The term with $|g_{E1}|^2+|g_{M1}|^2$ is very large for λ_1 or λ_2 at or near $\frac{1}{2}$, which corresponds to one of the electrons coming out in the same direction are the positron. This behavior of the energy spectrum of pairs coming from internal conversion when the real photon decay is allowed is also well known.[2] However, in view of the fact that $|g_{E1}|^2+|g_{M1}|^2$ is very small or zero, this term is probably unimportant.

We conclude that present information about $\mu\rightarrow 3e$ does not give such sensitive restrictions on the $\mu e\gamma$ form factors as other measurements, such as $\mu+p\rightarrow e+p$. However, they are consistent with the vanishing of all such form factors, and future searches for $\mu\rightarrow 3e$ may be sensitive enough to lower the limits on the form factors to those values predicted by the intermediate boson theory.

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Pion-Pion Interactions in τ and τ' Decays*

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The final-state interactions in τ and τ' decays of K^+ mesons are studied by means of a Mandelstam representation. It is assumed that only the S-wave pion-pion scattering amplitude is large enough to have an appreciable imaginary part. Coupled, linear, singular, integral equations are found for the amplitudes describing $K^++\pi\rightarrow\pi+\pi$ in the physical and unphysical regions. From the solutions to these equations the τ and τ' decay matrix elements may be constructed. In an approximation the equations are solved in terms of pion-pion phase shifts. Comparison with experiment is made using Chew and Mandelstam's solution of the corresponding nonlinear equations for $\pi-\pi$ scattering. Consistency with experiment is found for values of the coupling constant implying repulsive $T=0$ and $T=2$ phase shifts. Independently of the approximation the equations show that the implications in τ and τ' decay of the rule, $\Delta T=\frac{1}{2}$, $\frac{3}{2}$, are identical with those of the $\Delta T=\frac{1}{2}$ rule. Hence the energy spectra in τ' decay cannot be a critical test of the $\Delta T=\frac{1}{2}$ rule. Also, independently of the approximation, it is shown that the decay matrix element cannot be expanded in a series of integral powers of the pion kinetic energies.

1. INTRODUCTION

IN a recent analysis of 900 τ^+ decays of K^+ mesons, $K^+\rightarrow\pi^++\pi^++\pi^-$, it was found that the π^- energy spectrum differs significantly from that predicted from the density of states alone.[1] The minimum pion wavelength in this process is about one pion Compton wavelength. If the weak interaction proceeds in some way through the heavy fermion pairs and has a range determined by the intermediate mass, one would expect very little dependence upon the pion momenta in the matrix element, because the pion wavelength is large compared to the radius of interaction. Hence the observed deviation from a constant matrix element is some evidence for final-state interactions which extend the spatial region of interaction for the outgoing pions. This is an attractive possibility in view of the current interest in pion-pion interactions. The system of three low-energy pions should be an ideal analyzer for such effects.

Thomas and Holladay[2] have investigated the effect of an attractive, $T=2$, pion-pion interaction in τ decay by using Watson's final-state interaction formalism.[3] This method is suited to discussion of the final-state scattering of a single pair of particles. It is applicable to the problem of τ^+ decay with a $T=2$, $\pi-\pi$ force because the state of two π^+ is pure $T=2$ and the other pairs, $\pi^+\pi^-$, are predominantly $T=0$ in the S wave. If, however, there are strong $T=0$ pion-pion effects, then a formulation is required which is capable of dealing simultaneously with interactions between various pairs. The Mandelstam representation can serve as the basis for such an analysis, just as an ordinary single variable dispersion relation can be used to discuss a single final-state interaction.

In the present work Mandelstam representations are assumed for the τ and τ' decay amplitudes.[3a] It is assumed that only the S-wave pion-pion scattering amplitudes are large (have an imaginary part). Linear integral equations are obtained for the $\tau+\pi\rightarrow\pi+\pi$ scattering amplitudes in the physical and unphysical regions. These equations involve the pion-pion S-wave phase shifts. From the solutions the Mandelstam representations for τ and τ' decays may be constructed. In general, two new parameters are needed to characterize τ decay. The $\Delta T=\frac{1}{2}$, $\frac{3}{2}$ rule can be applied to yield a simpler set of equations with only one parameter, which, if solved,

* This work was supported in part by the U. S. Atomic Energy Commission and in part by the Research Committee of the University of Wisconsin with funds provided by the Wisconsin Alumni Research Foundation.

[1] S. McKenna, S. Natali, M. O'Connell, J. Tietge, and N. C. Varshneya, Nuovo cimento **10**, 763 (1958).

[2] B. S. Thomas and W. G. Holladay, Phys. Rev. **115**, 1329 (1958).

[3] Also, A. N. Mitra has investigated the effects of a $T=2$ pion-pion resonance in the τ decay [Nuclear Phys. **6**, 404 (1958)].

[3a] N. N. Khuri and S. B. Treiman have also considered this problem (Phys. Rev. to be published).

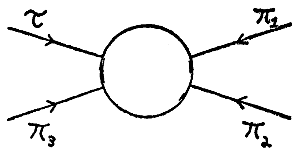

FIG. 1. The τ-decay interaction, $\tau+\pi\rightarrow\pi+\pi$.

should give τ and τ' spectra free of new parameters. The implications of $\Delta T=\frac{1}{2}$ and $\Delta T=\frac{1}{2}$, $\frac{3}{2}$ are identical in our approach. In an approximation these equations have been solved in terms of pion-pion phase shifts and integrals over pion-pion phase shifts. The phases themselves are obtained from nonlinear integral equations which follow from the same approach to the $\pi-\pi$ problem.[4] These equations (for large S waves) have been solved[5] by Chew, Mandelstam, and Noyes, and we have used their solutions in analyzing our results.

