Partial-Wave Dispersion Relations for Pion-Nucleon Scattering*

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Partial-wave dispersion relations for pion-nucleon scattering are derived from the Mandelstam representation. The symmetry of the representation is used to obtain expressions for the discontinuities across the unphysical branch cuts. These expressions involve scattering amplitudes for pion-nucleon scattering and for the process $\pi+\pi\rightarrow N+\bar{N}$. In the approximation of neglecting all but the nearest singularities it is shown that the Chew-Low effective-range formula is a solution to the equations.

I. INTRODUCTION

A PROGRAM of calculation of cross sections for strong interactions by means of partial-wave dispersion relations has been initiated by Chew and Mandelstam.[1] In this paper we carry out the initial stages of the application of their method to pion-nucleon scattering.[2] Most of the paper is concerned with using Mandelstam's representation[3] to locate the singularities of the partial-wave amplitudes and derive expressions for the discontinuities across the branch cuts. These tasks are necessary preliminaries to any calculations of pion-nucleon scattering. Finally, in the approximation of nelgecting all but the closest branch cut to the physical region, it is shown that the Chew-Low effective range formula[4] is a solution of the equations.

In Sec. II the variables are defined, and in Sec. III the Mandelstam representation for pion-nucleon scattering is discussed. In Sec. IV the singularities of the partial-wave amplitudes are located, and in Sec. V the discontinuities across the branch cuts are calculated. Expressions are given for these discontinuities in terms of scattering amplitudes for pion-nucleon scattering and the process $\pi+\pi\rightarrow N+\bar{N}$. Many complicating features of the problem are found which do not appear in the case of scattering of particles of equal mass. In Sec. VI the Chew-Low effective-range formula is shown to be an approximate solution of the partial-wave dispersion relations plus unitarity.

II. DEFINITION OF VARIABLES

Let the four-vector momenta of the incident and outgoing pion be q_1 and q_2, respectively, while those of the initial and final nucleon are p_1 and p_2. Define the variables

$$s=(p_1+q_1)^2, \tag{2.1a}$$

$$\bar{s}=(p_1-q_2)^2, \tag{2.1b}$$

$$t=(q_2-q_1)^2. \tag{2.1c}$$

Conservation of momentum leads to the relation

$$s+\bar{s}+t=2m^2+2\mu^2. \tag{2.2}$$

The Lorentz invariants defined by Eqs. (2.1a, b, c) are the squares of the energies in the barycentric system of the three reactions:

$$\text{I.}\quad p_1+q_1\rightarrow p_2+q_2\quad(\pi+N\rightarrow\pi+N), \tag{2.3a}$$

$$\text{II.}\quad p_1-q_2\rightarrow p_2-q_1\quad(\pi+N\rightarrow\pi+N), \tag{2.3b}$$

$$\text{III.}\quad q_1-q_2\rightarrow p_2-p_1\quad(\pi+\pi\rightarrow N+\bar{N}). \tag{2.3c}$$

These three reactions will all enter into the equations for pion-nucleon scattering if one uses the Mandelstam representation.

For an analysis of the kinematics of pion-nucleon scattering we refer the reader to Chew, Goldberger, Low, and Nambu[5] (hereafter CGLN), whose notation we shall employ herein. Similar considerations for the process $\pi+\pi\rightarrow N+\bar{N}$ are given by the authors.[6] Finally, defining k as the magnitude of the three-vector momentum and θ as the scattering angle in the barycentric system of reaction I, one finds

$$t=-2k^2(1-\cos\theta), \tag{2.4}$$

$$k^2=[s-(m+\mu)^2][s-(m-\mu)^2]/4s, \tag{2.5}$$

$$\bar{s}=\frac{1-\cos\theta}{2}\frac{(m^2-\mu^2)^2}{s}-\frac{1+\cos\theta}{2}(s-2m^2-2\mu^2). \tag{2.6}$$

III. MANDELSTAM REPRESENTATION

We assume that the invariant functions $A^{(\pm)}$ and $B^{(\pm)}$ defined in CGLN satisfy the spectral representation

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[2] This problem has been discussed by S. W. MacDowell, Phys. Rev. **116**, 774 (1960). We include for completeness and consistency of notation many formulas contained in this reference.

[3] S. Mandelstam, Phys. Rev. **112**, 1344 (1958), and Phys. Rev. **115**, 1741 and 1752 (1959).

[4] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[5] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[6] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).

proposed by Mandelstam:

$$B^{(\pm)}(s,\bar{s},t)=\frac{g_r^2}{m^2-s}\mp\frac{g_r^2}{m^2-\bar{s}}+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty}ds'\times\int_{(m+\mu)^2}^{\infty}d\bar{s}'\frac{b_{12}^{(\pm)}(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty}ds'\int_{4\mu^2}^{\infty}dt'\frac{b_{13}^{(\pm)}(s',t')}{(s'-s)(t'-t)}+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty}d\bar{s}'\int_{4\mu^2}^{\infty}dt'\frac{b_{23}^{(\pm)}(\bar{s}',t')}{(\bar{s}'-\bar{s})(t'-t)}. \quad (3.1)$$

The functions $A^{(\pm)}$ satisfy a similar representation, excluding the first two terms. It may be necessary to make subtractions in these representations, but this will not affect the considerations which follow.

