

Anisotropy of the Intrinsic Domain Magnetization of a Ferromagnet

S. H. CHARAP

Research Laboratory, International Business Machines Corporation, Poughkeepsie, New York

(Received September 14, 1959; revised manuscript received April 27, 1960)

The effect of pseudodipolar coupling on the intrinsic domain magnetization of a cubic ferromagnet at low temperatures is investigated. Besides the anisotropy in the magnetization resulting from the dependence of the thermal excitation of spin waves on magnetization direction in the crystal, the pseudodipolar coupling is capable of making an anisotropic contribution to the magnetization, even at $T=0^\circ\text{K}$, in the same approximation as its contribution to the first order anisotropy constant K_1 . Both effects are calculated and found to be in opposition to one another, tending to cancel in nickel at about 7.5°K .

I. INTRODUCTION

THE low-temperature behavior of an ideal ferromagnet was shown by Dyson¹ to be well described by a theory of noninteracting spin waves. The isotropic exchange between nearest neighbor spins on a cubic lattice was treated very precisely, but other energies were completely ignored. However, his work serves as a basis for including other terms in the Hamiltonian. Thus, Charap and Weiss² were able to treat the pseudodipolar coupling between nearest neighbor spins and show that the first order anisotropy constant, K_1 , as calculated for this interaction, varies at low temperatures as the tenth power of the magnetization. As was first pointed out by Van Vleck,³ the biquadratic angular dependence required to represent the cubic anisotropy associated with K_1 could be obtained from this perturbation in the second approximation. In such a case we may also expect to find, even at the absolute zero of temperature, an anisotropy of the intrinsic domain magnetization to the same order in the perturbation as the leading contribution to K_1 . It is the purpose of this paper to calculate the anisotropy in domain magnetization due to this pseudodipolar coupling.

The dipolar interaction has a twofold effect on the magnetization. In the first place, the magnetization associated with a given state, as defined by the set of spin-wave occupation numbers, varies with direction as stated above. That is, because of the perturbation, the low-lying states have admixed into them other states differing by the excitation of two spin waves of wave vectors \mathbf{k} and $-\mathbf{k}$. The second-order energy is depressed by this admixture by an amount dependent upon the magnetization direction in the crystal. Those directions for which the energy is depressed most are the easy directions. These are the directions for which the unperturbed states have the greatest admixture of excited states, that is, states of smaller magnetization. Thus, the effect is one which results in greater magnetization in the hard directions than in the easy ones. It is a temperature dependent effect, decreasing with

increasing temperature. Secondly, the spin-wave occupation numbers become dependent upon the direction of the magnetization with respect to the crystalline axes. Such an effect was described by Callen⁴ on a molecular field model. The spins are visualized as precessing in a narrower cone when the magnetization is in an easy direction than when it is in a hard one. In the spin-wave picture, we use the fact that the energy required to excite a spin wave is least when the magnetization is in a hard direction. Thus, at a given temperature, there will be more spin waves excited if the magnetization is in a hard direction than in an easy one. On either model we have an effect which vanishes at $T=0$ and results in greater magnetization in the easy directions. In the case of nickel we shall find that the two effects tend to cancel each other at about $T=7.5^\circ\text{K}$.

We omit the long-range magnetic dipole forces from the present calculation. Thus, as a calculation of the magnetization this work is incomplete. The result is, however, a complete description of its anisotropic part, to lowest order. On the other hand, we point out that the Holstein and Primakoff⁵ calculation may not be regarded as a complete treatment of the magnetic dipolar interaction. In I it is shown that certain terms in the dipolar interaction which are neglected by H-P make important contributions to the anisotropy. It should be noted also that the argument given by H-P concerning the relative effects of pseudodipolar and magnetic dipolar interactions on the magnetization does not apply to M_β (M_β is the H-P designation for the deviation from the simple Bloch law which survives at $T=0$). The important contributions to M_β come from the very short wavelength spin waves, for which the pseudodipolar interaction far outweighs the magnetic dipolar coupling. However, on the basis of the magnetic contribution alone, M_β was judged to be negligible and dropped by them.

In Sec. II we briefly present the low-lying energies to second order in the pseudodipolar interaction. The

¹ F. J. Dyson, Phys. Rev. **102**, 1217, 1230 (1956).

² S. H. Charap and P. R. Weiss, Phys. Rev. **116**, 1372 (1959). Henceforth, this paper shall be referred to as I.