2. THE MANDELSTAM REPRESENTATION

We begin by considering the problem of $\tau+\pi\rightarrow\pi+\pi$ scattering. The variables s_1, s_2, and s_3 are defined by drawing all three pion momenta inward (Fig. 1).

$$
\begin{aligned}
s_1&=-(p_2+p_3)^2,\\
s_2&=-(p_1+p_3)^2,\\
s_3&=-(p_1+p_2)^2.
\end{aligned}
\tag{1}
$$

The three pions are referred to as π_1, π_2, and π_3. The assignment of charge will be deferred until the next section. For the process $\tau+\pi_3\rightarrow\pi_1+\pi_2$ the variables s_1, s_2, and s_3 are expressed in terms of the initial three momentum, q', the final momentum, q, and a scattering angle, θ, in the center-of-mass coordinate system,

$$
\begin{aligned}
s_3&=4(q^2+\mu^2),\\
s_2&=(M_K^2/2)-(\mu^2/2)-2q^2+2qq'\cos\theta,\\
s_1&=(M_K^2/2)-(\mu^2/2)-2q^2-2qq'\cos\theta,
\end{aligned}
$$

with

$$
2(q^2+\mu^2)^{\frac{1}{2}}=(q'^2+\mu^2)^{\frac{1}{2}}+(q'^2+M_K^2)^{\frac{1}{2}}. \tag{2}
$$

Here M_K is the mass of K^+ and μ is the pion mass. We shall take, for the moment, $M_K=3\mu$. This will simplify the contributions to the unphysical region. It will eventually be argued that the τ-decay matrix element does not vary strongly with the K-meson mass. For $M_K=3\mu$ the physical region extends to $q=(3\mu^2)^{\frac{1}{2}}$. Between $q=(3\mu^2)^{\frac{1}{2}}$ and $q=0$, q' is imaginary.

The amplitude $A(s_1s_2s_3)$ for the reaction $\tau+\pi_3\rightarrow\pi_1+\pi_2$ (with s_3 the energy variable) is given by,

$$
\langle\pi_1\pi_2\,\text{out}|\pi_3\tau\,\text{in}\rangle
=i(2\pi)^5\delta^4(p_\tau+p_3-p_1-p_2)\frac{A(s_1,s_2,s_3)}{(\omega_\tau\omega_1\omega_2\omega_3)^{\frac{1}{2}}}. \tag{3}
$$

The continuation of this function $A(s_1,s_2,s_3)$ to all values of s_1, s_2, and s_3, constrained by the condition $s_1+s_2+s_3=3\mu^2+M_K^2\approx 12\mu^2$, provides amplitudes for other channels of $\tau+\pi\rightarrow\pi+\pi$ scattering and also for τ decay. Since s_1, s_2, and s_3 stand for energies of two-pion systems in their center-of-mass system, we expect the analytic structure to be the same as for $\pi-\pi$ scattering.[4] In the case of each s the lowest mass intermediate state is that of two pions. We write therefore,

$$
\begin{aligned}
A(s_1,s_2,s_3)
&=\frac{1}{\pi^2}\int_{(2\mu)^2}^{\infty}ds_1'\int_{(2\mu)^2}^{\infty}ds_2'\frac{\rho_{12}(s_1',s_2')}{(s_1'-s_1)(s_2'-s_2)}\\
&\quad+\frac{1}{\pi^2}\int_{(2\mu)^2}^{\infty}ds_2'\int_{(2\mu)^2}^{\infty}ds_3'\frac{\rho_{23}(s_2',s_3')}{(s_2'-s_2)(s_3'-s_3)}\\
&\quad+\frac{1}{\pi^2}\int_{(2\mu)^2}^{\infty}ds_3'\int_{(2\mu)^2}^{\infty}ds_1'\frac{\rho_{13}(s_1',s_3')}{(s_3'-s_3)(s_1'-s_1)}.
\end{aligned}
\tag{4}
$$

This representation may be rearranged to give

$$
\begin{aligned}
A(s_1,s_2,s_3)
&=\frac{1}{\pi^2}\int ds_1'\frac{1}{(s_1'-s_1)}\bigg[\int ds_2'\frac{\rho_{12}(s_1',s_2')}{s_1'+s_2'+s_3-3\mu^2-M_K^2}\\
&\qquad+\int ds_3'\frac{\rho_{13}(s_1',s_3')}{s_1'+s_3'+s_2-3\mu^2-M_K^2}\bigg]\\
&\quad+\frac{1}{\pi^2}\int ds_2'\frac{1}{(s_2'-s_2)}\bigg[\int ds_3'\frac{\rho_{23}(s_2',s_3')}{s_2'+s_3'+s_1-3\mu^2-M_K^2}\\
&\qquad+\int ds_1'\frac{\rho_{12}(s_2',s_1')}{s_2'+s_1'+s_3-3\mu^2-M_K^2}\bigg]\\
&\quad+\frac{1}{\pi^2}\int ds_3'\frac{1}{(s_3'-s_3)}\bigg[\int ds_1'\frac{\rho_{13}(s_1',s_3')}{s_1'+s_3'+s_2-3\mu^2-M_K^2}\\
&\qquad+\int ds_2'\frac{\rho_{23}(s_2',s_3')}{s_3'+s_2'+s_1-3\mu^2-M_K^2}\bigg].
\end{aligned}
\tag{5}
$$

From the region of nonvanishing ρ determined by Mandelstam[4] we see that the functions $\rho(x,y)$ vanish unless $x+y>36\mu^2$. For small s, therefore, the typical inside integral of (5),

$$
I(x,s)=\int dy\frac{\rho(x,y)}{x+y+s-3\mu^2-M_K^2},
$$

should be nearly independent of s. Applying this argument to each term in (5) we arrive at the approximate form,[6]

[4] G. F. Chew and S. Mandelstam, University of California Radiation Laboratory Report UCRL-8728 (unpublished).