As shown by Mandelstam, one can easily derive from Eq. (3.1) one-dimensional dispersion relations with either s, $\bar{s}$, or t held fixed. For example, the ordinary dispersion relation at fixed t is

$$B^{(\pm)}(s,\bar{s},t)=\frac{g_r^2}{m^2-s}\mp\frac{g_r^2}{m^2-\bar{s}}+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\frac{b_1^{(\pm)}(s',t)}{s'-s}+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}d\bar{s}'\frac{b_2^{(\pm)}(\bar{s}',t)}{\bar{s}'-\bar{s}}. \quad (3.2)$$

and similarly for $A^{(\pm)}$. The "absorptive parts" $b_i(x,y)$ are equal to ImB when the variables s, $\bar{s}$, and t are in the physical region for reaction i, as defined by Eq. (2.3). The variable x is the square of the energy corresponding to reaction i, and y is the corresponding momentum transfer. The two absorptive parts b_1 and b_2 are related by crossing symmetry, which requires that

$$A^{(\pm)}(s,\bar{s},t)=\pm A^{(\pm)}(\bar{s},s,t), \quad (3.3a)$$

$$B^{(\pm)}(s,\bar{s},t)=\mp B^{(\pm)}(\bar{s},s,t). \quad (3.3b)$$

Imposing these requirements on Eq. (3.2) and its counterpart for $A^{(\pm)}$, one finds that

$$a_2^{(\pm)}(s',t)=\pm a_1^{(\pm)}(s',t), \quad (3.4a)$$

$$b_2^{(\pm)}(s',t)=\mp b_1^{(\pm)}(s',t). \quad (3.4b)$$

In order to investigate the analytic properties of the pion-nucleon scattering partial-wave amplitudes, we need the representation which makes explicit the dependence on $\cos\theta$ at fixed energy:

$$B^{(\pm)}(s,\bar{s},t)=\frac{g_r^2}{m^2-s}\mp\frac{g_r^2}{m^2-\bar{s}}\mp\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}d\bar{s}'\frac{b_1^{(\pm)}(\bar{s}',\Sigma-s-\bar{s}')}{\bar{s}'-\bar{s}}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{b_3^{(\pm)}(t',s)}{t'-t}, \quad (3.5)$$

where the abbreviation $\Sigma=2m^2+2\mu^2$ has been introduced, and where Eq. (3.4b) has been used to eliminate b_2. The relation for $A^{(\pm)}$ does not contain the pole terms, and has the $\mp$ inverted. Now Eq. (3.1) shows that

$$b_1^{(\pm)}(\bar{s}',\Sigma-s-\bar{s}')=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\frac{b_{12}(s',\bar{s}')}{s'-s}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{b_{13}(\bar{s}',t')}{t'+s+\bar{s}'-\Sigma}. \quad (3.6)$$

The spectral functions b_{ij}, a_{ij} actually vanish over parts of the regions of integration in Eqs. (3.6) and (3.1). For details see reference 3 and Fig. 2 of reference 6.

IV. ANALYTIC CONTINUATION OF THE PARTIAL WAVE AMPLITUDES

A straightforward method for performing the decomposition in partial waves is available in the results of Jacob and Wick.[7] One can easily show by their method that, in the notation of CGLN,

$$f_{l\pm}(W)=\frac{1}{2}\int_{-1}^{1}d\cos\theta(P_l f_1+P_{l\pm1}f_2), \quad (4.1)$$

where the superscripts $(\pm)$, which refer to the isotopic spin decomposition, have been suppressed. In terms of the invariant amplitudes A and B one finds

$$f_{l\pm}(W)=\frac{1}{16\pi W}\{(E+m)[A_l+(W-m)B_l] \quad (E-m)[-A_{l\pm1}+(W+m)B_{l\pm1}]\}, \quad (4.2)$$

where W is the total energy in the barycentric system of reaction I, and E is the total energy of a nucleon in this system; i.e.,

$$W=s^{\frac{1}{2}} \quad \text{and} \quad E=(W^2+m^2-\mu^2)/2W. \quad (4.3)$$

We have also defined

$$[A_l(s);B_l(s)]\equiv\int_{-1}^{1}d\cos\theta P_l(\cos\theta)[A(s,t);B(s,t)]. \quad (4.4)$$

Now let us consider the analytic properties of the amplitudes $f_{l\pm}(W)$. In the case of scattering of particles of equal mass it has proved convenient to consider the partial-wave amplitudes as functions of the variable s.[1] In the present case, however, we will encounter considerable complexity in working in the s plane because the kinematical factors multiplying A_l and B_l in Eq. (4.2) have a branch point at $s=0$. We shall therefore concentrate our attention on the W plane.

