (without core).[a]

Potential	Reference	Strength parameter	Intrinsic range	Formula	x	$I_t(x)$
Singlet Gaussian	···	0.95	2.5	$-35\exp(-0.33r^2)$	2.44	···
Weak central triplet Gaussian	row 1	0.6	2.33	$-25\exp(-0.38r^2)$	2.27	···
Strong tensor Gaussian	row 1	1.199	2.59	$-[41\exp(-31r^2)]S_{12}$	2.52	0.0081
Strong tensor exponential	row 4	1.25	2.64	$-(134e^{-1.3r})S_{12}$	1.04	0.139
Strong tensor Yukawa	row 7	1.34	2.69	$-(57e^{-0.79r}/0.79r)S_{12}$	1.78	0.030
Strong central triplet Gaussian	row 3	1.0	1.80	$-71\exp(-0.64r^2)$	1.75	···
Weak tensor Gaussian	row 3	0.882	3.20	$-[20\exp(-0.20r^2)]S_{12}$	3.12	0.0045
Weak tensor exponential	row 6	0.891	3.44	$-(57e^{-1.03r})S_{12}$	1.36	0.0718
Weak tensor Yukawa	row 9	0.933	3.74	$-(21e^{-0.56r}/0.56r)S_{12}$	2.46	0.014

[a] The reference gives the row in Table 2.1, of Biedenharn et al., reference 16. The intrinsic range is given in fermis, and the formula for the potential in Mev. The parameter $x=k_f\beta$; and I_t is the integral of Eq. (21).

[27] K. A. Brueckner (private communication).

TABLE IX. Potentials used with core.[a]

Potential	Reference	Strength parameter	Intrinsic range	Formula	x	Integral
Singlet, shifted Gaussian	···	0.95	1.6	$-87\exp[-0.83(r-0.43)^2]$	1.54	$I=0.079$ $I_c=0.22$
Weak central triplet, shifted Gaussian	row 1	0.6	0.91	$-162\exp[-2.5(r-0.43)^2]$	0.89	$I=0.82$ $I_c=1.17$
Strong tensor, shifted Gaussian	row 1	0.849	2.07	$-\{45\exp[-0.48(r-0.43)^2]\}S_{12}$	2.03	$I_t=0.037$
Strong tensor, exponential	row 4	0.803	2.24	$-(121e^{-1.59(r-0.43)})S_{12}$	0.88	$I_t=0.20$
Strong central triplet, shifted Gaussian	row 3	1.0	0.88	$-\{290\exp[-2.7(r-0.43)^2]\}S_{12}$	0.86	$I=0.96$ $I_c=1.29$
Weak tensor, shifted Gaussian	row 3	0.599	2.96	$-\{16\exp[-0.24(r-0.43)^2]\}S_{12}$	2.88	$I_t=0.012$
Weak tensor, exponential	row 6	0.562	3.44	$-(36e^{-1.03(r-0.43)})S_{12}$	1.36	$I_t=0.072$
Gammel-Thaler, singlet	···	···	···	$-434e^{-1.45r}/1.45r$	0.97	$I=0.12$ $I_c=0.37$
Gammel-Thaler, central triplet	···	···	···	$-877e^{-2.09r}/2.09r$	0.67	$I=0.21$ $I_c=0.57$
Gammel-Thaler, tensor	···	···	···	$-(159e^{-1.045r}/1.045r)S_{12}$	1.34	$I_t=0.045$

[a] The reference for "row" gives the row number in Table 2.3 of Biedenharn et al., reference 16. The Gammel-Thaler potentials are from reference 12. The intrinsic range is given in fermis, and the formula for the potential in Mev. All attractive potentials are zero inside the core of radius of 0.43 FM for Biedenharn's potentials, and radius 0.40 FM for the Gammel-Thaler potentials. The parameter $x=k_f\beta$. The three integrals I, I_t, and I_c are given in Eqs. (14), (21), and (45), respectively.

Equation (10) gives

$$f^G(w)=(\sqrt{\pi}/4)[\cos wc'+2(c'/w)\sin\omega c']\exp(-\omega^2/4)+[(c'/w)\cos wc'-\tfrac{1}{2}\sin wc']\times\exp[-\omega^2/4]G(w/2)+\sin\omega c'/2w. \quad \text{(A4)}$$

The function[28]

$$G(x)=\int_0^x \exp(t^2)dt.$$

The tensor shifted Gaussian has the same radial dependence. The integral $f_t{}^G(w)$ was found by numerical integration for different values of w, and substituted numerically into the expression for I_t.

For f_t for the tensor exponential, with core, we used Eq. (A2) for the integral from zero to infinity, and made a small numerical correction (less than 4%) for the integral from zero to the core radius.

The Gammel-Thaler central Yukawa potential

$$v_Y(y)=e^{-y}/y,$$

gives the function

$$f^y(w)=-\frac{e^{-c'}}{w(1+w)^2}(w\cos\omega c'+\sin\omega c'). \quad \text{(A5)}$$

For calculation of f_t for the tensor Gammel-Thaler potential, we used Eq. (A3) for the integral from zero to infinity. Numerical corrections for the integral from zero to the core radius were about 1%.

The values of I, I_t, and I_c [Eqs. (14), (21), and (45)] calculated in this manner for the various attractive potentials with cores are tabulated in Table IX. These quantities are used in Eqs. (13), (20), and (44) to give the second order contributions to the energy quoted in Tables V, VI, and VIII.

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[28] E. Jahnke and F. Emde *Tables of Functions*, (Dover Publications, New York, 1945), 4th ed.