(B17) using (B15) as kernel can be rewritten as

$$P(\lambda)=[16(2+\lambda)]^{-1}\int_{a}^{\lambda+4(1+\lambda)^{\frac{1}{2}}-4}|\lambda-\lambda'|d\lambda'P(\lambda'), \quad \text{(B19)}$$

where

$$\begin{aligned} a&=0, && \text{for} \quad \lambda\leq 24 \\ &=\lambda-4(1+\lambda)^{\frac{1}{2}}-4 && \text{for} \quad \lambda\geq 24. \end{aligned}$$

In the region $\lambda\ll 1$ Eq. (B19) reduces to

$$P(\lambda)=\frac{1}{32}\int_{0}^{3\lambda}|\lambda-\lambda'|d\lambda'P(\lambda'), \qquad \text{(B20)}$$

a deceptively simple equation. To obtain the analytic dependence near $\lambda=0$, one might try $P(\lambda')=(\lambda')^{n}$. However, this yields $P(\lambda)\propto\lambda^{n+2}$, so that each iteration raises the exponent by two. Thus $P(\lambda)$ vanishes faster than λ^{n} for any finite n. It is indeed not obvious, at all, that there is any analytic behavior near $\lambda=0$ that will satisfy Eq. (B20).

We can, however, obtain the nature of the singularity at $\lambda=0$ by a physical argument. Equation (B20) indicates that the probability of each (downward) step is of the order λ, so that n downward steps occurs with a probability of the order λ^{n}. Since the binding energy can at most change from λ to 3λ, i.e., triple in each step, if $\lambda\ll 1$, the number of steps n to go from λ to a binding energy b of order unity is given by

$$b/\lambda=3^{n}, \quad n=\ln_{3}(1/\lambda)+\ln_{3}b, \qquad \text{(B21)}$$

so that we expect

$$P(\lambda)\propto\lambda^{n} \quad \text{with} \quad n=\ln_{3}b+\ln_{3}(1/\lambda),$$

as stated in Eq. (2.23) of the introduction.

The conjecture (B21) can be verified by converting Eq. (B20) to the differential difference equation

$$32P''(\lambda)=3P(3\lambda)+16\lambda P'(\lambda)+P(\lambda). \qquad \text{(B22)}$$

A transformation to the new variables $\varphi(y)$ and y

$$P=e^{-\varphi} \quad \text{and} \quad \lambda=e^{-y}, \qquad \text{(B23)}$$

permits (B22) to be rewritten after some rearrangement in the form

$$\varphi(y)-\varphi(y-\ln 3)=2y+\ln A, \qquad \text{(B24)}$$

where

$$A=(32/3)[\varphi'+(\varphi')^{2}-\varphi'']+(16/3)\varphi'e^{-2y}-e^{-2y}. \quad \text{(B25)}$$

We are concerned with the solution of (B24) in the region of large y. The term in $2y$ is dominant, and may be used to obtain an approximate solution $\varphi\approx y^{2}/\ln 3$ that already contains the dominant behavior described in Eq. (1.23). One iteration using the dominant term in A, i.e., $A\approx(32/3)(\varphi')^{2}$ leads to

$$\varphi(y)\approx y^{2}/\ln 3+(2/\ln 3)y\ln y+cy+\cdots, \qquad \text{(B26)}$$

where

$$c=\frac{1}{\ln 3}\left[\ln\left(\frac{128}{3(\ln 3)^{2}}\right)-2\right]\approx 3.52.$$

A return to the original variables gives $P\approx\lambda^{n}$ with

$$n\approx\ln(1/\lambda)/\ln 3+3.52+2\ln\ln(1/\lambda)/\ln 3. \quad \text{(B27)}$$

Equation (B27) verifies to good approximation our original conjecture Eq. (B21), and supplies us with an understanding of the nature of the essential singularity near $\lambda=0$.

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Anisotropic Ferromagnetic Resonance Linewidth in Ferrites*

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In disordered magnetic materials such as the ferrites, the dominant source of resonance linewidth can be attributed to processes involving only two elementary excitations: the destruction of a magnon with the creation either of another magnon or a phonon. We here consider only the primary magnon-magnon scattering process. We show that the random variation of the spin-orbit coupling parameters of the disordered ions leads to a resonance linewidth comparable to that observed in ferrites. The particular symmetry of the crystalline fields around the octahedrally coordinated sites causes an anisotropy in the linewidth with a minimum in the [100] directions and a maximum in the [111] directions. This anisotropy of linewidth is in general agreement with experimental observations on typical ferrites, as for example, the measurements of Schnitzler, Folen, and Rado on disordered lithium ferrite.

1. INTRODUCTION

THE source of resonance linewidth in disordered magnetic materials such as the ferrites has been discussed by Clogston, Suhl, Walker, and Anderson,[1] who pointed out the possibility of two excitation processes conserving energy but not momentum. The primary mechanism of magnon scattering was attributed to the random pseudodipolar interaction. However, the subsequent discovery by Folen and Rado[2] that the

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[1] A. M. Clogston, H. Suhl, L. R. Walker, and P. W. Anderson, J. Phys. Chem. Solids 1, 129 (1959).

[2] V. J. Folen and G. T. Rado, J. Appl. Phys. 29, 438 (1958).

magnetocrystalline anisotropy is single-ion additive provides an upper limit to the strength of the pseudodipolar interaction; this upper limit is too small to account for the observed linewidths.

That the pseudodipolar interaction is small has been suggested also by White[3] on the basis of observed paramagnetic linewidths above the Curie temperature, and by Yosida and Tachiki[4] on the basis of theoretical calculations of both the pseudodipolar and single-ion contributions to the magnetocrystalline anisotropy.

We show that a relatively strong loss mechanism exists in disordered ferrites because of the variation from ion to ion of the intra-atomic spin-spin and spin-orbit coupling parameters. These interactions contribute terms to the effective Hamiltonian in first- and second-order perturbation theory, respectively, and give strong magnon scattering from the homogeneous mode to the degenerate spin-wave modes.

The interactions which we consider are, of course, the same interactions which give rise to the one-ion anisotropy, as calculated by Yosida and Tachiki.[4] However, in an ordered structure the first and second-order perturbation terms reduce to constants when summed over all equivalent sites in a cubic unit cell, and the scattering and loss thereby disappear. The magnetocrystalline anisotropy arises in third order (two spin-orbit and one spin-spin interaction) or in fourth order (four spin-orbit interactions). It should perhaps be noted here that no simple correlation exists between our scattering mechanism and the magnetocrystalline anisotropy because of the different relative admixtures of the underlying spin-spin and spin-orbit interactions in each order of perturbation theory.

We find a contribution to the linewidth of the form

$$\gamma\Delta H\sim\lambda_{0k}=\lambda_{0k}^{(0)}+\lambda_{0k}^{(1)}(\alpha_1^2\alpha_2^2+\alpha_1^2\alpha_3^2+\alpha_2^2\alpha_3^2), \quad (1)$$

where α_1, α_2, α_3 are the direction cosines of the magnetization with respect to the crystal axes. The quantities $\lambda_{0k}^{(0)}$ and $\lambda_{0k}^{(1)}$ are coefficients which we calculate; they depend on the applied field, sample shape, spin-spin, and spin-orbit coupling parameters, and the strength and symmetry of the crystalline fields. The sign of $\lambda_{0k}^{(1)}$ is independent of the sign of the spin-orbit interaction (which usually dominates the spin-spin contribution) but depends on the symmetry of the crystalline fields. For the octahedral sites in the spinel structure, the crystalline fields have trigonal symmetry along the [111] axes, and $\lambda_{0k}^{(1)}$ is found to be positive. Thus the contribution to the linewidth is maximum in the [111] directions and minimum in the [100] directions. In a typical resonance experiment on a spherical ferrite sample $\lambda_{0k}^{(0)}/\gamma$ and $\lambda_{0k}^{(1)}/\gamma$ are calculated to be of the order of 1 and 45 oersteds, respectively, corresponding to a linewidth of 1 oersted in the [100] directions and 16 oersteds in the [111] directions.