It is actually possible to form a linear combination of

[7] M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959).

the f_l's which does not have a branch point at $s=0$, namely

$$\phi_{l+} \equiv (1/W)(f_{l+} + f_{(l+1)-}), \qquad (4.5a)$$

$$\phi_{l-} \equiv f_{l+} - f_{(l+1)-}. \qquad (4.5b)$$

These amplitudes, suggested by MacDowell,[2] are just the helicity amplitudes of Jacob and Wick.[7] Both have $J=L+\frac{1}{2}$; ϕ_{l+} describes scattering of a nucleon of positive helicity (spin along direction of motion) into a nucleon of positive helicity, whereas ϕ_{l-} describes scattering of a nucleon of positive helicity into a nucleon of negative helicity. Again using Eq. (4.1) and Eqs. (3.5) and (3.6) of CGLN, one finds that

$$\phi_{l-} = (1/16\pi s)[(s+m^2-\mu^2)(A_l - A_{l+1}) + m(s-m^2+\mu^2)(B_l - B_{l+1})], \qquad (4.6a)$$

$$\phi_{l+} = (1/16\pi s)[2m(A_l + A_{l+1}) + (s-m^2-\mu^2)(B_l + B_{l+1})]. \qquad (4.6b)$$

In spite of the comparative simplicity of these formulas, the f_l's will probably be more convenient for many applications. The usefulness of partial-wave dispersion relations has rested on the simple way in which the requirement of unitarity can be satisfied. Namely, in the elastic-scattering part of the physical region,

$$f_{l\pm} = \frac{e^{i\delta_{l\pm}} \sin\delta_{l\pm}}{k}, \qquad (4.7)$$

with δ real. The methods which have been used to solve partial-wave dispersion relations with this type of unitarity condition rely on the fact that

$$\mathrm{Im}(1/f_{l\pm}) = -k. \qquad (4.8)$$

No such simple relation holds for the $\phi_{l\pm}$.

Turning now to the complex W plane, we see that all the singularities of f_l come from singularities in the A_l's and B_l's (except for a pole at the origin which we shall remove in a moment). These singularities arise from the vanishing of denominators in Eqs. (3.5) and (3.6). The first term in Eq. (3.6) gives rise to the physical cuts; i.e., branch cuts in the regions $W \geq m+1$ and $W \leq -m-1$.[8] The former is the true physical region in which one can apply the unitarity condition, Eq. (4.7), below the inelastic threshold. The latter region (hereafter called the left-hand physical cut) is a novel feature of the W plane, whose meaning is elucidated by the symmetry relation

$$f_{l+}(-W) = -f_{(l+1)-}(W). \qquad (4.9)$$

This relation, which was pointed out by MacDowell,[2] can easily be verified by inspection of Eqs. (4.2) and (4.3). It permits one to apply unitarity on the left-hand physical cut. Namely, as one approaches the cut from above in the region $-m-2 \leq W \leq -m-1$, one finds from Eq. (4.9) that

$$-f_{l+}(W+i\epsilon) = \frac{e^{-i\delta_{(l+1)-}(-W)} \sin\delta_{(l+1)-}(-W)}{k}. \qquad (4.10)$$

Thus the two partial-wave amplitudes corresponding to a given total angular momentum are boundary values in different regions of the complex plane of a single analytic function $f_{l+}(W)$. Therefore we can limit our attention to f_{l+} and ignore f_{l-}.

It will probably be more convenient in the application of these analyticity properties not to work directly with the functions f_{l+} but rather with functions in which the threshold behavior is made manifest, namely,

$$h_l(W) \equiv \frac{W}{E+m} \frac{f_{l+}(W)}{k^{2l}} \qquad (4.11a)$$

$$= \frac{1}{16\pi}\left\{\frac{A_l}{k^{2l}} + (W-m)\frac{B_l}{k^{2l}} + (E-m)^2 \times \left[-\frac{A_{l+1}}{k^{2l+2}} + (W+m)\frac{B_{l+1}}{k^{2l+2}}\right]\right\}. \qquad (4.11b)$$

No singularity is introduced by division by k^{2l}, since it can be seen directly from the Mandelstam representation that A_l, $B_l \sim k^{2l}$ for $k^2 \approx 0$, and hence that $f_{l+} \sim k^{2l}$ for $W \approx m \pm 1$. At the other two points at which $k^2=0$, namely $W=-m\pm 1$, it follows from Eq. (4.2) that $f_{l+} \sim (E+m)k^{2l}$, thereby justifying the analyticity of $h_l(W)$. This latter consideration follows also from Eq. (4.9), which shows that at the left-hand physical threshold f_{l+} behaves like a partial wave of orbital angular momentum $l+1$. At this threshold $k^2 \sim E+m$, so that $h_l(W)$ approaches a constant at both the left- and right-hand physical thresholds. Finally, the factor W is introduced in h_l in order to avoid a singularity at the origin.