[5] G. F. Chew, S. Mandelstam, and H. P. Noyes, University of California Radiation Laboratory Report UCRL-9001 (unpublished).

[6] A representation similar to (6) has been used by M. Gourdin and A. Martin (CERN preprint) and has been attributed by them to M. Cini and S. Fubini.

$$A(s_1,s_2,s_3)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{\sigma_1(s')}{s'-s_1}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{\sigma_2(s')}{s'-s_2}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{\sigma_3(s')}{s'-s_3}. \quad (6)$$

This simplification in the case of pion-pion scattering leads to exactly the Chew-Mandelstam nonlinear integral equations for S-wave dominant amplitudes (that is, the equations in which the imaginary part is taken as pure S-wave). Chew and Mandelstam[5] have shown that within a wide range of pion-pion coupling constants consistent solutions with a small P wave exist.

Now defining, for the process $\tau+\pi_3\rightarrow\pi_1+\pi_2$

$$A_s(q^2)=\frac{1}{2}\int_{-1}^{+1}A(s_1,s_2,s_3+i\epsilon)d(\cos\theta), \quad (7)$$

we see that (in spite of the fact that s_1 and s_2 become complex in the region $4\mu^2<s_3<16\mu^2$),

$$\mathrm{Im}A_s(q^2)=\sigma_3[4(q^2+\mu^2)] \quad (8)$$

for all $q^2>0$. Similar expressions relate σ_1 and σ_2 to the other two S-wave scattering amplitudes. Therefore, knowledge of the S-wave amplitudes for the three processes, $\tau_1+\pi_1\rightarrow\pi_2+\pi_3$, $\tau_1+\pi_2\rightarrow\pi_1+\pi_3$, and $\tau_1+\pi_3\rightarrow\pi_1+\pi_2$, is sufficient for construction of the single-variable Mandelstam relation (6).

In the physical region, $q^2>3\mu^2$, the S-wave unitarity condition including only 2π intermediate states is,

$$\mathrm{Im}A_s(\tau\pi_3,\pi_1\pi_2)=[q^2/(q^2+1)]^{\frac{1}{2}}\sum A_s(\tau\pi_3,2\pi)t_s^{\dagger}(2\pi,\pi_1\pi_2), \quad (9)$$

where t_s is the S-wave pion-pion scattering amplitude. The sum is to be extended over all intermediate charge states. q is the pion momentum in the intermediate and final two-pion states. For an intermediate and final state of definite total isotopic spin, T, we have

$$t_s=[(q^2+1)/q^2]^{\frac{1}{2}}e^{i\delta_T(q)}\sin\delta_T(q).$$

To show the applicability of this unitarity condition also in the unphysical region, $q^2<3$, we consider the quantity,

$$\langle 0|K(0)|\pi_3\tau\ \mathrm{in}\rangle=\frac{i}{(2\omega_3)^{\frac{1}{2}}}\int e^{-ip_3\cdot x}d^4x\langle 0|T(K(0)J_3(x))|\tau\rangle,$$

where $K(0)$ is some operator such that $\langle 0|K(0)|2\pi\rangle\neq 0$. From standard methods it follows that

$$\mathrm{Im}\langle 0|K(0)|\pi_3\tau\ \mathrm{in}\rangle=[\pi/(2\omega_3)^{\frac{1}{2}}]\sum\delta^4(p_n-p_\tau-p_3)\times\langle 0|K(0)|n\rangle\langle n|J_3(0)|\tau\rangle, \quad (10)$$

where J_3 is the current associated with π_3. For $-(p_\tau+p_3)^2$ less than the four π-meson threshold the only intermediate states are those of two pions. The operator K may be chosen to connect with a two-pion state of any particular angular momentum or isotopic spin.

We note that $\langle 0|K(0)|2\pi\ \mathrm{out}\rangle_{T,J}$ has the phase $\exp[-i\delta_{TJ}(q)]$. If we require $\mathrm{Im}\langle 0|K(0)|\pi_3\tau\ \mathrm{in}\rangle$ to be real, we find from (10) that $\langle 2\pi\ \mathrm{out}|_{TJ}J_3(0)|\tau\rangle$ has the phase, $\exp[i\delta_{TJ}(q)]$. This implies the conditions (9).

3. τ^+ AND τ'^+ DECAY

For τ and τ' decays, π_1 and π_2 will be taken as the identical pions and π_3 as the odd pion. Thus for the τ^+-decay representation the process, $\tau+\pi_3\rightarrow\pi_1+\pi_2$, is $\tau^++\pi^+\rightarrow\pi^++\pi^+$. Then $\sigma_3(s')$ is determined from $\mathrm{Im}A_s(\tau^+\pi^+,\pi^+\pi^+)$. The weight functions σ_1 and σ_2 are identical by the Bose principle and are both equal to the imaginary part of $A_s(\tau^+\pi^-,\pi^+\pi^-)$. In the unitarity condition for $A_s(\tau^+\pi^-,\pi^+\pi^-)$ one encounters the intermediate amplitude $A_s(\tau^+\pi^-,\pi^0\pi^0)$. This must be determined from a separate dispersion relation for τ' decay.