The primary process of magnon scattering from the $\mathbf{k}=0$ mode to degenerate modes is found to be temperature independent. The secondary process of equilibration among the $\mathbf{k}\neq0$ magnons cannot proceed by two-excitation processes and is strongly temperature dependent. We assume the temperature to be sufficiently high that the secondary equilibration process proceeds much more rapidly than the primary process. At low temperatures, however, failure of the $\mathbf{k}\neq0$ magnons to relax among themselves can lead to a saturation of the primary transition and a decrease of the contribution to the linewidth.

Preparatory to the detailed calculations we briefly discuss the relationship of magnon scattering to ferromagnetic resonance linewidth in Sec. 2. In Sec. 3 we summarize the theory in such a way as to exhibit its main qualitative features. In Sec. 4 we compute the contribution to the linewidth, its dependence on the orientation of the external field, its dependence on the magnitude of the external field (or resonant frequency) and its dependence on sample shape. Finally, the results are discussed in Sec. 5.

2. MAGNON SCATTERING, DYNAMICAL RESPONSE, AND LINEWIDTH

The dynamical response of a ferromagnetic material to an applied field has been analyzed in terms of magnon scattering probabilities by Callen.[5] For an ellipsoidal sample symmetric around the applied magnetic field the dynamical equation follows from two basic results of spin-wave theory. Every magnon excited reduces the z component of magnetization by one unit ($\gamma\hbar$) of magnetic moment. That is

$$M_z=M_0-\gamma\hbar n, \quad (2)$$

where n is the total number of magnons present, and M_0 is the saturation magnetization. However, the *magnitude* of the magnetization M is reduced only by the magnons with nonzero wave vector;

$$M=M_0-\gamma\hbar n', \quad (3)$$

where n' is the number of magnons with $\mathbf{k}\neq0$. Subtracting these two equations yields the relation

$$M_z=M-\gamma\hbar n_0, \quad (4)$$

where $n_0\equiv n-n'$ is the number of magnons with $\mathbf{k}=0$.

Alternatively, n, n', and n_0 can be interpreted as the numbers of magnons in excess of the thermal equilibrium numbers; M_0 then represents the equilibrium magnetization at the given temperature. We adopt this interpretation henceforth.

Two-excitation scattering processes are represented in the Hamiltonian by products of a creation operator for one excitation and a destruction operator for the other excitation. Such terms give rise to simple first-order equations for the rates of change of the magnon numbers

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[3] R. L. White, Phys. Rev. **115**, 1519 (1959).

[4] K. Yosida and M. Tachiki, Progr. Theoret. Phys. (Kyoto) **17**, 3 (1957).

[5] H. Callen, J. Phys. Chem. Solids **4**, 256 (1958).

of the form

$$\dot{n}_0=\dot{n}_0^{(h)}-(\lambda_{0k}+\lambda_{0\sigma})n_0, \tag{5}$$

$$\dot{n}'=\lambda_{0k}n_0-\lambda_{k\sigma}n', \tag{6}$$

where $\dot{n}_0^{(h)}$ is the rate of creation of $\mathbf{k}=0$ magnons by the applied magnetic fields, and where λ_{0k} is the probability of magnon-magnon scattering from $\mathbf{k}=0$ to $\mathbf{k}\neq 0$. That is, λ_{0k} is the probability of destruction of a $\mathbf{k}=0$ magnon with the creation of a $\mathbf{k}\neq 0$ magnon. $\lambda_{0\sigma}$ is the probability of destruction of a $\mathbf{k}=0$ magnon with the creation of a phonon, and $\lambda_{k\sigma}$ is the probability of destruction of a $\mathbf{k}\neq 0$ magnon with the creation of a phonon.

Expressing n_0 and n' in terms of M_z and M through Eqs. (3) and (4) and substituting into Eqs. (5) and (6) gives immediately

$$\dot{M}_z=-\gamma(\mathbf{M}\times\mathbf{H}_i)_z+\lambda_{0\sigma}(M-M_z)+\lambda_{k\sigma}(M_0-M), \tag{7}$$

$$\dot{M}=-\lambda_{0k}(M-M_z)+\lambda_{k\sigma}(M_0-M). \tag{8}$$

In the first term of Eq. (7) $\mathbf{H}_i$ is the sum of external and demagnetizing fields, and this term describes the precession of the magnetization around the field $\mathbf{H}_i$. The part of this term involving the dc field gives a pure loss-free precession which is not represented in Eqs. (5) or (6), and which has been added in accordance with the familiar loss-free quantum mechanical equations of motion. The part which involves precession around the rf field is responsible for the absorption of power from the rf generator and is represented by the term $\dot{n}_0^{(h)}$ in Eq. (5).

It is useful to regroup the terms in Eqs. (7) and (8) as follows:

$$\begin{aligned}\dot{M}_z=-\gamma(\mathbf{M}\times\mathbf{H}_i)_z&+\lambda_{0\sigma}(M_0-M_z)\\&+(\lambda_{k\sigma}-\lambda_{0\sigma})(M_0-M),\end{aligned} \tag{9}$$

$$\dot{M}=-\lambda_{0k}(M_0-M_z)+(\lambda_{0k}+\lambda_{k\sigma})(M_0-M). \tag{10}$$

An alternative pair of equations can be written in terms of M_z and M_t, the transverse component of $\mathbf{M}$, defined by

$$M_t^2+M_z^2=M^2, \tag{11}$$

whence

$$M-M_z\simeq M_t^2/2M\simeq M_t^2/2M_0. \tag{12}$$

To first-order equations (7), (8), and (12) give

$$\begin{aligned}\dot{M}_z=-\gamma(\mathbf{M}\times\mathbf{H}_i)_z&+(\lambda_{0\sigma}-\lambda_{k\sigma})M_t^2/2M_0\\&+\lambda_{k\sigma}(M_0-M_z),\end{aligned} \tag{13}$$

$$\dot{M}_t=-\gamma(\mathbf{M}\times\mathbf{H}_i)_t-(\lambda_{0k}+\lambda_{0\sigma})/2M_t. \tag{14}$$

Finally, these two equations can be represented by a single vector equation

$$\begin{aligned}\frac{\partial\mathbf{M}}{\partial t}=-\gamma(\mathbf{M}\times\mathbf{H}_i)&-(\lambda_{0k}+\lambda_{0\sigma})\\&\times[\mathbf{H}_0\times(\mathbf{M}\times\mathbf{H}_0)]/2H_0^2+[\lambda_{k\sigma}(M_0-M)\\&+(MH_0-\mathbf{M}\cdot\mathbf{H}_0)\lambda_{0\sigma}/H_0]\mathbf{H}_0/H_0,\end{aligned} \tag{15}$$

where H_0 is the externally applied magnetic field in the z direction. It is to be recalled that M_0 is the equilibrium magnitude of the magnetization at the given temperature, whereas M is the instantaneous magnitude.

Each of the terms in the dynamical equation (15) can be interpreted in a direct physical way. The first term is, of course, the loss-free precessional term. The second term has the direction $\mathbf{H}_0\times(\mathbf{M}\times\mathbf{H}_0)$, which is perpendicular to the z axis in the plane containing $\mathbf{M}$. This term therefore describes the relaxation of the transverse component of $\mathbf{M}$. It is proportional to $(\lambda_{0k}+\lambda_{0\sigma})$, or to the rate of destruction of $\mathbf{k}=0$ magnons. This is reasonable since the $\mathbf{k}=0$ magnons decrease M_z without decreasing M; that is, they "tilt" the magnetization and account for its transverse component. The final term in Eq. (15) describes the relaxation of the z component of the magnetization. This relaxation arises from two mechanisms; the change in the magnitude of $\mathbf{M}$ and the change in the tilt. The two terms in the square brackets of Eq. (15) represent the sum of these two mechanisms, the contribution of λ_{0k} to each cancelling out as it did in the derivation of Eq. (7).