We actually cannot be completely certain about the behavior of h_l at the origin, since the question of the asymptotic behavior of the scattering amplitude is involved. This fact can be seen from Eqs. (2.4) and (2.5), which show that k^2 is infinite at $W=0$ and therefore that if $\cos\theta$ is restricted to physical values, then also t is infinite at $W=0$. Then A_l and B_l are regular at $W=0$ as long as $A(s,t)$ and $B(s,t)$ are bounded as $t \to \infty$. Although it is not known whether this condition is true in general, it has been found to be true in lowest-order perturbation theory.[3] We shall assume hereafter that h_l is regular at the origin, in which case all the singularities of $h_l(W)$ come from singularities in the A_l's and B_l's.

Let us now continue the enumeration of these singularities. We have already seen that the first term in Eq. (3.6) gives rise to the physical cuts. The second term, having been introduced artifically through the

[8] Hereafter we use units in which the pion mass is unity. We have set $\hbar=c=1$ throughout, and use the coupling constants $f^2=g^2/4m^2 \approx 0.08$ and $g^2=g_r^2/4\pi$.

separation into partial fractions of one of the terms in Eq. (4.1), actually does not give rise to a singularity in the scattering amplitude. A similar term which occurs in b_3 cancels the apparent singularity when both terms are substituted in Eq. (3.5).

The rest of the singularities of A_l and B_l come from vanishing denominators in Eq. (3.5). The first term, the pole in s, contributes only to B_0 (and hence only to h_0) since it is independent of $\cos\theta$. The other terms on the right-hand side of Eq. (3.5) produce singularities in all B_l. Let us consider first the denominators containing $\bar{s}$. From the defining equation (4.4) for B_l we find after a change of variable in the integration over $\cos\theta$ that these terms make the following contribution to B_l:

$$B_l^{(\pm)}(s)=\pm\frac{1}{2k^2}\int_{L_1(s)}^{L_2(s)}d\bar{s}\,P_l\left(1+\frac{\Sigma-s-\bar{s}}{2k^2}\right)\left[\frac{g_r^2}{m^2-\bar{s}}+\frac{1}{\pi}\int_{(m+1)^2}^{\infty}d\bar{s}'\frac{b_1^{(\pm)}(\bar{s}',\Sigma-s-\bar{s}')}{\bar{s}'-\bar{s}}\right]+\cdots,\quad(4.12)$$

where $L_1(s)$ and $L_2(s)$ are the limiting curves corresponding to $\cos\theta=-1$ and $\cos\theta=1$, respectively. These curves are shown in Fig. 1. One finds from Eq. (2.6) that their form is:

$$L_1(s)=(m^2-1)^2/s,\quad(4.13a)$$

$$L_2(s)=2m^2+2-s.\quad(4.13b)$$

It can easily be shown that the denominators in Eq. (4.12) can vanish only for real s. The location of the zeros, and hence of the singularities of B_l, can be seen from Fig. 1. To find the branch cut in $B_l(s)$ corresponding to a given value of $\bar{s}'$, find the intersection of the line $\bar{s}=\bar{s}'$ with the shaded region. In this manner one finds that the pole at $\bar{s}=m^2$ gives a branch cut in the regions $s<0$ and $m^2-2+1/m^2\leq s\leq m^2+2$. The

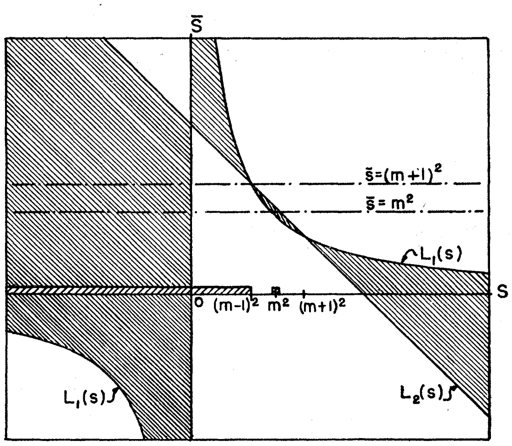

FIG. 1. The real $s\bar{s}$ plane for pion-nucleon scattering. The shaded area is the region in which $-1\leq\cos\theta\leq1$. The curve $L_1(s)$ corresponds to $\cos\theta=-1$; the curve $L_2(s)$, to $\cos\theta=+1$. The intersection of this area with the region of integration in $\bar{s}$ in the Mandelstam representation $[\bar{s}=m^2,\,\bar{s}\geq(m+1)^2]$ gives rise to the branch cuts shown along the s axis.

continuum beginning at $\bar{s}'=(m+1)^2$ gives a branch cut in the region $s\leq(m-1)^2$. In the W plane these cuts map into cuts along the entire imaginary axis and in the following regions:

$$-(m^2+2)^{\frac{1}{2}}\leq W\leq -m+1/m,\quad(4.14a)$$

$$-m+1\leq W\leq m+1,\quad(4.14b)$$

$$m-1/m\leq W\leq(m^2+2)^{\frac{1}{2}}.\quad(4.14c)$$

Finally, there are the singularities coming from the vanishing of the denominator of the last term in Eq. (3.5), $t'-t$. In the k^2 complex plane these are very simple; namely, a branch cut in the region $k^2\leq-1$. In the s plane this cut maps into a cut along the negative real axis plus a cut along a circle centered at the origin and with radius r^2, where

$$r^2=m^2-1.\quad(4.15)$$

In the W plane the cuts then lie along the imaginary axis plus a circle about the origin having radius r.