³ J. H. Van Vleck, Phys. Rev. **52**, 1178 (1937).

⁴ E. R. Callen, *Fifth Annual Conference on Magnetism and Magnetic Materials, Detroit, Michigan, 1959* [J. Appl. Phys. **31**, 149S (1960)].

⁵ T. Holstein and H. Primakoff, Phys. Rev. **58**, 1107 (1940). Henceforth, this paper shall be referred to as H-P.

notation is thereby established for the magnetization calculation of Sec. III. In Sec. IV we give an approximate calculation of the anisotropic M_β . Finally, in Sec. V, the results are discussed for nickel and for iron.

II. SPECTRUM OF LOW-LYING LEVELS

We treat a cubic lattice of N identical spins. Each spin is coupled only with its nearest neighbors by way of the exchange (J) and short-range pseudodipolar (D) interaction. The spin-wave Hamiltonian may be obtained by use of the Dyson¹ formalism. This approach was used in the anisotropy calculation of I for $S=\frac{1}{2}$ only. The extension to arbitrary S is straightforward. Alternatively, one may use the prescription of Oguchi⁶ for handling the Holstein-Primakoff⁵ formalism. This treatment may be found in the analysis of the pseudodipolar anisotropy problem by Keffer and Oguchi.⁷ The spin-wave Hamiltonians resulting from the two procedures are somewhat different in form. However, the energy levels which have been calculated by means of these are identical. For the purposes of this paper, those spin-wave interactions are neglected which conserve the total number of spin waves. Of these the most important is Dyson's¹ dynamical interaction.

The dipolar term is regarded as a perturbation on the exchange and, to second order, the energy of the state with spin-wave occupation numbers $N_k, N_{k'}, N_{k''}, \dots$ is, except for constants,

$$E = -\frac{1}{4} \sum_k \frac{|B_k|^2}{A_k} + \sum_k N_k e_k. \quad (1)$$

The first term is the anisotropic part of the ground-state energy, E_g , and the second is the energy associated with the excitation of spin waves. For small \mathbf{k} ,

$$e_k = A_k + E_k - \frac{1}{2} \frac{|B_k|}{A_k} + \frac{1}{N} \sum_{k'} \frac{1}{A_{k'}} \left(\frac{1}{2S} |B_{k'}|^2 - |F_{k'}|^2 \right). \quad (2)$$

Much of the notation used here is that of Keffer and Oguchi.⁷ We have

$$A_k = 2SJ \sum_l [1 - \exp(i\mathbf{k} \cdot \mathbf{l})], \quad (3)$$

$$E_k = -\frac{1}{2} SD \sum_l [1 - 3(l^z/l)^2] \exp(i\mathbf{k} \cdot \mathbf{l}), \quad (4)$$

$$B_k = -\frac{3}{2} SD \sum_l (l^-/l)^2 \exp(i\mathbf{k} \cdot \mathbf{l}), \quad (5)$$

and

$$F_k = -(\frac{1}{2}S) \frac{3}{2} D \sum_l (l^z l^-/l^2) \exp(i\mathbf{k} \cdot \mathbf{l}). \quad (6)$$

In these expressions, the sums on \mathbf{l} range over all the nearest neighbor vectors of the lattice. The z direction is the axis of quantization and

$$l^\pm = l^x \pm il^y.$$

⁶ T. Oguchi, Phys. Rev. **117**, 117 (1960).

⁷ F. Keffer and T. Oguchi, Phys. Rev. **117**, 718 (1960).

In I, the pseudodipolar coupling was measured by the parameter ϵ , which is $-\frac{3}{4}D$. The effect of a magnetic field, H , along the z axis is taken into account by the usual Zeeman connections; to E_g we add $-g\mu_B N S H$, to A_k we add $g\mu_B H$.