The dynamical equation (15), or the equivalent pairs of scalar equations (9) and (10), or (13) and (14), provide a basis for the analysis of magnetic response. Three independent constants appear in the equations, in contrast to a single constant in the Landau-Lifshitz[6] equation and to two constants in the Bloch-Bloembergen[7] equation.

We consider specifically a ferromagnetic resonance experiment in a spherical sample, with an rf field applied in the x-y plane. If the components of the field are h_x and h_y we define h_+, h_-, and $\mathbf{m}$ by

$$h_\pm=h_x\pm ih_y, \tag{16}$$

$$\mathbf{M}=M_0\mathbf{H}_0/H_0+\mathbf{m}, \tag{17}$$

also

$$m_\pm=m_x\pm im_y. \tag{18}$$

Then, the dynamical equation becomes

$$\dot{m}_+=[i\gamma H_0-\tfrac{1}{2}(\lambda_{0\sigma}+\lambda_{0k})]m_+-i\gamma M_0h_+, \tag{19}$$

$$\begin{aligned}\dot{m}_z=-i\gamma(m_+h_--m_-h_+)/2&\\+m_+m_-(\lambda_{0\sigma}-\lambda_{k\sigma})/2&-\lambda_{k\sigma}m_z.\end{aligned} \tag{20}$$

If we let

$$h_+=h_0e^{i\omega t},\quad m_+=m_te^{i\omega t}. \tag{21}$$

The solution of Eq. (19) is

$$m_t=-\gamma M_0h_0/[\omega-\gamma H_0-(i/2)(\lambda_{0k}+\lambda_{0\sigma})], \tag{22}$$

whence the half-width of the resonant response is

$$\gamma\Delta H=\lambda_{0k}+\lambda_{0\sigma}. \tag{23}$$

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[6] L. Landau and E. Lifshitz, Physik. Z. Sowjetunion **8**, 153 (1935).

[7] F. Block, Phys. Rev. **70**, 460 (1946); N. Bloembergen, Phys. Rev. **78**, 572 (1950).

Solutions of Eq. (20) have been given by Seiden[8] in connection with an analysis of the "roll-off" region of decreasing rf susceptibility with increasing rf power.

From Eq. (23) it is seen that measurements of resonance linewidth evaluate only the sum of λ_{0k} and $\lambda_{0\sigma}$, the total scattering probability of the $\mathbf{k}=0$ magnons. However, Fletcher, Le Craw, and Spencer[9] have used both Eqs. (13) and (14), which they rederive by an exactly equivalent energy argument, as the basis of an experimental method for measuring λ_{0k}, $\lambda_{0\sigma}$, and $\lambda_{k\sigma}$ separately. They modulate the rf power in a resonance experiment, observe $\dot{M}_t$ in the usual way, and observe $\dot{M}_z$ by an additional pickup coil with its axis parallel to the applied dc field.

In this paper we are concerned only with a theory of the magnon-magnon scattering parameter λ_{0k} in ferrites. However, the method of Fletcher, Le Craw, and Spencer has not yet been applied to the measurement of λ_{0k} in ferrites, and we will therefore be able to compare our results only with the general magnitude and angular dependence of the observed resonance linewidth.

3. GENERAL CONSIDERATIONS

In this section we discuss the general features of magnon scattering processes and of magnetic loss preparatory to the detailed calculation of Sec. 4.

In a finite ferromagnetic sample the $\mathbf{k}=0$ mode is degenerate with modes of very small $\mathbf{k}$, or very long wavelength. In a spherical sample the $\mathbf{k}=0$ mode has a frequency of $\omega(0)=\gamma H_0$, whereas modes of wave vector $\mathbf{k}$ have frequencies $\omega(\mathbf{k})=\gamma[H_0-(4\pi/3)M_0+H_e a^2k^2]$, where H_e is the exchange field and a is the inter-atomic distance. This latter dispersion equation actually applies only to modes with $\mathbf{k}$ parallel to z, but for the purpose of our present qualitative discussion we can ignore the directional dependence of the dispersion law. The mode degenerate with the $\mathbf{k}=0$ mode then has

$$ka=\left(\frac{4\pi M_0}{3H_e}\right)^{\frac{1}{2}}\simeq\left(\frac{3000}{3\times10^7}\right)^{\frac{1}{2}}\simeq10^{-2}, \tag{24}$$

or $\lambda\simeq600a$ for a typical ferrite.

The transition probability from the $\mathbf{k}=0$ mode to the degenerate modes is given by the standard expression

$$\lambda_{0k}=(2\pi/\hbar)|M_{0k}|^2\rho(E), \tag{25}$$

where M_{0k} is the matrix element of the scattering perturbation and $\rho(E)$ is the density-in-energy of final states. Since the number of states in the Brillouin zone is N, and its volume is $(2\pi/a)^3$, the density is

$$\rho(E)=\frac{N4\pi k^2}{(2\pi/a)^3}\frac{dk}{dE}=\frac{Nka}{4\pi^2\gamma\hbar H_e}, \tag{26}$$

where $dk/dE=dk/\hbar d\omega$ is evaluated from the given dispersion law. If Eqs. (24) and (26) are inserted into (25) the result is

$$\lambda_{0k}=\frac{N|M_{0k}|^2}{\pi\hbar^2\gamma}\left(\frac{4\pi M_0}{12H_e^3}\right)^{\frac{1}{2}}. \tag{27}$$

Let us suppose that the underlying perturbation which acts on the spins and causes the magnon scattering can be represented by an effective magnetic field $H(\mathbf{r}_i)$ which is a random function of position. Let $H(\mathbf{k})$ be the $\mathbf{k}$th Fourier component of this random magnetic field

$$H(\mathbf{k})=N^{-\frac{1}{2}}\sum_i H(\mathbf{r}_i)e^{i\mathbf{k}\cdot\mathbf{r}_i}. \tag{28}$$

The perturbing energy acting on the ith spin is $\gamma\hbar SH(\mathbf{r}_i)$ and the $\mathbf{k}$th Fourier component of the perturbing energy is $\gamma\hbar SH(\mathbf{k})$. The matrix element between the $\mathbf{k}=0$ spin wave and a spin wave of wave vector $\mathbf{k}$ is therefore $N^{-\frac{1}{2}}\gamma\hbar SH(\mathbf{k})$, whence

$$\lambda_{0k}=\frac{\gamma S^2|H(\mathbf{k})|^2}{\pi}\left(\frac{4\pi M_0}{12H_e^3}\right)^{\frac{1}{2}}. \tag{29}$$

If we are to find a linewidth $\Delta H=\lambda_{0k}/\gamma$ of the order of 10 oersteds, Eq. (29) requires a perturbation $H(\mathbf{k})$ of the order of 10^5 oersteds.

The central problem of the theory now becomes evident when we translate this requirement on $H(\mathbf{k})$ back to the spatial representation. Let us suppose, first, that the fluctuating field has no correlation from ion to ion. Then from Eq. (28)

$$H(\mathbf{k})H^*(\mathbf{k})=N^{-1}\sum_{i,j} H(\mathbf{r}_i)H^*(\mathbf{r}_j)e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)}, \tag{30}$$

and if we take an ensemble average

$$\langle H(\mathbf{r}_i)H^*(\mathbf{r}_j)\rangle_{\text{av}}=\langle H_i^2\rangle_{\text{av}}\delta_{ij}, \tag{31}$$

whence, for any $\mathbf{k}$,

$$|H(\mathbf{k})|^2=\langle H_i^2\rangle_{\text{av}}. \tag{32}$$

Here $\langle H_i^2\rangle_{\text{av}}$ is the mean square of the random field acting on a representative ion. If we insert Eq. (32) into Eq. (29), we obtain directly the result of Clogston, Suhl, Walker, and Anderson; their more detailed analysis simply replaces our numerical factor $S^2/\pi(12)^{\frac{1}{2}}$ by a lattice sum with a value of the order of 3/20. The conclusion to be drawn from this analysis is that whatever the perturbation, be it pseudodipolar or otherwise, it must have an rms value of 10^5 oersteds if it fluctuates without correlation from ion to ion.