In Fig. 2 the location of all the singularities in the W plane is shown, along with the appropriate contour of integration to be used in writing a partial-wave dispersion relation. We include the contour inside the circle, even though the integral around it vanishes, because it will turn out that the discontinuity across the circle is related to the process $\pi+\pi\rightarrow N+\bar{N}$. We would not know how to evaluate the integral around the outside of the circle only. Similar considerations lead us to the inclusion of contours in both left and right half-planes.

The foregoing statements about the regions of analyticity of h_l are equivalent to the following dispersion relation:

$$h_l(W)=\frac{1}{\pi}\int_{m+1}^{\infty}dW'\frac{\mathrm{Im}h_l(W')}{W'-W}+\frac{1}{\pi}\int_{-\infty}^{-m-1}dW'\frac{\mathrm{Im}h_l(W')}{W'-W}-\frac{g^2\delta_{l0}}{2(m+W)}+\frac{1}{\pi}\int_{-(m^2+2)^{\frac{1}{2}}}^{-m+1/m}dW'\frac{\alpha_1^l(W')}{W'-W}+\frac{1}{\pi}\int_{m-1/m}^{(m^2+2)^{\frac{1}{2}}}dW'\frac{\alpha_1^l(W')}{W'-W}+\frac{1}{\pi}\int_{-m+1}^{m-1}dW'\frac{\alpha_2^l(W')}{W'-W}+\frac{r}{\pi}\int_0^{2\pi}id\phi\frac{\alpha_3^l(\phi)}{We^{-i\phi}-r}+\frac{1}{\pi}\int_{-\infty}^{\infty}dy\frac{\alpha_4^l(y)}{y+iW},\quad(4.16)$$

provided the function behaves properly at infinity. Along the positive real axis this is guaranteed by unitarity. The discontinuities $\alpha_i^l(W)$ will be evaluated in the next section. As usual, the superscripts $(\pm)$ have been suppressed.

V. DISCONTINUITIES ACROSS THE BRANCH CUTS

All the integrations on the right-hand side of Eq. (4.16) except for the first two terms extend over unphysical regions of the W plane. Therefore, in

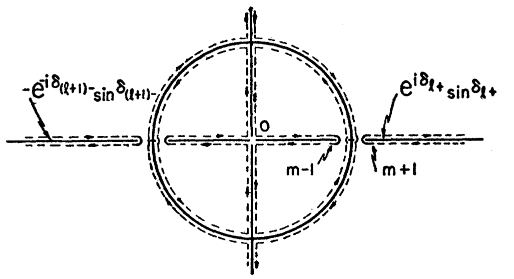

FIG. 2. The position of the singularities in the W plane of the partial-wave amplitudes for pion-nucleon scattering. The branch cuts are indicated by heavy lines; the dotted line is the contour of integration for the partial-wave dispersion relation.

order to apply the partial-wave dispersion relations we must evaluate the numerators of the integrands, which are the discontinuities across the various branch cuts. This can be done by examination and interpretation of Eq. (3.5) and the corresponding equation for $A(s,t)$.

Consider the discontinuity $\alpha_1(W)$ across the short branch cuts arising from the pole at $\bar{s}=m^2$. This term depends only on the pion-nucleon coupling constant. From Eq. (4.12) and the corresponding equation for A_l one can see that in the region of the short branch cuts [see Eqs. (4.14a) and (4.14b)]

$$\mathrm{Im}A_l^{(\pm)}(W)=0,$$
$$\mathrm{Im}B_l^{(\pm)}(W)=\mp\epsilon(W)\pi g_r^2P_l[x_0(W)]/2k^2, \quad (5.1)$$

where

$$\epsilon(W)=\ \ 1 \quad \text{for} \quad W>0$$
$$=-1 \quad \text{for} \quad W<0,$$

and where

$$x_0(W)=1+(\Sigma-W^2-m^2)/2k^2. \quad (5.2)$$

From the definition of h_l we then find that

$$\alpha_1^{l(\pm)}(W)=\mp[\epsilon(W)\pi g^2/8k^2][(W-m)k^{-2l}P_l(x_0)$$
$$+(E-m)^2(W+m)k^{-2l-2}P_{l+1}(x_0)]. \quad (5.3)$$