Setting $\sigma_1=\sigma_2=\sigma$, $\sigma_3=\rho$, we write

$$A^{\tau}(s_1,s_2,s_3)=\lambda+\frac{s_3-s_0}{\pi}\int ds'\frac{\rho(s')}{(s'-s_3)(s'-s_0)}+\frac{s_2-s_0}{\pi}\int ds'\frac{\sigma(s')}{(s'-s_2)(s'-s_0)}+\frac{s_1-s_0}{\pi}\int ds'\frac{\sigma(s')}{(s'-s_1)(s'-s_0)}, \quad (11)$$

and

$$A^{\tau'}(s_1,s_2,s_3)=\lambda'+\frac{s_3-s_0}{\pi}\int ds'\frac{\rho'(s')}{(s'-s_3)(s'-s_0)}+\frac{s_2-s_0}{\pi}\int ds'\frac{\sigma'(s')}{(s'-s_2)(s'-s_0)}+\frac{s_1-s_0}{\pi}\int ds'\frac{\sigma'(s')}{(s'-s_1)(s'-s_0)}. \quad (12)$$

In the case of τ' decay, s_3 stands for the energy in the center-of-mass system of the two π^0. A subtraction has been here inserted into the Mandelstam relation itself, rather than into the partial wave dispersion relations as in reference 4. This is essential since our aim is to discuss the continuation to τ decay by constructing the complete Mandelstam relation. Solution of the S wave, $\tau+\pi\rightarrow\pi+\pi$ problem will give only the densities ρ and σ. The point s_0 is chosen as the symmetrical point in pion-pion scattering, $s_1=s_2=s_3=\frac{4}{3}\mu^2$. It is of no importance that this point does not obey the constraint between s_1, s_2, and s_3 in τ decay. For any given τ mass the function A is uniquely defined as an independent function of s_1, s_2, and s_3, if one demands the analytic properties in (6). Equation (11) then results from defining λ as $A(s_0,s_0,s_0)$.

In determination of ρ, σ, ρ', σ', the S-wave amplitudes of four processes will be used,

$$\begin{aligned} A_1(q^2)&=A(\tau^+\pi^+,\pi^+\pi^+)_s, & \mathrm{Im}A_1(q^2)&=\rho[4(q^2+\mu^2)],\\ A_2(q^2)&=A(\tau^+\pi^-,\pi^+\pi^-)_s, & \mathrm{Im}A_2(q^2)&=\sigma[4(q^2+\mu^2)],\\ A_3(q^2)&=A(\tau^+\pi^-,\pi^0\pi^0)_s, & \mathrm{Im}A_3(q^2)&=\rho'[4(q^2+\mu^2)],\\ A_4(q^2)&=A(\tau^+\pi^0,\pi^+\pi^0)_s, & \mathrm{Im}A_4(q^2)&=\sigma'[4(q^2+\mu^2)]. \end{aligned} \tag{13}$$

Here q is always the momentum in the two-pion system. The unitarity conditions for the four amplitudes are given from (9) in terms of pion-pion S-wave amplitudes $t_0(q^2)$ and $t_2(q^2)$. They are

$$\begin{aligned} &[(q^2+\mu^2)/q^2]^{\frac{1}{2}}\rho[4(q^2+\mu^2)]=A_1t_2^\dagger,\\ &[(q^2+\mu^2)/q^2]^{\frac{1}{2}}\sigma[4(q^2+\mu^2)]\\ &\qquad=A_2(\tfrac{1}{3}t_2^\dagger+\tfrac{2}{3}t_0^\dagger)+A_3(-\tfrac{1}{3}t_2^\dagger+\tfrac{1}{3}t_0^\dagger),\\ &[(q^2+\mu^2)/q^2]^{\frac{1}{2}}\rho'[4(q^2+\mu^2)]\\ &\qquad=A_2(\tfrac{1}{3}t_2^\dagger-\tfrac{1}{3}t_0^\dagger)+A_3(\tfrac{2}{3}t_2^\dagger+\tfrac{1}{3}t_0^\dagger),\\ &[(q^2+\mu^2)/q^2]^{\frac{1}{2}}\sigma'[4(q^2+\mu^2)]=A_4t_2^\dagger. \end{aligned} \tag{14}$$

From (12) we derive S-wave dispersion relations for the various A. When the angular integrations are performed an amplitude is expressed in terms of a subtraction constant, an integral which gives a cut from $q^2=0$ to $q^2=\infty$, and an integral which has singularities to the left of the imaginary axis. These contributions can be expressed in terms of functions $R(\rho;\nu)$ and $L(\rho;\nu)$ with $\nu=(q^2/\mu^2)$,

$$R(\rho;\nu)=\frac{\nu-\nu_0}{\pi}\int_0^\infty d\nu'\frac{\rho[4(\nu'+1)\mu^2]}{(\nu'-\nu)(\nu'-\nu_0)},$$

$$L(\rho;\nu)=\frac{\nu-\nu_0}{\pi}\frac{1}{\nu}\left(\frac{\nu+1}{\nu-3}\right)^{\frac{1}{2}}\int_0^\infty d\nu'\frac{\rho[4(\nu'+1)\mu^2]}{(\nu'-\nu_0)} \times\ln\frac{\nu'+\frac{1}{2}\nu+\frac{1}{2}\nu[(\nu-3)/(\nu+1)]^{\frac{1}{2}}}{\nu'+\frac{1}{2}\nu-\frac{1}{2}\nu[(\nu-3)/(\nu+1)]^{\frac{1}{2}}}. \tag{15}$$

The dispersion relations become,

$$\begin{aligned} A_1(\nu)&=\lambda+R(\rho;\nu)+2L(\sigma;\nu),\\ A_2(\nu)&=\lambda+R(\sigma;\nu)+L(\rho+\sigma;\nu),\\ A_3(\nu)&=\lambda'+R(\rho';\nu)+2L(\sigma';\nu),\\ A_4(\nu)&=\lambda'+R(\sigma';\nu)+L(\rho'+\sigma';\nu). \end{aligned} \tag{16}$$

Taken with the unitarity conditions, (14), these comprise a set of four coupled linear integral equations for the four amplitudes A.