If the Fourier spectrum of the perturbation were not "white," or constant through $\mathbf{k}$ space, but were peaked in the vicinity of those $\mathbf{k}$'s corresponding to $\lambda\simeq600a$, the required rms field would be much smaller. Suppose, for instance, some impurities were to strain the lattice and through magnetostriction were to produce a random anisotropy field. If these ions had a mean separation of $600a$, the Fourier spectrum would be peaked at

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[8] P. E. Seiden and H. J. Shaw, *Fifth Conference on Magnetism and Magnetic Materials, Detroit, Michigan, 1959* [J. Appl. Phys. **31**, 225S (1960)].

[9] R. C. Fletcher, R. C. Le Craw, E. G. Spencer, Bell Telephone Laboratories Technical Report (unpublished).

$k_p \simeq 2\pi/600a$. Let us suppose the half-width of the Fourier spectrum to be Δk. Then by writing the equation analogous to Eq. (30) for $H(\mathbf{r}_i)H^*(\mathbf{r}_i)$ we find

$$|H(\mathbf{r}_i)|^2 = (1/N)\sum_{\mathbf{k}}|H(\mathbf{k})|^2 \simeq |H(\mathbf{k}_p)|^2 a^3 k_p^2 \Delta k/2\pi^2, \quad (33)$$

or putting $\Delta k \simeq k_p$

$$|H(\mathbf{r}_i)|^2 \simeq |H(\mathbf{k}_p)|^2 (k_p a)^3/2\pi^2 \simeq 500. \quad (34)$$

Thus, if the Fourier spectrum were peaked in the vicinity of $\lambda \simeq 600a$, with a half-width of the same order, an rms fluctuating field of only 20 or 30 oersteds would be sufficient to give the observed loss. Local anisotropy fields of this magnitude can be produced magnetostrictively by the strains in the vicinity of impurity atom clusters, dislocation lines, or other such defects, and it is plausible that these contribute a structure-sensitive background to the observed linewidths.

The model of scattering which we propose in this paper is a single-ion theory in which the perturbing field is not correlated from ion to ion. As mentioned in Sec. 1, the perturbing field is obtained in first- and second-order perturbation theory from the intra-atomic spin-spin and spin-orbit interactions, respectively. The second-order spin-orbit contribution is of the order of $\lambda^2L^2S^2/\delta$ where $\lambda\mathbf{L}\cdot\mathbf{S}$ is the spin-orbit interaction and δ is the energy difference of the orbital states split by the crystalline field. Equating this energy to $\gamma\hbar SH_i$ we find the order of magnitude of the effective perturbing field to be

$$H_i \simeq \lambda^2L^2S^2/\gamma\hbar\delta \simeq 2\times10^5 \text{ oersteds}, \quad (35)$$

where we take the values $\lambda \simeq 10^2$ cm^{-1} and $\delta \simeq 10^3$ cm^{-1} from Yosida and Tachiki's calculation for Fe^{2+} ions on the octahedrally coordinated sites of ferrites. Thus, on the basis of these qualitative arguments, it is plausible that the random variation of intra-atomic spin-spin and spin-orbit interactions will lead to the observed magnitude of resonance linewidth, as we shall corroborate in detail.

The angular dependence of the loss can be understood by a semiclassical symmetry argument as well. Consider a single ion in an octahedrally-coordinated site with spin-spin and spin-orbit parameters which deviate from the average values. By perturbation theory we obtain terms in the effective spin Hamiltonian which are second order in the spin operators—the first-order terms vanishing by inversion symmetry. These terms must reflect the symmetry of the crystalline field. The crystalline field at an octahedrally-coordinated site in a ferrite has trigonal symmetry about a body diagonal. If we choose a site with trigonal field along the [111] direction, the perturbing term therefore must be of the form $CS^2_{[111]}$, where $S_{[111]}$ is the component of $\mathbf{S}$ along [111]. Let us first suppose that the average magnetization is along the [111] direction, so that the $\mathbf{k}=0$ spin-wave mode consists of all spins precessing in phase around the [111] axis. The perturbation $CS^2_{[111]}$ is then equivalent to an additional field acting on the particular ion, which consequently precesses with a different frequency and destroys the coherence of the $\mathbf{k}=0$ mode. Quantum-mechanically this corresponds to scattering from the $\mathbf{k}=0$ mode to other spin-wave modes. Now let us suppose that the average magnetization is along the [001] direction, and that the $\mathbf{k}=0$ spin wave consists of all spins precessing in phase around the [001] axis. If the perturbation is expressed in terms of S_x, S_y, and S_z, it has the form

$$CS^2_{[111]} = \tfrac{1}{3}C[S(S+1)+2S_xS_y+2S_xS_z+2S_yS_z],$$

of which the classical average over a precessional cycle is zero. This perturbation therefore produces no shift in the energy levels nor in the precessional frequency and does not scatter the $\mathbf{k}=0$ mode when the magnetization is in the [001] direction. We conclude that the total loss vanishes in a $\langle 100\rangle$ direction; it must be fourth order in the components of the magnetization and hence must have the form $\lambda_{0k}^{(1)}(\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2)$ with a positive value of $\lambda_{0k}^{(1)}$. The constant term $\lambda_{0k}^{(0)}$ in Eq. (1) would presumably be zero but in our more detailed calculation we actually find a small but nonzero value of $\lambda_{0k}^{(0)}$ which results from the effect of magnetic dipole interactions in a finite sample. This long-range interaction makes the $\mathbf{k}=0$ mode less simple than the pure homogeneous rotation assumed above. From the formal point of view the Holstein-Primakoff transformation diagonalizes the magnetic dipole interaction by admixing shorter wavelength spin waves in the homogeneous mode. Nevertheless, the essential features of the above symmetry argument hold true.

4. CALCULATION OF λ_{0k}

To simplify the calculations, we adopt the simplest model of a crystal structure which has the relevant features of the true ferrite structure. We assume a simple cubic array of magnetic ions situated on four different kinds of sites which are distinguished only by the direction of the crystalline field acting on each type of site. The crystalline fields have trigonal symmetry about the four separate body diagonal directions. In obvious analogy to the ionic distribution on the octahedral sites of a partially inversed ferrite we assume the four interpenetrating lattices to be populated at random by two types of ions; A ions with fractional concentration c, and B ions with fractional concentration $1-c$.

The Hamiltonian of the system consists of the interaction of each spin with the external field, the exchange interactions between nearest neighboring spins, the dipolar interaction, the spin-orbit coupling $\lambda\mathbf{L}\cdot\mathbf{S}$ of each ion, the intra-atomic spin-spin interaction $-\rho[(\mathbf{L}\cdot\mathbf{S})^2+\frac{1}{2}\mathbf{L}\cdot\mathbf{S}-\frac{1}{3}L^2S^2]$ of each ion,[4,10] and the crystalline field potential. The complicating feature of this Hamiltonian

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[10] A. Abragam and M. H. L. Pryce, Proc. Roy. Soc. (London) **A205**, 135 (1951).

is the difference between the A and B ions, and the disorder of the spatial distribution. We consequently proceed by introducing an "unperturbed" or average Hamiltonian, in which we ignore the difference between A and B ions and replace exchange integrals, spin, and magnetic dipole moments by their average values. The perturbing terms in the Hamiltonian arise from the deviations of the ionic parameters from these average values.

Note Added in Proof.—The contribution to the linewidth from variations in spin magnitude is clearly isotropic and these variations consequently do not contribute to the anisotropic linewidth here calculated. Actually it can be shown that variations in spin magnitude make a completely negligible contribution even to the isotropic component of the linewidth.