The discontinuity $\alpha_2^l(W)$ in the region $-m+1\leq W\leq m-1$ can be expressed in terms of pion-nucleon scattering cross sections by means of crossing symmetry. One finds from Eq. (4.12) that in this region[9]

$$\mathrm{Im}B_l^{(\pm)}(W)=\mp\frac{\epsilon(W)}{2k^2}\int_{L_1(s)}^{L_2(s)}d\bar{s}$$
$$\times P_l[x(W,\bar{s})]b_1^{(\pm)}(\bar{s},\Sigma-W^2-\bar{s}), \quad (5.4)$$

where

$$x(W,\bar{s})=1+(\Sigma-W^2-\bar{s})/2k^2. \quad (5.5)$$

[9] Strictly speaking, we should write $\mathrm{Re}b_1$ in Eq. (5.4), and similarly for all the absorptive parts in the remainder of this section. As mentioned in Sec. IV, the absorptive parts can become complex, but their imaginary parts do not contribute to the discontinuities $\alpha_i^l(W)$. This fact is not of immediate practical importance, because the Legendre polynomial expansions discussed below do not converge when the absorptive part becomes complex. We are indebted to Professor M. L. Goldberger for calling our attention to this point.

The same equation with the $\mp$ inverted holds when the B_l and b_1 are replaced by A_l and a_1. Then for $\alpha_2^l(W)$ we find

$$\alpha_2^{l(\pm)}(W)=\pm\frac{\epsilon(W)}{32\pi k^2}\int_{L_1(s)}^{L_2(s)}d\bar{s}\{k^{-2l}P_l(x)$$
$$\times[-a_1^{(\pm)}+(W-m)b_1^{(\pm)}]+(E-m)^2$$
$$\times k^{-2l-2}P_{l+1}(x)[a_1^{(\pm)}+(W+m)b_1^{(\pm)}]\}, \quad (5.6)$$

where the arguments of a_1 and b_1 are the same as in Eq. (5.4).

The quantities a_1 and b_1 have a direct physical meaning, since $\bar{s}$ and $\Sigma-s-\bar{s}$ are restricted by the limits of integration to values which are physically accessible in pion-nucleon scattering. For $\bar{s}$, the square of the energy, this is obvious from the limits. For $\Sigma-s-\bar{s}$, the momentum transfer, it can be seen by consideration of the cosine of the corresponding scattering angle $\bar{\theta}$; i.e.,

$$\cos\bar{\theta}=1+(\Sigma-s-\bar{s})/2\bar{k}^2, \quad (5.7)$$

where $\bar{k}^2$ is obtained from k^2 by substituting $\bar{s}$ for s in Eq. (2.5). One can easily show that $-1\leq\cos\bar{\theta}\leq1$ as long as $s>0$ and $\bar{s}$ lies within the limits of integration in Eq. (5.6).

If the energy and momentum transfer are in the physical region, it follows that $a_1=\mathrm{Im}A$, as discussed in Sec. III. One can then expand in partial waves, using the formulas of CGLN, to obtain

$$[a_1^{(\pm)}(\bar{s},\Sigma-s-\bar{s});b_1^{(\pm)}(\bar{s},\Sigma-s-\bar{s})]$$
$$=4\pi\left\{\frac{[\bar{W}+m;1]}{\bar{E}+m}\sum_{l=0}^{\infty}[\mathrm{Im}f_{l+}^{(\pm)}(\bar{W})P_{l+1}'(\cos\bar{\theta})\right.$$
$$-\mathrm{Im}f_{l-}^{(\pm)}(\bar{W})P_{l-1}'(\cos\bar{\theta})]-\frac{[\bar{W}-m;-1]}{\bar{E}-m}$$
$$\times\sum_{l=1}^{\infty}P_l'(\cos\bar{\theta})[\mathrm{Im}f_{l-}^{(\pm)}(\bar{W})$$
$$\left.-\mathrm{Im}f_{l+}^{(\pm)}(\bar{W})]\right\}, \quad (5.8)$$

where $\bar{W}$ and $\bar{E}$ are related to $\bar{s}$ in the same way as W and E are related to s. Utilization of Eqs. (5.8) and (5.6) now gives us a formula for $\alpha_2^l(W)$, but not an explicit one. It involves the amplitudes $f_{l\pm}$ which are, of course, the quantities which we are trying to determine. Thus a complete treatment of the pion-nucleon problem will involve coupled integral equations similar to those encountered in pion-pion scattering.[1]

A serious complication of the pion-nucleon problem appears in Eq. (5.6). The quantity $L_1(W^2)$ becomes infinite at $W=0$, permitting contributions to α_2^l from intermediate states of arbitrarily high energy. In the equal-mass case this does not occur for finite W.