It is a rather good approximation to neglect the second and third terms on the right-hand side of (2), since E_k and B_k vanish for $\mathbf{k}=0$, and $D \ll J$. The sums on \mathbf{k} appearing in Eqs. (1) and (2) have been treated in some detail in I. It may be shown that

$$\frac{1}{N} \sum_k \frac{1}{A_k} \left(\frac{2}{S} |B_k|^2 + |F_k|^2 \right)$$

is an isotropic quantity, i.e., a simple number, which we may conveniently denote by γ . The spin-wave energy (2) may be rewritten

$$e_k \approx A_k + \frac{1}{N} \sum_{k'} \frac{5}{2S} \frac{|B_{k'}|^2}{A_{k'}} - \gamma. \quad (7)$$

We insert this expression into Eq. (1), noting that the anisotropy is entirely contained in the sums on B_k , so that we may immediately write for the first-order anisotropy constant in terms of the thermally excited spin-wave occupation numbers $\langle N_k \rangle(T)$,

$$K_1(T) = K_{10} [1 - (10/NS) \sum_k \langle N_k \rangle], \quad (8)$$

Since, according to simple theory, the magnetization, M , obeys the equation

$$M(T) = M_0 [1 - (1/NS) \sum_k \langle N_k \rangle], \quad (9)$$

then, for the low-lying states

$$\frac{K_1(T)}{K_{10}} \approx \left[\frac{M(T)}{M_0} \right]^{10}, \quad (10)$$

which is the well-known tenth power law.

III. THE MAGNETIZATION

The magnetization is calculated from the partition function, Z , according to the relation

$$M = \frac{k_B T}{V} \frac{\partial}{\partial H} \ln Z. \quad (11)$$

Here k_B is Boltzmann's constant, the volume of the crystal is V , and

$$Z = \sum_E \exp(-E/k_B T).$$

For the problem being treated here,

$$M = -\frac{1}{V} \frac{\partial E_g}{\partial H} - \frac{1}{V} \frac{\partial e_k}{\partial H} \sum_k \langle N_k \rangle. \quad (12)$$

In the simple theory, $B_k = F_k = 0$, and we have

$$-\frac{1}{V} \frac{\partial E_g}{\partial H} = g\mu_B SN/V = M_0, \quad (13)$$

and

$$-\frac{1}{V} \frac{\partial e_k}{\partial H} = -\frac{g\mu_B}{V} = -\frac{M_0}{NS}. \quad (14)$$

These results lead immediately to Eq. (9). The thermal spin-wave excitation numbers are given by the Bose function

$$\langle N_k \rangle = [\exp(e_k/k_B T) - 1]^{-1}. \quad (15)$$

In the presence of the pseudodipolar interaction two anisotropic effects may be distinguished in Eq. (12). The first of these is the deviation of the derivatives with respect to H from the values given by (13) and (14). The second is the variation of the expectation values $\langle N_k \rangle$ as given by Eq. (15), with the direction of the magnetization in the lattice. The deviation from Eq. (13) due to the pseudodipolar coupling in second order is

$$-\frac{1}{V} \frac{\partial}{\partial H} \left(-\frac{1}{4} \sum_k \frac{|B_k|^2}{A_k} \right) = -\frac{g\mu_B}{4V} \sum_k \frac{|B_k|^2}{A_k^2}, \quad (16)$$

and to Eq. (14) we must add,

$$\frac{g\mu_B}{NV} \frac{5}{2S} \sum_k \frac{|B_k|^2}{A_k^2} + \frac{1}{V} \frac{\partial \gamma}{\partial H}. \quad (17)$$

In the simple theory e_k includes only the exchange energy which is approximated by its leading term, quadratic in \mathbf{k} . This leads to the well-known Bloch T^3 law, i.e.,

$$(1/NS) \sum_k \langle N_k \rangle = CT^3.$$

The addition of other terms, independent of \mathbf{k} , to the spin-wave energy modifies this result. We may make use of an expansion due to Robinson⁸ to show that

$$(1/NS) \sum_k \langle N_k \rangle = CT^3 (1 - 1.32\alpha^3 + 0.559\alpha - 0.0398\alpha^2 + 0.00163\alpha^3 + \dots), \quad (18)$$

where for our problem, according to Eq. (7),

$$k_B T \alpha = g\mu_B H + \frac{1}{N} \sum_k \frac{5}{2S} \frac{|B_k|^2}{A_k} - \gamma. \quad (19)$$

Robinson's expansion of the Bose-Einstein integral converges for $|\alpha| \leq 2\pi$ and the terms given in Eq. (19) are accurate to at least 1% for $\alpha \leq 1$. For nickel the latter condition will be satisfied in zero field for temperatures above about 0.6°K.