The average, or unperturbed, Hamiltonian can be reduced to an effective spin Hamiltonian by the method of Yosida and Tachiki.[4] We assume that the orbital states of the ions are qualitatively similar to the orbital states of the Fe^{2+} ion, so that a perturbation treatment of the spin-orbit and intra-atomic spin-spin interactions yields a one ion anisotropy, and hence an unperturbed Hamiltonian of the form

$$\begin{aligned}\mathcal{H}_0 = &-\mathbf{M}\cdot\mathbf{H} - J\textstyle\sum_{\text{nei}}\mathbf{S}_i\cdot\mathbf{S}_j \\ &+\tfrac{1}{2}\textstyle\sum_{i,j}D_{ij}[\mathbf{S}_i\cdot\mathbf{S}_j-(3/r_{ij}^2)(\mathbf{S}_i\cdot\mathbf{r}_{ij})(\mathbf{S}_j\cdot\mathbf{r}_{ij})] \\ &+(a/12)\textstyle\sum_j(S_{j\xi}^4+S_{j\eta}^4+S_{j\rho}^4). \qquad (36)\end{aligned}$$

The quantity D_{ij} is the dipole interaction strength, equal to the product of the magnetic moments divided by the cube of the distance between the ith and jth ions. The vector $\mathbf{r}_{ij}$ is the vector from the ith to the jth ion. The quantity a is the (average) single-ion anisotropy coefficient and is a function of λ, ρ, and the crystalline field splitting parameter δ.[3] The crystallographic axes are designated by ξ, η, ρ. We assume the sample to be an ellipsoid of revolution. We designate this symmetry axis by z', and the two orthogonal axes by x' and y'. Finally, we introduce a third coordinate system x, y, z, with the z axis parallel to the equilibrium direction of the magnetization $\mathbf{M}$, as determined by the externally applied field.

To summarize: ξ, η, ρ are along crystallographic [100], [010], and [001] directions. z' is along the symmetry axis of an ellipsoidal sample. z is parallel to equilibrium direction of $\mathbf{M}$.

The rotation matrices which relate these coordinate systems are defined by

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\beta_1'' & \beta_2'' & \beta_3''\\ \beta_1' & \beta_2' & \beta_3'\\ \beta_1 & \beta_2 & \beta_3\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix} =\begin{pmatrix}\alpha_1'' & \alpha_2'' & \alpha_3''\\ \alpha_1' & \alpha_2' & \alpha_3'\\ \alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}\xi\\ \eta\\ \rho\end{pmatrix}. \qquad (37)$$

The direction cosines of the equilibrium magnetization with respect to the crystal axes then are α_1, α_2, α_3. Also

$$\begin{pmatrix}S_\xi\\S_\eta\\S_\rho\end{pmatrix}=\begin{pmatrix}\alpha_1'' & \alpha_1' & \alpha_1\\ \alpha_2'' & \alpha_2' & \alpha_2\\ \alpha_3'' & \alpha_3' & \alpha_3\end{pmatrix}\begin{pmatrix}S_x\\S_y\\S_z\end{pmatrix}. \qquad (38)$$

We introduce the spin-wave variables by the substitution

$$S_j^+=S_{jx}+iS_{jy}=(2S/N)^{\frac{1}{2}}\textstyle\sum_{\mathbf{k}} a^+(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}_j}, \qquad (39)$$

$$S_j^-=S_{jx}-iS_{jy}=(2S/N)^{\frac{1}{2}}\textstyle\sum_{\mathbf{k}} a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}_j}, \qquad (40)$$

$$S_{jz}=S-S_j^-S_j^+/2S. \qquad (41)$$

If these equations are inserted into the Hamiltonian (36) and only those terms not greater than second order in the spin-wave operators are kept, the result, expressed in the notation of Van Kranendonk and Van Vleck,[11] becomes

$$\begin{aligned}\mathcal{H}_0=\tfrac{1}{4}\textstyle\sum_{\mathbf{k}}\{&A(\mathbf{k})[a^+(\mathbf{k})a(\mathbf{k})+a(\mathbf{k})a^+(\mathbf{k})] \\ &+B(\mathbf{k})a(\mathbf{k})a(-\mathbf{k})+B^*(\mathbf{k})a^+(\mathbf{k})a^+(-\mathbf{k})\}, \qquad (42)\end{aligned}$$

where

$$\begin{aligned}A(\mathbf{k})=2\gamma\hbar(H_z+H_A)&+4JSa^2k^2 \\ &+4\pi\gamma\hbar M_0[A_{xx}(\mathbf{k})+A_{yy}(\mathbf{k})] \\ &+(8\pi V/NS)\mathbf{M}_0\cdot(\tfrac{1}{3}\mathbf{I}-\mathbf{N})\cdot\mathbf{M}_0, \qquad (43)\end{aligned}$$

$$B(\mathbf{k})=4\pi\gamma\hbar M_0[A_{xx}(\mathbf{k})-A_{yy}(\mathbf{k})+2iA_{xy}(\mathbf{k})], \qquad (44)$$

$$\begin{aligned}A(0)=2\gamma\hbar(H_z+H_A)+(4\pi V/NS)M_0^2(\tfrac{1}{3}\mathbf{I}-\mathbf{N}) \\ \cdot(2\boldsymbol{\beta}\boldsymbol{\beta}-\boldsymbol{\beta}'\boldsymbol{\beta}'-\boldsymbol{\beta}''\boldsymbol{\beta}''), \qquad (45)\end{aligned}$$

$$\begin{aligned}B(0)=(4\pi VM_0^2/NS)(\mathbf{N}-\tfrac{1}{3}\mathbf{I}) \\ \cdot(\boldsymbol{\beta}'\boldsymbol{\beta}'+\boldsymbol{\beta}''\boldsymbol{\beta}''+2i\boldsymbol{\beta}'\boldsymbol{\beta}''), \qquad (46)\end{aligned}$$

where H_z is the z component of the external field, H_A is the effective anisotropy field, $\mathbf{N}$ and $\mathbf{I}$ are the sample demagnetizing tensor and the unit tensor, respectively, $\boldsymbol{\beta}$ is the vector with components β_1, β_2, β_3 as defined in Eq. (37) and $\boldsymbol{\beta}'$ and $\boldsymbol{\beta}''$ are defined similarly to $\boldsymbol{\beta}$. The quantity $\boldsymbol{\beta}\boldsymbol{\beta}$ is the dyadic tensor product. The tensor components $A_{xx}(\mathbf{k})$, $A_{yy}(\mathbf{k})$ and $A_{xy}(\mathbf{k})$ are lattice sums which have been computed by Cohen and Keffer,[12] who show that for wavelengths small compared to crystal dimensions

$$A_{xx}(\mathbf{k})=k_x^2/k^2-\tfrac{1}{3}, \qquad (47)$$

$$A_{yy}(\mathbf{k})=k_y^2/k^2-\tfrac{1}{3}, \qquad (48)$$

$$A_{xy}(\mathbf{k})=k_xk_y/k^2. \qquad (49)$$

The Holstein-Primakoff transformation which diagonalizes the Hamiltonian is

$$a^+(\mathbf{k})=\cosh(\lambda_k/2)\sigma_1^+(\mathbf{k})-\sinh(\lambda_k/2)e^{2i\phi_k}\sigma_1(-\mathbf{k}), \qquad (50)$$

$$a(\mathbf{k})=\cosh(\lambda_k/2)\sigma_1(\mathbf{k})-\sinh(\lambda_k/2)e^{-2i\phi_k}\sigma_1^+(-\mathbf{k}), \qquad (51)$$

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[11] J. Van Kranendonk and J. H. Van Vleck, Revs. Modern Phys. **30**, (1958).