4. ΔT RULES IN τ AND τ' DECAY

The $\Delta T=\frac{1}{2}$ rule, which relates τ' to τ decay, would reduce this set of four equations to two independent equations. For our amplitudes, A, the $\Delta T=\frac{1}{2}$ rule implies the relations,

$$A_4=A_1/2,\quad A_3=A_2-A_1/2. \tag{17}$$

Essentially equivalent relations have been given by Weinberg.[7]

The four equations, (16), have been written down in order to show that the implications of $\Delta T=\frac{1}{2}$, $\frac{3}{2}$ are the same as those of $\Delta T=\frac{1}{2}$. To prove this we first note that $\Delta T=\frac{1}{2}$, $\frac{3}{2}$ implies that $\lambda'=\lambda/2$. This follows from the definitions of λ as $A^\tau(s_0,s_0,s_0)$ and λ' as $A^{\tau'}(s_0,s_0,s_0)$. When A is divided into a completely symmetric function of the three variables plus a function of mixed symmetry as in reference 7, one sees that the function of mixed symmetry vanishes at the symmetric point. For the completely symmetrical function we need only $\Delta T<\frac{5}{2}$ to prove $A_{\mathrm{sym}}{}^{\tau'}=\frac{1}{2}A_{\mathrm{sym}}{}^\tau$ as in reference 7. Hence, $\lambda'=\lambda/2$.

When $\lambda'=\lambda/2$ is inserted into the integral equations (16) and when the unitarity conditions (14) are invoked it is possible to see that the substitutions,

$$A_4=A_1/2,\quad A_3=A_2-A_1/2,$$

make the last two equations linearly dependent upon the first two. Hence, if the four equations have unique solutions (assuming particular behavior at infinity) and if $\lambda'=\lambda/2$ the solutions must be such that the relations, (17), are satisfied. $\Delta T=\frac{1}{2}$, $\frac{3}{2}$ therefore implies the same conclusions as does $\Delta T=\frac{1}{2}$. Henceforth we shall adopt the relations (17) implied by the $\Delta T=\frac{1}{2}$, $\frac{3}{2}$ rule.

5. THE RELATION TO PION-PION SCATTERING

Under the conditions (17) our equations become quite similar in form to the nonlinear equations describing pion-pion S-wave scattering. This is apparent if we write the two independent equations in terms of the combinations,

$$B_2=A_1,\quad B_0=3A_2-\tfrac{1}{2}A_1.$$

Equations (16) become (using 14),

$$\begin{aligned} B_2(\nu)&=\lambda+R(B_2t_2^\dagger;\nu)+L(\tfrac{1}{3}B_2t_2^\dagger+\tfrac{2}{3}B_0t_0^\dagger;\nu),\\ B_0(\nu)&=\tfrac{5}{2}\lambda+R(B_0t_0^\dagger;\nu)+L(10/3B_2t_2^\dagger+\tfrac{2}{3}B_0t_0^\dagger;\nu). \end{aligned} \tag{18}$$

This structure reflects the fact that under the $\Delta T=\frac{1}{2}$ rule the K^+ must decay into a state with $T=1$ and therefore the K^+ couples to the pions with exactly the same coefficients as would π^+. This enables us to define our "$T=2$" and "$T=0$" combinations of τ and π.

Equations (18) may be converted into nonsingular integral equations by well-known methods.[8] However, these equations are of little value unless one knows pion-pion phase shifts accurately at all energies and is willing to do extensive numerical work. Instead, we look for further approximations.

The nonlinear equations for $\pi-\pi$ scattering[4] have exactly the form of (18), but with B_0 replaced by t_0, B_2 by t_2, and λ replaced by $-2\lambda_\pi$, λ_π being the coupling constant of reference 4. The form of the function R is identical in the two problems but the function L is

[7] S. Weinberg, Phys. Rev. Letters 4, 87 (1960).
[8] R. Omnes, Nuovo cimento 8, 316 (1958).

different. The function $L(2\sigma;\nu)$ is given, e.g., for $T=2$, in the case of τ decay by

$$L(2\sigma;\nu)=\frac{\nu-\nu_0}{\pi}\frac{1}{\nu}\left(\frac{\nu+1}{\nu-3}\right)^{\frac{1}{2}}\int_0^\infty d\nu'\frac{1}{\nu'-\nu_0}\times\left[\ln\frac{\nu'+\frac{1}{2}\nu+\frac{1}{2}\nu[(\nu-3)/(\nu+1)]^{\frac{1}{2}}}{\nu'+\frac{1}{2}\nu-\frac{1}{2}\nu[(\nu-3)/(\nu+1)]^{\frac{1}{2}}}\right]\times[\tfrac{1}{3}B_2(\nu')t_2^\dagger(\nu')+\tfrac{2}{3}B_0(\nu')t_0^\dagger(\nu')]\left(\frac{\nu'}{\nu'+1}\right)^{\frac{1}{2}}. \quad (19a)$$

and in the case of $\pi-\pi$ scattering by

$$\bar{L}(2\sigma;\nu)=\frac{\nu-\nu_0}{\pi}\frac{1}{\nu}\int_0^\infty\frac{d\nu'}{\nu'-\nu_0}\left(\frac{\nu'}{\nu'+1}\right)^{\frac{1}{2}}\times[\tfrac{1}{3}t_2(\nu')t_2^\dagger(\nu')+\tfrac{2}{3}t_0(\nu')t_0^\dagger(\nu')]\ln\frac{\nu'+1+\nu}{\nu'+1}. \quad (19b)$$

If $\bar{L}(\rho;\nu)$ were the same function of ρ and ν as $L(\rho;\nu)$ then the solutions of Eqs. (18) could be expressed in terms of pion-pion solutions as

$$B_2=-\tfrac{1}{2}(\lambda/\lambda_\pi)t_2,\qquad B_0=-\tfrac{1}{2}(\lambda/\lambda_\pi)t_0. \quad (20)$$

We shall take these solutions as a basis for further analysis. With given $\pi-\pi$ phase shifts one may determine whether $\bar{L}(\rho;\nu)$ is indeed a good approximation to $-2\lambda^{-1}\lambda_\pi L(\rho;\nu)$ when the relations (20) are assumed. We have investigated this question using integrable phase shifts and the results are discussed in the Appendix.