At the absolute zero of temperature, then, the magnetization consists of two parts;

$$M_0 + M_\beta,$$

where M_β is given by Eq. (16) and is anisotropic. Finally we have

$$M = M_0 + M_\beta - \left(M_0 + 10M_\beta - \frac{NS}{V} \frac{\partial \gamma}{\partial H} \right) \frac{1}{NS} \sum_k \langle N_k \rangle. \quad (20)$$

We neglect the isotropic term $(NS/V) \partial \gamma / \partial H$ in comparison with M_0 and separate the magnetization into two parts,

$$M(T) = M_1(T) + M_2(T), \quad (21)$$

where

$$M_1(T) = M_0 [1 - (1/NS) \sum_k \langle N_k \rangle], \quad (22)$$

and

$$M_2(T) = M_\beta [1 - (10/NS) \sum_k \langle N_k \rangle]. \quad (23)$$

In the spirit of the approximations of this paper, the value of $\sum_k \langle N_k \rangle$ to be used in Eq. (22) is that of Eq. (18), but in Eq. (23) the value at $\alpha = 0$, i.e., the Bloch value is appropriate. Thus the two anisotropic effects are completely separated in the formalism. Note also that M_2 , like K_1 , obeys the tenth power law;

$$\frac{M_2(T)}{M_\beta} \approx \left[\frac{M_1(T)}{M_0} \right]^{10}. \quad (24)$$

IV. CALCULATION OF M_β

The evaluation of

$$M_\beta = -\frac{g\mu_B}{4V} \sum_k \frac{|B_k|^2}{A_k^2} \quad (25)$$

is made in a manner similar to that used for the energy in I. Thus we write

$$M_\beta = -\frac{g\mu_B}{4V} \frac{9}{4} S^2 D^2 \sum_{l_1, l_2} \left(\frac{l_1^+ l_2^-}{l^2} \right)^2 d_{l_1 l_2}, \quad (26)$$

where

$$d_r \equiv \sum_k A_k^{-2} \exp(i\mathbf{k} \cdot \mathbf{r}). \quad (27)$$

The expression for M_β (Eq. 26) is anisotropic by virtue of the fact that l^+ and l^- are defined in terms of a coordinate system whose z axis is the magnetization direction. Except for some constant factors, the definition (27) is a direct analog of the b_r used in I. By d_{fgh} we shall mean (as in I) that d_r associated with the lattice vector whose components along the cube edges are, in units of the length of a component of a nearest neighbor vector, the integers fgh . Thus, for the face-centered cubic, that d_r which is associated with a nearest neighbor vector is $d_{110} = d_{101} = d_{011}$. The d_r arise in combinations and we define

$$\begin{aligned} f_A &= d_{000} + d_{200} - 2d_{110}, & \text{simple cubic,} \\ &= d_{000} + d_{222} - d_{022} - d_{200}, & \text{body-centered cubic,} \\ &= d_{000} + d_{220} - d_{211} - d_{011}, & \text{face-centered cubic,} \end{aligned} \quad (28)$$

⁸ J. E. Robinson, Phys. Rev. **83**, 678 (1951).

and for the face-centered cubic only

$$f_B = 2d_{200} - d_{211} - d_{011}. \quad (29)$$

Also, for the face-centered cubic only, we define the vector \mathbf{l}^* as one of the nearest neighbor vectors which is normal to \mathbf{l} . Then by the techniques of appendix A in I,

$$M_\beta = -\frac{g\mu_B}{4V} \frac{9}{4} S^2 D^2 \left[f_A \sum_1 \left(\frac{l+l^-}{l^2} \right)^2 + f_B \sum_1 \left(\frac{l+l^{*-}}{l^2} \right)^2 \right]. \quad (30)$$

In terms of $\Gamma = \alpha_1^2 \alpha_2^2 + \alpha_1^2 \alpha_3^2 + \alpha_2^2 \alpha_3^2$, α_1 , α_2 , and α_3 being the direction cosines of the magnetization with respect to the crystal axes,

$$\sum_1 \left(\frac{l+l^-}{l^2} \right)^2 = \begin{cases} 4(1-\Gamma) & \text{sc} \\ (32/9)(1+\Gamma) & \text{bcc}, \\ 2(3+\Gamma) & \text{fcc} \end{cases} \quad (31)$$

and

$$\sum_1 \left(\frac{l+l^{*-}}{l^2} \right)^2 = 2(1-3\Gamma) \text{ fcc}. \quad (32)$$