[12] M. H. Cohen and F. Keffer, Phys. Rev. **99**, 1128 and 1135 (1955).

where

$$\hbar\omega(\mathbf{k})\cosh\lambda_k=\tfrac{1}{2}A(\mathbf{k}), \tag{52}$$

$$\hbar\omega(\mathbf{k})\sinh\lambda_k=\tfrac{1}{2}|B(\mathbf{k})|, \tag{53}$$

$$B(\mathbf{k})=|B(\mathbf{k})|e^{2i\phi_k}, \tag{54}$$

$$\hbar\omega(\mathbf{k})=\tfrac{1}{2}[A^2(\mathbf{k})-B(\mathbf{k})B^*(\mathbf{k})]^{\frac{1}{2}}. \tag{55}$$

The average Hamiltonian thereby reduces to

$$\mathcal{H}_0=\textstyle\sum_{\mathbf{k}}[\sigma_1(\mathbf{k})\sigma_1^+(\mathbf{k})+\tfrac{1}{2}]\hbar\omega(\mathbf{k}), \tag{56}$$

where $\sigma_1(\mathbf{k})\sigma_1^+(\mathbf{k})$ is the number operator for magnons of wave vector $\mathbf{k}$.

We now consider the scattering effect of the random deviations of the Hamiltonian from its average value. The variations in dipolar interactions give rise to linewidths of the order of a tenth of an oersted, as can be seen easily from the discussion of Sec. 3, and hence, will be ignored. Variations in exchange interactions contribute isotropic terms which commute with the total spin and which therefore cannot scatter the $\mathbf{k}=0$ mode; we therefore do not consider these variations. The remaining perturbations arise from variations in spin-spin, spin-orbit, or crystalline field parameters. For an ion with crystalline field axis along [111] the first- and second-order perturbation of spin-spin and spin-orbit interaction terms, respectively, contribute terms to the Hamiltonian of the form $CS^2_{[111]}$, as discussed in Sec. 3. Yosida and Tachiki,[4] by a direct perturbation treatment of the orbital states of an Fe^{2+} ion split by the cubic and trigonal fields appropriate to an octahedrally coordinated site, show that C has the value

$$C=-(3\rho+7.5\lambda^2/10\delta), \tag{57}$$

where ρ is defined in the second paragraph of Sec. 4. For Fe^{2+} the parameter ρ has a value of approximately 0.95 cm^{-1} whereas $\lambda\simeq10^2$ cm^{-1} and $\delta\simeq10^3$ cm^{-1}, so that the spin-orbit contribution to C is larger than the spin-spin contribution for this ion. In the case of disordered ferrites we must consider two values of C appropriate to the two different types of ions; i.e., C_A and C_B. In order to write these terms in the Hamiltonian explicitly, we define several useful quantities: $\mathbf{E}_1$ is a unit vector along [111], $\mathbf{E}_2$ is a unit vector along $[\bar{1}\bar{1}1]$, $\mathbf{E}_3$ is a unit vector along $[1\bar{1}\bar{1}]$, $\mathbf{E}_4$ is a unit vector along $[\bar{1}1\bar{1}]$, $P_{tj}=+1$ if the site j has crystalline field parallel to $\mathbf{E}_t$; zero otherwise, and $q_j=+1$ if the site j contains an A atom, and $q_j=0$ if the site j contains a B atom.

The total Hamiltonian can then be written as

$$\mathcal{H}=\mathcal{H}_0+\textstyle\sum_{j,t}[q_jC_A+(1-q_j)C_B]P_{tj}(\mathbf{E}_t\cdot\mathbf{S}_j)^2 \\ -\textstyle\sum_{j,t}[cC_A+(1-c)C_B]P_{tj}(\mathbf{E}_t\cdot\mathbf{S}_j)^2, \tag{58}$$

or

$$\mathcal{H}=\mathcal{H}_0+(C_A-C_B)\textstyle\sum_{j,t}(q_j-c)P_{tj}(\mathbf{E}_t\cdot\mathbf{S}_j)^2. \tag{59}$$

The third term in Eq. (58) is simply the average value of the second term and is obtained by replacing q_j by its average value c; this term must be subtracted in order that the average value of $\mathcal{H}$ shall be $\mathcal{H}_0$, as defined. In Eq. (59) we see that the perturbation terms vanish if the two ions become identical ($C_A=C_B$) and that it is the deviation from the average distribution (q_j-c) which is significant.

From Eqs. (37) it is evident that $\boldsymbol{\alpha}''$, with components α_1'', α_2'', α_3'' in the ξ, η, ρ coordinate system, is a unit vector along the x axis; a similar relation exists for $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}$ so that we may write

$$\mathbf{S}_j=\boldsymbol{\alpha}''S_{jx}+\boldsymbol{\alpha}'S_{jy}+\boldsymbol{\alpha}S_{jz}, \tag{60}$$

and then from Eq. (41)

$$\mathbf{S}_j=\tfrac{1}{2}\boldsymbol{\alpha}''(S_j^++S_j^-)+(1/2i)(S_j^+-S_j^-)\boldsymbol{\alpha}' \\ +\boldsymbol{\alpha}(S-S_j^-S_j^+/2S). \tag{61}$$

If this equation is introduced into Eq. (59), the result is

$$\begin{aligned}\mathcal{H}=\mathcal{H}_0&+(C_A-C_B)\textstyle\sum_{j,t}P_{tj}(q_j-c)\\ &\times\{S^2(\mathbf{E}_t\cdot\boldsymbol{\alpha})^2+\tfrac{1}{4}[\mathbf{E}_t\cdot(\boldsymbol{\alpha}''-i\boldsymbol{\alpha}')S_j^+]^2-S_j^-S_j^+(\mathbf{E}_t\cdot\boldsymbol{\alpha})^2\\ &+\tfrac{1}{4}[\mathbf{E}_t\cdot(\boldsymbol{\alpha}''+i\boldsymbol{\alpha}')S_j^-]^2\\ &+\tfrac{1}{2}[1-(\mathbf{E}_t\cdot\boldsymbol{\alpha})^2](S_j^+S_j^-+S_j^-S_j^+)\\ &+S(\mathbf{E}_t\cdot\boldsymbol{\alpha})(\mathbf{E}_t\cdot\boldsymbol{\alpha}'')(S_j^++S_j^-)\\ &\qquad+iS(\mathbf{E}_t\cdot\boldsymbol{\alpha})(\mathbf{E}_t\cdot\boldsymbol{\alpha}')(S_j^--S_j^+)\}.\end{aligned} \tag{62}$$

If the spin-wave variables are introduced through Eqs. (39)–(41) and the subsequent Holstein-Primakoff transformation (50)–(51), the resulting Hamiltonian contains terms of zero, first, and second order in the spin-wave variables. The first-order terms are easily eliminated by the canonical transformation

$$\sigma_1^+(\mathbf{k})+d^*(\mathbf{k})=\sigma^+(\mathbf{k}), \tag{63}$$

$$\sigma_1(\mathbf{k})+d(\mathbf{k})=\sigma(\mathbf{k}), \tag{64}$$

where $d(\mathbf{k})$ is a constant.

Finally, after considerable algebra, the introduction of the final spin-wave variables defined in (63) and (64) transforms the Hamiltonian to the form

$$\begin{aligned}\mathcal{H}=\mathcal{H}_0+\textstyle\sum_{\mathbf{k},\mathbf{k}'}\{&f_1(\mathbf{k},\mathbf{k}')[\sigma^+(\mathbf{k})\sigma(\mathbf{k}')+\sigma(\mathbf{k}')\sigma^+(\mathbf{k})]\\ &+\sigma^+(\mathbf{k})\sigma^+(\mathbf{k}')f_2^*(\mathbf{k},\mathbf{k}')\\ &+\sigma(\mathbf{k})\sigma(\mathbf{k}')f_2(\mathbf{k},\mathbf{k}')\},\end{aligned} \tag{65}$$