These are the solutions we would have obtained had we begun with a K meson of mass μ. That is, in our approximation when the τ-decay amplitude is expressed in terms of the pion variables s_1, s_2, and s_3 in the standard way, the weight functions are independent of the τ mass.

There is a diagrammatic interpretation that independently suggests this result. Consider all diagrams of the general form of Fig. 2, where one pair of pions interacts locally. The contribution of this diagram is a function only of the variable $s_3=-(p_1+p_2)^2$. The sum of such diagrams is of the form of Eq. (6). We note that each such diagram depends only upon the center-of-mass energy of some one pair of pions, and not upon the τ mass.

Substituting (20) and the unitarity conditions (14) into the subtracted representation (11) we find, in terms of $s_i=4(\nu_i^2+1)\mu^2$,

$$A^\tau(\nu_1,\nu_2,\nu_3)=-\frac{\lambda}{2\lambda_\pi}\Bigg[-2\lambda_\pi+\frac{\nu_1-\nu_0}{\pi}\times\int_0^\infty d\nu'\frac{\frac{1}{6}t_2(\nu')t_2^\dagger(\nu')+\frac{1}{3}t_0(\nu')t_0^\dagger(\nu')}{(\nu'-\nu_1)(\nu'-\nu_0)}\left(\frac{\nu'}{\nu'+1}\right)^{\frac{1}{2}}+\frac{\nu_2-\nu_0}{\pi}\int_0^\infty d\nu'\frac{\frac{1}{6}t_2(\nu')t_2^\dagger(\nu')+\frac{1}{3}t_0(\nu')t_0^\dagger(\nu')}{(\nu'-\nu_2)(\nu'-\nu_0)}\left(\frac{\nu'}{\nu'+1}\right)^{\frac{1}{2}}+\frac{\nu_3-\nu_0}{\pi}\int_0^\infty d\nu'\frac{t_2(\nu')t_2^\dagger(\nu')}{(\nu'-\nu_3)(\nu'-\nu_0)}\left(\frac{\nu'}{\nu'+1}\right)^{\frac{1}{2}}\Bigg]. \quad (21)$$

For physical τ decay the ν_1, ν_2, ν_3 are the three relative pion squared momenta in the final state,

$$\nu_1=|\mathbf{p}_2-\mathbf{p}_3|^2/4,\quad \nu_2=|\mathbf{p}_1-\mathbf{p}_3|^2/4,\quad \nu_3=|\mathbf{p}_1-\mathbf{p}_2|^2/4.$$

The pion-pion scattering equations may be used once more to express the integrals in (21) in terms of $\pi-\pi$ phase shifts at the momenta ν_i, subtraction constants, and left-hand cut integrals. These left-hand cut contributions are real. We obtain, finally,

$$-[(2\lambda_\pi)/\lambda]A^\tau(\nu_1,\nu_2,\nu_3)=4\lambda_\pi+t_2(\nu_3)+\tfrac{1}{6}[t_2(\nu_1)+t_2(\nu_2)]+\tfrac{1}{3}[t_0(\nu_1)+t_0(\nu_2)]+(7/6)[I_2(\nu_1)+I_2(\nu_2)]+\tfrac{1}{3}[I_0(\nu_1)+I_0(\nu_2)]+\tfrac{1}{3}[I_2(\nu_3)+2I_0(\nu_3)], \quad (22)$$

where

$$I_{2,0}(\nu)=\frac{1}{\pi}\int_0^\infty d\nu'\left(\frac{\nu'+1}{\nu'}\right)^{\frac{1}{2}}\sin^2\delta_{2,0}(\nu')\times\frac{1}{\nu'-\nu_0}\left[1+\frac{\nu_0-\nu'}{\nu}\ln\left(1+\frac{\nu}{\nu'+1}\right)\right].$$

A similar expression for τ' decay follows from (17).

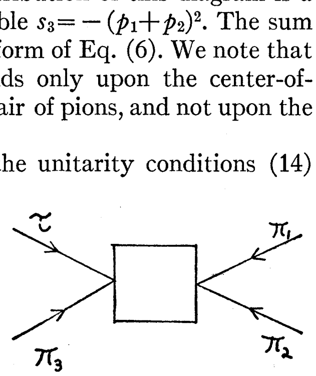

FIG. 2. One type of τ-decay diagram.

6. NUMERICAL RESULTS

Chew and Mandelstam[5] have numerically solved the S-wave dominant pion-pion equations. We have used their solutions for $\lambda_\pi=-0.1$, $+0.1$, $+0.3$ in order to analyze our Eq. (22). The integrals, I (which are nonsingular) have been expanded to first order in ν and have been evaluated using Gauss' three-point technique, the points being read directly off the curves of reference 5. To evaluate the $t(\nu)$ terms a linear fit to the curves of reference 5 was made for the quantities, $[\nu/(\nu+1)]^{\frac{1}{2}}\cot\delta(\nu)$. Then the $t(\nu)=[(\nu+1)/\nu]^{\frac{1}{2}}\exp[i\delta(\nu)]\sin\delta(\nu)$ were expanded in powers of $(\nu)^{\frac{1}{2}}$, with terms to order ν retained. The resulting π^- and π^+ energy spectra in τ decay were calculated and the results were compared with the spectra predicted on the basis of a constant matrix element.