We shall now calculate the discontinuity across the circle $W=re^{i\phi}$, where $r^2=m^2-1$, in terms of the process $\pi+\pi\rightarrow N+\bar{N}$. This discontinuity arises from the last term in Eq. (3.5), whose contribution to A_l is

$$A_l^{(\pm)}=\frac{1}{\pi}\int_{-1}^{1}dx\int_4^{\infty}dt'\frac{P_l(x)a_3^{(\pm)}(t',W^2)}{t'+2k^2(1-x)}+\cdots, \quad (5.9)$$

where k^2 is, of course, a function of W. From Eq. (2.5) one finds that for any point on the circle, $k^2(W)$ takes on the value

$$k^2(re^{i\phi})=\tfrac{1}{4}(2r^2\cos2\phi-\Sigma), \quad (5.10)$$

whereas at points just inside or outside the circle $k^2(W)$ has the value

$$k^2[(r\pm\epsilon)e^{i\phi}]=k^2(re^{i\phi})\pm i\epsilon\sin2\phi. \quad (5.11)$$

Now from Eq. (5.9) one finds that for $k^2\leq -1$,

$$\frac{1}{2i}[A_l^{(\pm)}(k^2+i\epsilon)-A_l^{(\pm)}(k^2-i\epsilon)]=\frac{1}{2k^2}\int_4^{-4k^2}dt\,P_l\left(1+\frac{t}{2k^2}\right)a_3^{(\pm)}(t,W^2). \quad (5.12)$$

The corresponding equation with B_l and b_3 is identical. Using the two equations above, we are now able to calculate the discontinuity $\alpha_3^l(\phi)$ across the circle, where

$$\alpha_3^{l(\pm)}(\phi)\equiv\frac{1}{2i}\{h_l^{(\pm)}[(r+\epsilon)e^{i\phi}]-h_l^{(\pm)}[(r-\epsilon)e^{i\phi}]\}. \quad (5.13)$$

We find that

$$\alpha_3^{l(\pm)}(\phi)=\frac{\epsilon(\sin2\phi)}{32\pi k^2}\int_4^{-4k^2}dt\left\{k^{-2l}P_l\left(1+\frac{t}{2k^2}\right)\times[a_3^{(\pm)}(t,W^2)+(W-m)b_3^{(\pm)}(t,W^2)]+(E-m)^2k^{-2l-2}P_{l+1}\left(1+\frac{t}{2k^2}\right)\times[-a_3^{(\pm)}(t,W^2)+(W+m)b_3^{(\pm)}(t,W^2)]\right\}. \quad (5.14)$$

It should be remembered that in this equation the quantities k^2, E, and W are functions of ϕ, since $W=re^{i\phi}$.

We evaluate $a_3(t,re^{i\phi})$ by recognizing that for the range of the variables in Eq. (5.14), it follows from the considerations of Sec. III that $a_3=\mathrm{Im}A$ for process III, $\pi+\pi\rightarrow N+\bar{N}$. This process has been calculated in terms of phase shifts for pion-pion scattering in reference 6, for the low-energy range of t. In order to use these results directly, one must make an expansion of a_3 in states of definite total angular momentum and helicity. From reference 6 we find, retaining the notation defined there, that

$$a_3^{(\pm)}(t,s)=\sum_{J=0}^{\infty}(J+\tfrac{1}{2})(8\pi/p_-^2)(ip_-q)^J\times[P_J(\cos\theta_3)\,\mathrm{Im}f_+^J(t)-[J(J+1)]^{-\frac{1}{2}}\times\cos\theta_3 mP_J'(\cos\theta_3)\,\mathrm{Im}f_-^J(t)], \quad (5.15)$$

$$b_3^{(\pm)}(t,s)=8\pi\sum_{J=1}^{\infty}(J+\tfrac{1}{2})[J(J+1)]^{-\frac{1}{2}}(ip_-q)^{J-1}\times P_J'(\cos\theta_3)\,\mathrm{Im}f_-^J(t),$$

where θ_3 is the scattering angle for process III, namely,

$$\cos\theta_3=(s-p_-^2+q^2)/2ip_-q. \quad (5.16)$$

For the superscript (+), the sum runs over even J; for (−), odd J.

For W on the circle, $\cos\theta_3$ is complex, and we must investigate the convergence of the series in Eq. (5.15). Since a function $f(x)$ that is analytic inside an ellipse with foci at $x=\pm1$ can be expanded in Legendre polynomials, we must find out from the Mandelstam representation which singularity limits the size of the ellipse. This singularity can be seen to come from the vanishing of the denominator $s'-s$ in the region where $a_{13}(s,t)\neq0$. The boundary of this region is given by Eq. (4.10) and Fig. 2 of reference 6. We find after a numerical calculation that the expansion converges on the circle only for those values of W for which $-33°\lesssim\phi\lesssim33°$ or for which $-33°\lesssim\pi-\phi\lesssim33°$. The expansion would, however, be of little use even if it were to converge in a larger region, since we can see from the upper limit of integration in Eq. (5.14) that at $\phi\sim33°$ energies as high as $t\sim m^2$ become important. The range of the angle ϕ in which a given value of t contributes to the discontinuity across the circle is given by