where

$$\begin{aligned}f_1(\mathbf{k},\mathbf{k}')=&\textstyle\sum_{j,t}(S/2N)P_{tj}(q_j-c)\\ &\times(C_A-C_B)e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_j}\\ &\times\{G_t[\sinh(\lambda_k/2)\cosh(\lambda_{k'}/2)e^{-2i\phi_k}\\ &+\cosh(\lambda_k/2)\cosh\lambda_{k'}/2\\ &+\cosh(\lambda_k/2)\sinh(\lambda_{k'}/2)e^{2i\phi_{k'}}\\ &+\sinh(\lambda_k/2)\sinh(\lambda_{k'}/2)e^{2i\phi_{k'}-2i\phi_k}]\\ &+L_t[-\cosh(\lambda_{k'}/2)\sinh(\lambda_k/2)e^{-2i\phi_k}\\ &+\cosh(\lambda_k/2)\cosh\lambda_{k'}/2\\ &-\cosh(\lambda_k/2)\sinh(\lambda_{k'}/2)e^{2i\phi_{k'}}\\ &+\sinh(\lambda_k/2)\sinh(\lambda_{k'}/2)e^{2i\phi_{k'}-2i\phi_k}]\\ &+2iQ_t[\sinh(\lambda_k/2)\cosh(\lambda_{k'}/2)e^{-2i\phi_k}\\ &\qquad-e^{2i\phi_{k'}}\cosh(\lambda_k/2)\sinh\lambda_{k'}/2]\},\end{aligned} \tag{66}$$

$$f_2(\mathbf{k},\mathbf{k}')=\frac{S}{2N}\sum_{j,t}P_{tj}(q_j-c)(C_A-C_B)e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_j}$$
$$\times\{[G_t+L_t][\cosh(\lambda_k/2)\cosh\lambda_{k'}/2$$
$$+2\cosh(\lambda_k/2)\sinh(\lambda_{k'}/2)e^{-2i\phi_{k'}}$$
$$+\sinh(\lambda_k/2)\sinh(\lambda_{k'}/2)e^{-2i\phi_k-2i\phi_{k'}}]$$
$$+2iQ_t[\cosh(\lambda_k/2)\cosh\lambda_{k'}/2$$
$$-\sinh(\lambda_k/2)\sinh(\lambda_{k'}/2)e^{-2i\phi_k-2i\phi_{k'}}]\},\quad(67)$$

and

$$G_t=(\mathbf{E}_t\cdot\boldsymbol{\alpha}'')^2-(\mathbf{E}_t\cdot\boldsymbol{\alpha})^2,\quad(68)$$

$$L_t=(\mathbf{E}_t\cdot\boldsymbol{\alpha}')^2-(\mathbf{E}_t\cdot\boldsymbol{\alpha})^2,\quad(69)$$

$$2iQ_t=(\mathbf{E}_t\cdot\boldsymbol{\alpha}')(\mathbf{E}_t\cdot\boldsymbol{\alpha}'').\quad(70)$$

The matrix element for scattering from the mode $\mathbf{k}=0$ to $\mathbf{k}'$ is simply $2f_1(0,\mathbf{k}')$, whence

$$M_{0k}=(S/N)(C_A-C_B)\textstyle\sum_{j,t}P_{tj}(q_j-c)e^{i\mathbf{k}\cdot\mathbf{r}_j}$$
$$\times[\cosh(\lambda_k/2)(G_t+L_t)$$
$$+\sinh(\lambda_k/2)e^{2i\phi_k}(G_t-L_t-2iQ_t)].\quad(71)$$

Squaring M_{0k}, summing over $r_j-r_{j'}=l$ and taking the ensemble average with the assumption that there is no correlation of the q_j's from ion to ion, gives

$$\langle|M_{0k}|^2\rangle_{\mathrm{av}}=(S^2/4N)(C_A-C_B)^2c(1-c)$$
$$\times\textstyle\sum_t[\cosh^2(\lambda_k/2)(L_t+G_t)^2$$
$$+|G_t-L_t-2iQ_t|^2\sinh^2\lambda_k/2$$
$$+\cos2\phi_k\cosh(\lambda_k/2)\sinh(\lambda_k/2)(G_t^2-L_t^2)$$
$$+4\sin2\phi_k\sinh(\lambda_k/2)\cosh(\lambda_k/2)$$
$$\times Q_t(G_t+L_t)].\quad(72)$$

It is evident immediately from (72) that the loss depends on the concentration solely through the factor $c(1-c)$; the loss therefore goes to zero for $c=0$ or 1, as it should, and is a maximum at $c=\frac{1}{2}$, which is to be expected.

If we now insert the specific values of $\mathbf{E}_t$ into (72) via Eqs. (68)–(70), we find λ_{0k} reduces to the expected form, Eq. (1). That is, from (25)

$$\lambda_{0k}=\lambda_{0k}^{(0)}+\lambda_{0k}^{(1)}(\alpha_1^2\alpha_2^2+\alpha_1^2\alpha_3^2+\alpha_2^2\alpha_3^2),\quad(73)$$

where

$$\lambda_{0k}^{(0)}=\frac{S^2V(c-c^2)(C_A-C_B)^2}{9\pi^2\hbar^2N}$$
$$\times\int_0^{2\pi}\int_0^{\pi}\left[\sinh^2\frac{\lambda_k}{2}k^2\frac{\partial k}{\partial\omega(\mathbf{k})}\right]_{\omega(\mathbf{k})=\omega(0)}$$
$$\times\sin\theta_kd\theta_kd\phi_k,\quad(74)$$

$$\lambda_{0k}^{(1)}=\frac{S^2V(c-c^2)(C_A-C_B)^2}{\pi^2\hbar^2N}$$
$$\times\int_0^{2\pi}\int_0^{\pi}\left[\left(\cosh^2\frac{\lambda_k}{2}+\tfrac{1}{9}\sinh^2\frac{\lambda_k}{2}\right)\right.$$
$$\left.\times k^2\frac{\partial k}{\partial\omega(\mathbf{k})}\right]_{\omega(\mathbf{k})=\omega(0)}\sin\theta_kd\theta_kd\phi_k,\quad(75)$$

$$\theta_k=\sin^{-1}[(k_x^2+k_y^2)/k^2]^{\frac{1}{2}},\quad(76)$$

$$\phi_k=\tan^{-1}k_x/k_y.\quad(77)$$

Those parts of Eq. (72) containing terms linear in $\sin2\phi_k$ and $\cos2\phi_k$ vanish when integrated over the variable ϕ_k and so have been omitted in Eqs. (74) and (75).

Of particular interest are the dependence of λ_{0k} on static field direction for a spherical sample, and the dependence of λ_{0k} on sample shape and resonant frequency for a field along the symmetry axis of the sample. When the field is along a symmetry axis Eqs. (43) and (44) become

$$\cosh\lambda_k=\frac{A(\mathbf{k})}{2\hbar\omega(\mathbf{k})}$$
$$=\left[\frac{2\gamma\hbar(H_z+H_A)+4JSa^2k^2+4\pi\gamma\hbar M_0\sin^2\theta_k}{2\hbar\omega(\mathbf{k})}\right.$$
$$\left.-\frac{4\pi\gamma\hbar M_0N_z}{\hbar\omega(\mathbf{k})}\right],\quad(78)$$

$$\sinh\lambda_k=\frac{|B(\mathbf{k})|}{2\hbar\omega(\mathbf{k})}=\frac{2\pi\gamma\hbar M_0\sin^2\theta_k}{\hbar\omega(\mathbf{k})}.\quad(79)$$