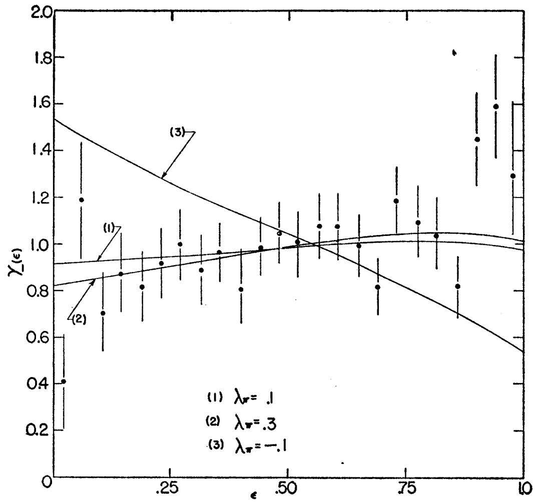

FIG. 3. The π^- distribution in τ^+ decay as a function of energy ϵ (in units of the maximum π^- energy). The experimental points are from reference 1. The curves are theoretical curves for indicated values of λ_π.

Defining the ratios of the π^+, π^- spectra predicted from (22) to the density of states spectra as $r_{+,-}(\omega)$, we plot the results for $r_-(\omega)$ for $\lambda_\pi=-0.1$, $+0.1$, $+0.3$ in Fig. 3. The experimental points are taken from reference 1.

The two repulsive values of λ_π, 0.1 and 0.3, are seen to give a $r_-(\omega)$ which increases with energy through most of the range of ω, in qualitative agreement with experiment. $\lambda_\pi=0.3$ is a better fit to the data. The scattering lengths for $\lambda_\pi=0.3$ are about $a_2=-0.48\mu^{-1}$, $a_0=-0.8\mu^{-1}$.

The spectrum ratio, $r_+(\omega)$, is plotted in Fig. 4 for $\lambda_\pi=0.1$ and $\lambda_\pi=0.3$. It decreases with increasing energy, also in agreement with experiment.

The attractive value $\lambda_\pi=-0.1$ gives π^- and π^+ spectra sloping violently the wrong way.

The energy spectrum for the π^+ in τ' decay has been calculated for $\lambda_\pi=0.1$ and $\lambda_\pi=0.3$ and shows a decrease (Fig. 5) with increasing π^+ energy, compared to the density of states prediction. This is qualitatively in accord with Weinberg's prediction[7]; however, the curvature of the plots for $r(\omega)$ are not compatible with Weinberg's analysis. Unfortunately, there are not enough experimental points near the high-energy end of the π^- spectrum to check the markedly curved aspect of our $r(\omega)$ in that region.

Weinberg[7] has analyzed the effect of the $\Delta T=\frac{1}{2}$ rule in the τ and τ' decay energy spectra on the basis of a development of the matrix elements in a power series in the invariants. He takes the invariants essentially to be the individual pion kinetic energies. They can as well be taken as our three ν's, being the squares of the pion relative momenta. Weinberg takes the first two terms in an expansion of the matrix element in integral powers of $\mathbf{p}_1^2$, $\mathbf{p}_2^2$, $\mathbf{p}_3^2$ and shows that they lead to a linear form for $r_{+,-}(\omega)$ (up to relativistic corrections). Our deviation from this linear form is due to the presence of terms in $|A|^2$ which cannot be so expanded. These terms arise from the fact that $(\nu)^{-\frac{1}{2}}\exp[i\delta(\nu)]\sin\delta(\nu)$ is an even function of $\nu^{\frac{1}{2}}$ only insofar as it is real. There are imaginary terms of the form $i\nu^{\frac{1}{2}}$ in an expansion. Since we have several phases entering (22), when we form $|A|^2$ and expand for small ν's we find terms of the form, e.g.,

$$(\nu_1\nu_2)^{\frac{1}{2}}=(|\mathbf{p}_2-\mathbf{p}_3|\,|\mathbf{p}_1-\mathbf{p}_3|)^{\frac{1}{2}}.$$

This kind of term is not expansible in powers of $\mathbf{p}_1^2$, $\mathbf{p}_2^2$, $\mathbf{p}_3^2$ and this effect is seen to be a somewhat important one in our analysis.

It is probably true in general that matrix elements leading to three-particle states with strong interactions cannot be expanded in integral powers of the invariants, $\mathbf{p}_\alpha\cdot\mathbf{p}_\beta$. In this problem we may see this property without making our approximation of neglecting the dependence upon the mass by looking back to the Mandelstam representation (12). For s_1, s_2, s_3 in the physical region (as in τ decay) each integral may be expressed as a sum of subtraction terms, an amplitude $A_s(\nu_i)$ for $\tau+\pi\rightarrow\pi+\pi$ at the momentum, $(\nu_i)^{\frac{1}{2}}$, and a real contribution from left-hand cuts. Since the amplitudes $A_s(\nu_i)$ have the pion-pion phases we obtain, as in (22) a sum of terms with various pion-pion phases plus some real terms. The threshold behavior of the phases alone invalidates Weinberg's expansion.

7. CONCLUSION

We have seen how Chew and Mandelstam's S-wave-dominant equations may serve as a basis for predicting the pion energy spectra in τ and τ' decay, under the assumption that final-state interactions are responsible for the shapes of these spectra. Linear integral equations have been derived for functions from which the τ and τ' decay matrix elements may be constructed.