In the energy anisotropy calculation, functions analogous to f_A and f_B were evaluated by calculation of the individual b_i involved. In the present work we calculate only the combinations f_A and f_B in the following approximation. The Zeeman contribution to A_k is neglected. The numerators and denominators are separately expanded in powers of \mathbf{k} and the leading terms are proportional to k^4 . Only the leading terms are kept and the sums converted to integrals. In this way we find

$$f_A \approx \frac{N}{(2SJ)^2} \begin{cases} 1/15 & \text{sc} \\ 2/45 & \text{bcc} \\ 1/80 & \text{fcc} \end{cases} \quad (33)$$

and

$$f_B \approx -\frac{N}{(2SJ)^2} \frac{1}{240} \text{ fcc}. \quad (34)$$

Finally then

$$M_\beta \approx -\frac{M_0}{S} \left(\frac{D}{J} \right)^2 \begin{cases} (3/80)(1-\Gamma) & \text{sc} \\ (1/45)(1+\Gamma) & \text{bcc}, \\ (3/1280)(4+3\Gamma) & \text{fcc} \end{cases} \quad (35)$$

We include a summary of results from I for the purpose of evaluation of the quantity α in Eq. (19). These are,

$$\frac{1}{2NS} \sum_k \frac{|B_k|^2}{A_k} = \frac{D^2}{J} \begin{cases} 0.476(1-\Gamma) & \text{sc} \\ 0.268(1+\Gamma) & \text{bcc}, \\ 0.0223(12+5\Gamma) & \text{fcc} \end{cases} \quad (36)$$

and

$$\gamma = \frac{D^2}{J} \begin{cases} 1.899 & \text{sc} \\ 1.612 & \text{bcc}, \\ 0.644 & \text{fcc} \end{cases} \quad (37)$$

For the case of nickel a numerical estimate of the pseudodipolar coupling parameter was made. Comparison with the experimental ground-state value of K_1 gives $D/J \approx 0.12$, with $J \approx 230k_B$.

V. DISCUSSION

We have calculated the anisotropic contributions to the magnetization of a cubic ferromagnet, using as their source the pseudodipolar coupling. There are two effects. One of these, the anisotropic thermal excitation of spin waves, is contained in that portion of the magnetization which is denoted by M_1 . The other, denoted by M_2 , is the anisotropy of the moment of a given spin-wave state and is essentially given by the magnetic field dependence of the anisotropy coefficient K_1 . Contributions to M_2 arising from a magnetic field dependence of the coupling parameter D have been neglected. We note that the terms calculated here are of magnitude $(\mu_B/kT_c)K_1$, where T_c is the Curie temperature. We may argue on the anisotropic exchange model described by Van Vleck³ that the neglected terms have the order of magnitude $(\mu_B/h\nu)K_1$, where $h\nu$ is of the order of magnitude of the crystal field splitting. Thus the neglected term is smaller by a factor $kT_c/h\nu (\sim \frac{1}{10})$ than what we have calculated here.

It is instructive to consider the relative magnitude of the effect of the pseudodipolar interaction in $M_1(T)$ versus that in $M_2(T)$. For the purpose of this comparison we note that the temperature dependence of M_2 is quite negligible at low temperatures so that we need only consider M_β . It is also necessary to suppose that a magnetic field of the order of magnitude of the "effective anisotropy field" (~ 3000 oe for nickel) is applied so that the magnetization may be equally well oriented in hard and easy directions. In the presence of such a field, for the face centered cubic we have

$$k_B T \alpha \approx 0.56 \frac{D^2}{J} \Gamma. \quad (38)$$

This field will have a negligible effect on M_β because it is the short wavelength spin waves excited in the intermediate states that are most important for its calculation. Unless an even larger field is applied we always have $\alpha \lesssim 1$ for $T \gtrsim 0.6^\circ\text{K}$.

For nickel (fcc, $s = \frac{1}{2}$) we calculate from Eq. (35) for the easy directions

$$M_\beta \approx -3.4 \times 10^{-4} M_0. \quad (39)$$

The change in M_β , which results from the rotation of the magnetization from a hard to an easy direction, is $\frac{1}{5}$ of the value given by Eq. (39) and is comparable to the change in magnetization resulting from heating the sample from 3°K to 6°K . The deviation of $M_1(T)$ from the $T^{\frac{3}{2}}$ law value when $\alpha = 1$ is, using $C = 8.6 \times 10^{-6} (^\circ\text{K})^{-\frac{3}{2}}$

³ M. Fallot, Ann. phys. 6, 305 (1936).

for nickel, approximately $3.4 \times 10^{-5} M_0$. As the temperature is increased the $\alpha^{\frac{1}{2}}$ term will tend to dominate among the deviations in M_1 , and grows linearly with T . The two anisotropic effects tend to cancel each other at about 7.5°K. At temperatures above this the M_1 effect dominates, i.e., the greater magnetization occurs in the easy directions. Below this temperature the greater magnetization occurs in the hard directions.