To evaluate the integrals in Eqs. (74) and (75) we rewrite these equations in the following form:

$$\frac{\lambda_{0k}^{(0)}}{\gamma}=S^2(c-c^2)\left(\frac{C_A-C_B}{\gamma\hbar}\right)^2\left(\frac{4\pi M_0}{He^3}\right)^{\frac{1}{2}}$$
$$\times\left\{\int_0^{\pi}\left(\frac{He^3}{4\pi M_0}\right)^{\frac{1}{2}}\frac{2V\gamma}{15\pi N}\left[\left(\cosh^2\frac{\lambda_k}{2}+\tfrac{2}{3}\sinh^2\frac{\lambda_k}{2}\right)\right.\right.$$
$$\left.\times\left(\frac{k^2dk}{\partial\omega(\mathbf{k})}\right)\right]_{\omega(\mathbf{k})=\omega(0)}\sin\theta_kd\theta_k-\int_0^{\pi}\left(\frac{He^3}{4\pi M_0}\right)^{\frac{1}{2}}$$
$$\left.\times\frac{2V\gamma}{15\pi N}\left[k^2\frac{dk}{\partial\omega(\mathbf{k})}\right]_{\omega(\mathbf{k})=\omega(0)}\sin\theta_kd\theta_k\right\}$$
$$\equiv S^2(c-c^2)\left(\frac{C_A-C_B}{\gamma\hbar}\right)^2\left(\frac{4\pi M_0}{He^3}\right)^{\frac{1}{2}}(I_1-I_2),\quad(80)$$

$$\frac{\lambda_{0k}^{(1)}}{\gamma}=S^2(c-c^2)\left(\frac{C_A-C_B}{\gamma\hbar}\right)^2\left(\frac{4\pi M_0}{He^3}\right)^{\frac{1}{2}}$$
$$\times\left\{\int_0^{\pi}\left(\frac{He^3}{4\pi M_0}\right)^{\frac{1}{2}}\frac{4V\gamma}{3\pi N}\left[\left(\cosh^2\frac{\lambda_k}{2}+\tfrac{2}{3}\sinh^2\frac{\lambda_k}{2}\right)\right.\right.$$
$$\left.\times\left(k^2\frac{\partial k}{\partial\omega(\mathbf{k})}\right)\right]_{\omega(\mathbf{k})=\omega(0)}\sin\theta_kd\theta_k$$
$$\left.+\int_0^{\pi}\left(\frac{He^3}{4\pi M_0}\right)^{\frac{1}{2}}\frac{2V\gamma}{3\pi N}k^2\frac{\partial k}{\partial\omega(\mathbf{k})}\sin\theta_kd\theta_k\right\}$$
$$=S^2(c-c^2)\left(\frac{C_A-C_B}{\gamma\hbar}\right)^2\left(\frac{4\pi M_0}{He^3}\right)^{\frac{1}{2}}(10I_1+5I_2).\quad(81)$$

The integrals I_1 and I_2 are dimensionless functions depending only on sample shape and on $\omega(0)/4\pi\gamma M_0$,

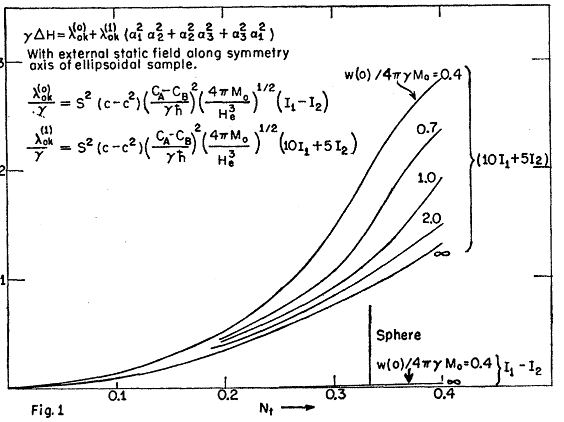

FIG. 1. Dependence of the resonance linewidth, ΔH, on the transverse demagnetizing coefficient of a sample which is an ellipsoid of revolution.

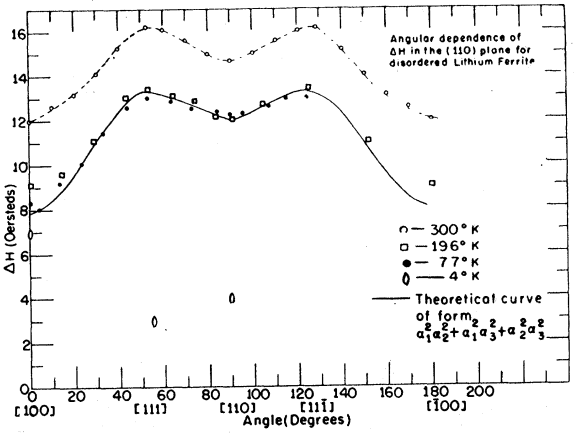

FIG. 2. Dependence of the resonance linewidth, ΔH, on direction of the applied static field in the (110) plane for spherical samples of disordered lithium ferrite.

where $\omega(0)$ is the resonant frequency. The first integral, I_1, has been evaluated numerically by Clogston, Suhl, Walker, and Anderson[1]; our integral I_1, is equal to $I/45\pi$ in their notation. The integral I_2 has been evaluated by us[5] in terms of Legendre elliptic functions of the first, second, and third kind. The relevant quantities (I_1-I_2) and $(10I_1+5I_2)$ are plotted in Fig. 1 as a function of reduced resonant frequency, $\omega(0)/4\pi\gamma M_0$, and of the transverse demagnetizing coefficient of an ellipsoidal sample with applied static field along the symmetry axis. These two graphs, together with Eqs. (1), (80), and (81), then determine λ_{0k} directly.

With the representative values $S=2$, $c=\frac{1}{2}$, $(4\pi M_0)^{\frac{1}{2}}=70$, $H_e=1.8\times10^6$, and $C_A-C_B=2\times10^{-15}$ ergs, the ordinate 0.1 in Fig. 1 corresponds approximately to a value of $\lambda_{0k}{}^1/\gamma$ of 34 oersteds, where it is to be recalled that S is the average spin on a lattice site, c is the fractional concentration of A ions, H_e is the exchange field, and C_A, C_B are coupling parameters for A ions and B ions respectively, and are defined in Eq. (57). With these values of the sample parameters and for a spherical sample with $\omega(0)/4\pi\gamma M_0=2.0$, the quantities $\lambda_{0k}{}^{(0)}/\gamma$ and $\lambda_{0k}{}^{(1)}/\gamma$ have the values 0.3 oersted and 36 oersteds, respectively, so that the line width is approximately 12.3 oersteds in the [111] direction and 0.3 oersted in the [001] direction.

5. DISCUSSION

In Fig. 2 we show the experimental observations of Schnitzler, Folen, and Rado[13] on disordered lithium ferrite, and a theoretical curve of the form $\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2$ which has been fitted to the data obtained at 77°K to compare the predicted angular dependence. By fitting $\lambda_{0k}{}^{(1)}$ to the anisotropy of the experimental data at each temperature, we find

$$\frac{\lambda_{0k}{}^{(1)}}{\gamma} \simeq \begin{cases} 15 & \text{oersteds at } 77°\text{K} \\ 14 & \text{oersteds at } 196°\text{K} \\ 14 & \text{oersteds at } 300°\text{K}. \end{cases}$$

At 4°K the data does not fit the predicted curve, and it is probable that other processes dominate in this region, as discussed in Sec. 1. At the higher temperatures the results are reasonably constant with temperature, as theoretically predicted. The experimental values of $\lambda_{0k}{}^{(1)}$ are in good agreement with the value of $\lambda_{0k}{}^{(1)}$ estimated by taking the parameter C_A for the Fe^{3+} ion as equal to that of Fe^{2+} ions (because each has the same order of magnetocrystalline anisotropy), taking $C_B=0$ for lithium ions, and $c=0.8$; in this way we estimate for a spherical sample with an external field $H_0=8\pi M_0$ (approximately 8×10^3 oersteds)

$$\frac{\lambda_{0k}{}^{(1)}}{\gamma}(\text{theor.}) \simeq 16 \text{ oersteds.}$$

Paradoxically the linewidth of ordered lithium ferrite is comparatively large, rather than being small (comparable to that in yttrium iron garnet) as we would expect. However, the cubically symmetric component of the linewidth does vanish with the onset of long-range ionic order.

Further comparison of the results with experiment is difficult until some method such as that of Fletcher, Le Craw, and Spencer is used in ferrites to measure λ_{0k}, rather than merely ΔH; a direct test of the results shown in Fig. 1 should be feasible, as should a test of the predicted concentration dependence of $\lambda_{0k}{}^{(1)}$. Nevertheless, the good agreement with experiment of the general order of magnitude of the loss, and particularly its predicted directional dependence, seem to indicate that this effect is the dominant source of loss in many disordered ferrites.

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