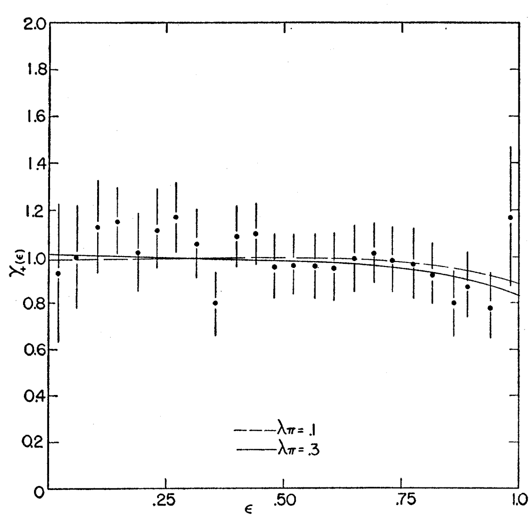

FIG. 4. The π^+ energy distribution in τ^+ decay. The experimental points are from reference 1. The curves are theoretical ones for indicated values of λ_π.

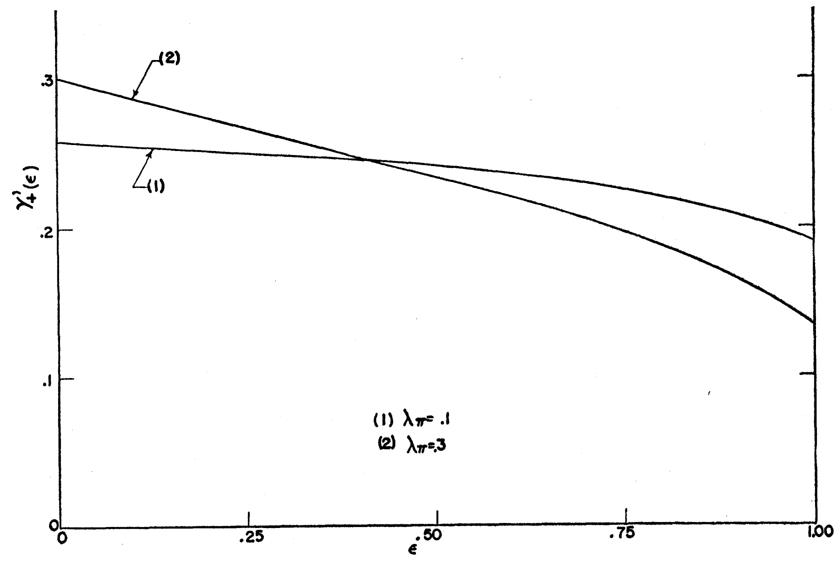

FIG. 5. The π^+ energy distribution in τ'^+ decay.

Under the rather drastic assumptions we have made in approximating the solution of the integral equations, it would seem that a repulsive S-wave force is required to explain the observed features of the τ^+-decay spectrum. Since we are using the Chew-Mandelstam approach to τ decay we must take seriously the similar equations for $\pi-\pi$ scattering. For this reason we may not take arbitrary phase shifts, but must take the ones from reference 5. Thus we cannot duplicate Thomas and Holladay's result[2] for an attractive $T=2$ force, because this is seen to imply an attractive $T=0$ force of greater strength.

From the original equations, without approximation, we have derived some results. We have shown that the implications of $\Delta T=\frac{1}{2}$, $\frac{3}{2}$ are identical to those of $\Delta T=\frac{1}{2}$. This is in spite of the fact that the matrix element deviates markedly from a constant. (A constant matrix element picks out the $T=1$ or $T=3$ final states and makes $\Delta T=\frac{3}{2}$ unobservable.)

The qualitative explanation for this result is that one may imagine a matrix element in the absence of final state interactions which is a constant. $\Delta T=\frac{1}{2}$, $\frac{3}{2}$ then implies a pure $T=1$ final state. The $\pi-\pi$ coupling is now turned on and may induce some momentum dependence in the matrix element; but it preserves the $T=1$ nature of the final state. Unfortunately this means that the spectrum in τ' decay, like the branching ratio, cannot be a critical test for $\Delta T=\frac{1}{2}$.

The other result of our equations which is independent of approximation is that the matrix element may not be expanded in integral powers of the invariants as is assumed in reference 7.

ACKNOWLEDGMENT

We wish to thank Professor W. G. Holladay for interesting conversations.

APPENDIX

The integrals in (19a) and (19b) have been performed, using (20), for a particular $t(\nu)$ corresponding to a scattering length approximation to the phase shift [omitting the relativistic factor $(\nu+1)^{\frac{1}{2}}$]

$$t(\nu)t^\dagger(\nu)=a^2/(a^2\nu+1).$$

The results for $0<\nu<3$ are

$$-2(\lambda_\pi/\lambda)L(\nu)=\frac{2}{\nu}\left(\frac{\nu+1}{\nu-3}\right)^{\frac{1}{2}}\frac{a^2}{1-\nu_0a^2}\times\left(\frac{1}{a}\ln\frac{1+ar}{a+ar^*}-(\nu_0)^{\frac{1}{2}}\ln\frac{\nu_0^{\frac{1}{2}}+r}{\nu_0^{\frac{1}{2}}+r^*}\right),$$

with

$$r=\left(\frac{\nu}{2}\right)^{\frac{1}{2}}\left[1+\left(\frac{\nu-3}{\nu+1}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

and

$$\bar{L}(\nu)=\frac{2}{\nu}\frac{a^2}{1-\nu_0a^2}\left(\frac{1}{a}\ln\frac{1+a(1+\nu)^{\frac{1}{2}}}{1+a}-(\nu_0)^{\frac{1}{2}}\ln\frac{\nu_0^{\frac{1}{2}}+(1+\nu)^{\frac{1}{2}}}{\nu_0^{\frac{1}{2}}+1}\right).$$

We have examined these results for values of a between 0.2 and 1 in units of the pion Compton wavelength. In the region $1<\nu<3$ there is in every case a 20% to 30% difference between $-2\lambda_\pi\lambda^{-1}L(\nu)$ and $\bar{L}(\nu)$. In the region $0<\nu<1$ the discrepancy is much larger and depends on the value of a. We conclude that the approximation leading to (20), while crude, does not neglect altogether the contributions from $L(\nu)$.