$$\cos2\phi\leq(\Sigma-t)/2r^2. \quad (5.17)$$

Finally, let us calculate the discontinuity across the imaginary axis, $W=iy$. In this region both the last two terms in Eq. (3.5) contribute, giving

$$\frac{1}{2i}[A_l^{(\pm)}(iy+\epsilon)-A_l^{(\pm)}(iy-\epsilon)]=\epsilon(y)\epsilon(y^2-r^2)\frac{1}{2k^2}\int_4^{-4k^2}dt\,P_l\left(1+\frac{t}{2k^2}\right)\times a_3^{(\pm)}(t,-y^2)\pm\frac{\epsilon(y)}{2k^2}\int_{(m+1)^2}^{L_2(-y^2)}d\bar{s}\times P_l\left(1+\frac{\Sigma+y^2-\bar{s}}{2k^2}\right)a_1^{(\pm)}(\bar{s},\Sigma+y^2-\bar{s}), \quad (5.18)$$

where

$$k^2=-[y^2+(m+1)^2][y^2+(m-1)^2]/4y^2.$$

The same holds for B_l, but with the $\pm$ on the last term of Eq. (5.17) inverted. We could now substitute into Eq. (4.11b) to obtain an expression for the discontinuity $\alpha_4{}^l(y)$, defined as

$$\alpha_4{}^l(y)=(1/2i)[h_l(iy+\epsilon)-h_l(iy-\epsilon)]. \quad (5.19)$$

We shall not write down the resulting formula, because it is difficult to see how to put it to any immediate practical use. One difficulty is that the momentum transfer variables in both a_1 and a_3 lie in unphysical regions. Analytic continuation by Legendre polynomial expansions is impossible, because an analysis of the type described above shows that the expansions of Eqs. (5.8) and (5.15) converge only for $y^2 \lesssim 3$. A second difficulty is that the energy variables in both a_1 and a_3 range through very high values, at which we have no reliable means of calculation.

VI. CHEW-LOW EFFECTIVE-RANGE FORMULA

One approximate solution of the partial-wave dispersion relations can be written down very easily. We designate by $h_{33}(W)$ the amplitude for the (3,3) state; i.e.,

$$h_{33}(W)\equiv h_1{}^{(+)}(W)-h_1{}^{(-)}(W). \quad (6.1)$$

Let us consider the low-energy approximation of neglecting all but the closest singularities to the physical region; i.e., the physical cut and the short branch cut coming from the pole in $\bar{s}$. The resulting truncated dispersion relation is then

$$h_{33}(W)\approx\frac{1}{\pi}\int_{m-1/m}^{(m^2+2)^{\frac{1}{2}}} dW'\frac{\alpha_1{}^{1(+)}(W')-\alpha_1{}^{1(-)}(W')}{W'-W} +\frac{1}{\pi}\int_{m+1}^{\infty} dW'\frac{\text{Im}h_{33}(W')}{W'-W}. \quad (6.2)$$

Moreover we observe, following CGLN, that the short branch cut can be approximated by a pole:

$$h_{33}\approx\frac{\Gamma}{m-W}+\frac{1}{\pi}\int_{m+1}^{\infty} dW'\frac{\text{Im}h_{33}(W')}{W'-W}. \quad (6.3)$$

where

$$\Gamma=\frac{1}{\pi}\int_{m-1/m}^{(m^2+2)^{\frac{1}{2}}} dW'[\alpha_1{}^{1(+)}(W')-\alpha_1{}^{1(-)}(W')] \approx -\tfrac{2}{3}f^2, \quad (6.4)$$

neglecting corrections of order $1/m$.

Following Chew and Mandelstam,[1] we set $h_{33}(W) = N(W)/D(W)$, and require $N(W)$ to contain the pole while $D(W)$ contains the physical cut. Then writing a dispersion relation with three subtractions for $D(W)$, and using the unitarity condition to determine the imaginary part, one finds

$$\frac{2}{3}f^2\frac{k^3\cot\delta_{33}}{\omega}\frac{E+m}{W}=1-\frac{\omega}{\omega_r}+P\omega^2 +\frac{\Gamma\omega^3}{\pi}\mathcal{P}\int_{m+1}^{\infty} dW'\frac{k'^3(E'+m)}{W'(W'-m)^4(W'-W)}, \quad (6.5)$$

where $\omega=W-m$. In the static limit the first two terms on the right-hand side give the Chew-Low effective-range formula[4]:

$$\frac{4}{3}f^2\frac{k^3\cot\delta_{33}}{\omega}=1-\frac{\omega}{\omega_r}. \quad (6.6)$$

It will, of course, be possible to go beyond this approximation, by including singularities which are farther from the physical region. In principle it will be possible to derive improved effective-range formulas and to give theoretical explanations of such quantities as the S-wave scattering lengths and the position of the (3,3) resonance. This paper is intended primarily as a tool to be used in such investigations, which are now in progress.

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