There is an interesting application of this work to the case of iron. Although the pseudodipolar coupling may be present, the problem of the anisotropy in iron has been treated more easily than that in nickel because of the possibility of using a pseudoquadrupolar interaction, which yields K_1 in first order.¹⁰ This interaction must be measured by a parameter whose magnitude is $\sim D^2/J$, if it is to explain completely the anisotropy

in iron. The anisotropy in magnetization resulting from pseudoquadrupolar coupling again has the two parts. The contribution to M_1 will be exactly the same as in the dipolar case. However, the contribution to M_2 must occur in a higher order of approximation than the contribution to K_1 . That is, in second order, we may have a term in M_2 of order of magnitude $(D/J)^4$ coming from the pseudoquadrupolar coupling. According to our discussion, such a term cannot be observed at any reasonable temperature. Thus, we can imagine the possibility of determining, experimentally, to what extent the anisotropy in iron is due to pseudodipolar coupling by observation of magnetization anisotropy.

¹⁰ See reference 3; also C. Zener, *Phys. Rev.* **96**, 1335 (1954), and F. Keffer, *Phys. Rev.* **100**, 1692 (1955).

Magnetic Scattering of Neutrons by Exchange-Coupled Lattices*

A. W. SÁENZ

U. S. Naval Research Laboratory, Washington, D. C.

(Received September 21, 1959)

The magnetic scattering of slow neutrons of arbitrary initial polarization by an extensive class of magnetically-coupled lattices is treated by a time-dependent operator approach for the case of complete orbital quenching of the magnetic ions. This magnetic scattering is carefully divided into purely magnetic and magneto-vibrational scattering, the types thereof involving, respectively, only zero-phonon processes and solely nonzero-phonon processes, and general formulas for these two types of scattering are obtained. These formulas are applied in temperature regions which are sufficiently large (I) or sufficiently small (II) compared with the temperature above which paramagnetism obtains. In region I, for the purely magnetic scattering and under certain invariance requirements on the above magnetic coupling, we analyze the energy spectrum of outgoing initially unpolarized neutrons of sufficiently high incident energy by a moment method. We thus

obtain general formulas for the energy-integrated effective differential cross section defined in this paper and for the moments of energy transfer defined therein. These formulas involve certain spin averages, explicit equations being given for a wide range of these averages for exchange-coupled lattices. These results are illustrated numerically and compared with experiment for the case of polycrystalline MnF_2 . In region II, we discuss certain broad features of the purely magnetic one-magnon scattering of arbitrarily polarized neutrons by exchange-coupled lattices of the class alluded to previously and by more complex ones, studying this scattering in detail for ferromagnets and certain antiferromagnets. A new spin-wave effect is pointed out for polarized neutrons incident on ferromagnets. Brief treatments of the magnetovibrational scattering in regions I and II are also given.

I. INTRODUCTION

IN this paper, we shall investigate the magnetic scattering of slow neutrons by a wide class of lattices having magnetic ions which are orbitally quenched and are magnetically coupled with one another. Our objective is twofold. First, we want to derive formulas which are general enough to encompass the case of neutrons of arbitrary initial polarization f incident on lattices of the above class at any temperature T . Second, we desire to employ these formulas in a detailed study of magnetic neutron scattering by certain lattices of this class, and particularly by exchange-coupled

ones,¹ for temperatures T which are sufficiently high or low compared with the temperatures T_c above which they are paramagnetic, a restriction which permits us to base our results on reliable quantum-statistical methods.

In what follows, the neutron magnetic scattering processes of interest, in which the initial and final lattice states are the same or different with regard to their vibrational quantum numbers, shall be defined as corresponding to purely magnetic or to magneto-vibrational scattering, respectively. In general, the purely magnetic scattering is of greater physical

* The main results of this paper were first reported in *Bull. Am. Phys. Soc.* **2**, 49 (1957); **3**, 203 (1958).

¹ The term exchange is employed in this paper to denote both ordinary exchange and superexchange, it being hoped that no confusion will be caused by this usage.